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From public good index to public value. An axiomatic approach and generalization

by

Manfred J. Holler and Xiaoguang Li

Institute of Economics,
University of Hamburg,
Von-Melle-Park 5,
D-20146 Hamburg,
Germany

Holler (1982) and Holler and Packel (1983) introduced the public good index and developed an axiomatization of it. In this paper, the public value, which measures the absolute power of a player in deciding on a public good whereas the public good index measures the relative power, is introduced through an alternative axiomatic approach for simple games. Using this axiomatic approach, the concept of the public value is further extended to general games. Then, two examples are given to illustrate the application of the public value to analyse the distribution of power among players in both simple and general games. Finally the feature of the public value and its relation to the public good index is discussed.

Keywords: Public value, simple game, HP-axiom, L-axiom, mergeability, MWC, RGC, public good index.

1. Introduction

It is standard to study the distribution of power in voting bodies by the Shapley value, the Banzhaf-Coleman index or variations of such measures. In Holler (1982), it has been argued that those measures are inadequate inasmuch as the collective decisions of a voting body result in collective (i.e., public) goods. As a consequence of this argument, the public good index (PGI) has been proposed. Holler and Packel (1983) give an axiomatization of this index in order to analyse its characteristics. Unfortunately, the axiom of mergeability, which is essential for this axiomatization, is not very lucid, and hence the application of the public good index to voting games (a special case of simple games) is insufficient in that the results cannot be straightforwardly interpreted with respect to the properties implicit to the mergeability axiom. The extension of the PGI to general games will encounter some difficulties if we follow the axiomatic approach developed by Holler and Packel (1983). In the following section, two new axioms are introduced in order to define an alternative measure of the power of a player to decide public goods, termed the public value, which is explicitly based on the definition of public goods. In Section 3, the public value is extended to general games. Section 4 discusses the features of the public value and its relation to the public good index.

2. An alternative measure of power – the public value

As the starting point of our study, the public good index (PGI), introduced by Holler (1982) and Holler and Packel (1983) for simple games, should be first reviewed. In order to do this and to prepare the introduction of the alternative measure of power – the public value, we consider the following concepts and the comments on them.

DEFINITION 2.1 *Let (N, v) be a simple game. That is, $v(S) = 1$ or $v(S) = 0$ for all $S \subseteq N$, and $v(\emptyset) = 0$. If a coalition $S \subseteq N$ satisfies $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$, then S is called a minimum winning coalition (MWC). If player i does not belong to any MWC, then i is called a dummy in the game v.*

The collection of all MWCs of the game v is denoted by $M(v)$.

The MWC concept was presented by Deegan and Packel (1979). In a simple game, any non-critical player is not decisive for the winning of his coalition, so he has no power to make a decision, and further he has no incentive to be a member of that coalition. Thus only MWCs are likely to form intentionally (not by luck), and only the players who are the members of some MWCs are supposed to have decisive power. Furthermore, the more MWCs of which a player is a member, the greater power he has in this game. Therefore how much power a player has depends upon how many MWCs he belongs to.

REMARK 2.1

(a) *For any $T \subseteq N$, if $v(T) = 1$, and $T \notin M(v)$, then there exists an $S \in M(v)$ such that $S \subset T$. This means that T can be contracted to an MWC S.*
(b) *For any $S \in M(v)$ and any player $i \in S$, we know $v(S) = 1$, and $v(S \setminus \{i\}) = 0$. Thus each $S \in M(v)$ contains only decisive players and no dummies.*

To analyse the situation of deciding on public goods, a measure of power, as argued by Holler and Packel (1983), should reflect the public good aspect of the outcome (value) of a coalition. **The players** in a coalition $S \in M(v)$ **consider the value $v(S)$ as a public good, therefore, they do not divide this value but jointly enjoy it.** Every player in S takes advantage of using this public good, $v(S)$, without rivalry in consumption.

The public good power index (PGI) measuring the distribution of power between players in deciding on public goods has been so far known as the only measure which is consistent with the above argument. The following restates the standard definition of it.

DEFINITION 2.2 *Let $M(u)$ be the collection of all MWCs in the simple game u. The public good index (PGI) for player i is defined by*

$$h_i(u) = \frac{c_i}{\sum_{j=1}^{n} c_j},$$

where c_i and c_j are the numbers of the MWCs containing players i and j, respectively.

The definition of PGI is based on intuitive understanding of the public good concept. Holler and Packel (1983) also characterized this measure by means of a set of axioms.

We now repeat, briefly, their axiomatic approach to PGI.

DEFINITION 2.3 *The sum $u \oplus v$ of two simple games u and v is defined as follows:*

$$u \oplus v(S) = \begin{cases} 1 & \text{if } u(S) = 1 \text{ or } v(S) = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for any } S \subseteq N.$$

DEFINITION 2.4 *Two simple games u and v are said to be mergeable if $S_1 \in M(u)$ and $S_2 \in M(v)$ implies $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$.*

REMARK 2.2 *It is easy to see that if u and v are mergeable, then $c_i(u \oplus v) = c_i(u) + c_i(v)$ where $c_i(u)$ and $c_i(v)$ are the numbers of MWCs containing i in the games u and v, respectively.*

DEFINITION 2.5 *(HP-axioms) Let h be a map from the collection of all n-person simple games to n-dimensional Euclidean space $\mathbf{R}^n$. the Holler-Packel axioms are defined by*

HP1. If player i is a dummy, then $h_i(v) = 0$.

HP2. $\sum_{i=1}^{n} h_i(v) = 1$

HP3. If π is a permutation, then $h_{\pi(i)}(\pi v) = h_i(v)$, where πv is the game defined by $\pi v(S) = v(\pi(S))$ for any $S \subseteq N$.

HP4. If u and v are mergeable, then for any $i \in N$, we have $h_i(u \oplus v) = [c(u)h_i(u) + c(v)h_i(v)]/[c(u) + c(v)]$ where $c(u) = \sum_{i=1}^{n} c_i(u)$ and $c(v) = \sum_{i=1}^{n} c_i(v)$.

Holler and Packel (1983) proved that the PGI is the unique power index satisfying the axioms HP1 through HP4. The Holler-Packel axioms characterize the PGI fully, and so they can be regarded as an alternative, equivalent way of defining the PGI for simple games.

We now ask how to measure the power of a player in producing public goods in general situations. In order to answer this question we need to establish a new measure of power. Axiomatization will still be an appropriate method to accomplish this task.

Before we begin to define this new measure, the axiomatization of the PGI should be reexamined. It is clear that the public good characteristics of the PGI are implicitly contained in the formula in Definition 2.5, but not expressed explicitly in the HP-axioms. HP1 says that any non-decisive player has no power. HP2 indicates that the PGI is a normalized value. The normality of the PGI is a strong requirement that may make the process of generalization of the PGI to general games through axiomatization very complicated. HP3 means that the measure of power should be independent of the labels of the players. Besides HP2, HP4 may be difficult to interprete. In Holler and Packel (1983), no thorough interpretation of the mergeability axiom (HP4) was given, the authors "offer no compelling story for its plausibility" (see, p. 27 of the paper mentioned). HP4 looks quite complex and somewhat *ad hoc.* The complex form of HP4 could also make it very difficult to generalize the PGI from simple games to general games by means of the HP-axioms.

In order to develop a new measure of power of a player in deciding on public goods for general games, we need an alternative axiomatic approach to define a new concept of value. Precisely speaking, in our axiomatic approach, HP2 is no longer relevant, and HP4 needs to be amended.

In this section we first define the new value concept through axiomatization for simple games. Two new axioms (L-axioms) are introduced as follows.

DEFINITION 2.6 *(L-axioms) Let p be a map from the collection of all n-person simple games to n-dimensinal Euclidean space $\mathbf{R}^n$. The L-aximos are defined by*

L1. If a simple game v has only one $S \in M(v)$, then $\max_{i \in S}\{p_i(u)\} = v(S)$ (=1 for simple games).

L2. If two simple games u and v are mergeable, then for any $i \in N$, $p_i(u \oplus v) = p_i(u) + p_i(v)$.

We will now introduce the concept of the public value.

DEFINITION 2.7 *Let p be a map from the collection of all n-person simple games to n-dimensional Euclidean space $\mathbf{R}^n$, i.e., for any simple game (N, v), $p(v) = (p_1(v), \ldots, p_n(v)) \in \mathbf{R}^n$. The map p is called a p-map if p satisfies the axioms HP1, HP3, L1, and L2. The vector value $p(v) = (p_1(v), \ldots, p_n(v))$ is called a public value (p-value) of the game v.*

The main ideas underlying the public value and its axiomatization are as follows:

(a) All the members of a coalition $S \in M(v)$ should be assigned the value of S to express their powers for the winning of S. Furthermore, the value $v(S)$ should be regarded as a public good for the members of S.

The above requirement is met by the axiom L1. If $p_i(u)$ is supposed to be individual consumption of player i, and $v(S)$ represents the aggregate consumption of the coalition S as a whole, the formula $\max_{i \in S}\{p_i(u)\} = v(S)$ means that the change in the individual consumption of a player in S will not affect the consumption of any other player in S, e.g., an increase in $p_i(u)$ does not cause any decrease in $p_k(u)$, where i, $k \in S$. By the theory of public good, the value $v(S)$ is exactly a pure non-rival good, and so a public good for all members of S (see Cowell, 1986, p.117, for a standard definition of a public good).

(b) The axiom L2 is a substitute for HP4. The mergeability of two games u and v is useful for our axiomatic approach. The sum game $u \oplus v$ can be decomposed into two component games u and v, and then they can be analysed separately. Imagine that the game $u \oplus v$ will be played in two stages, in the first stage u is played, in the second, v is played, and the MWCs in u are no longer MWCs in v, and *vice versa*. The axiom L2 just decribes a plain and reasonable story that the power of a player in $u \oplus v$ should be equal to the sum of his powers in the two games u and v. Moreover, L2 has a simple and clean form. As we shall see later, it is easier to generalize the p-value than the PGI to general games through our axiomatic approach.

The following theorem will fully characterize, and give the explicit expression of, the p-value.

THEOREM 2.1 *Let c_i be the number of all MWCs containing player i. Then $p_i(v)$ is the p-value for i if and only if $p_i(v) = c_i$.*

To prove this theorem, we need to prove some lemmas first.

LEMMA 2.1 *From any simple game (N, v) and $S \in M(v)$, we can define a new simple game v_s by*

$$v_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \text{ and } v(T) = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for any } T \subseteq N.$$

Then the following result is true:

$$p_i(v_S) = \begin{cases} 1 & \text{if } \ i \in S \\ 0 & \text{if } \ i \in N \setminus S \end{cases}$$

Proof: First, it is easy to see that S is the unique MWC in v_S. From HP1, we can further know that if $i \in N \setminus S$, then $p_i(v_s) = 0$. From L1, we have $\max_{i \in S}\{p_i(v_s)\} = v(S)(= 1)$. Assume now that the maximum value of the set $\{p_i(v_S)\}$ is taken at $p_{i^*}(v_S)$, where $i^* \in S$. We can now prove that for any player $i \in S$, $p_i(v_S) = p_{i^*}(v_S)$. Consider the following specific permutation π' of N:

$$\pi'(j) = \begin{cases} i & \text{if } \ j = i^* \\ i^* & \text{if } \ j = i \\ j & \text{if } \ j \in N \setminus \{i^*, i\} \end{cases}$$

Note that for this permutation π', we have $\pi'(S) = S$. Given any $T \subset N$, take the subset $T' \subset N$ satisfying $\pi'(T') = T$. Then it is true that $S \subseteq T$ if and only if $S \subseteq T'$. Moreover we have

$$\pi' v_S(T) = v_S(T') = \begin{cases} 1 & \text{if } S \subseteq T' \text{ and } v(T') = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } S \subseteq T \text{ and } v(T) = 1 \\ 0 & \text{otherwise} \end{cases} = v_S(T).$$

This means that $\pi' v_s = v_s$. Therefore the following result is true for any arbitrarily chosen $i \in S$:

$$p_i(v_S) = p_i(\pi' v_S) = p_{\pi'(i^*)}(\pi' v_s) = p_{i^*}(v_S)(= 1).$$

■

It should be noticed that the game V_S is substantially important not only for our axiomatic approach but for Holler-Packel's approach. Holler and Packel (1993) do not mention how the component game v_j corresponcing to the MWC S_j is defined, and hence one is not convinced of the existence of such a v_j and of the verity of the equality $v = v_1 \oplus v_2 \cdots \oplus v_{k-1} \oplus v_k$. These questions will be completely answered in this section.

LEMMA 2.2 *Assume that in a simple game v, there are only two MWCs S_1 and S_2. We can also define a game v_i with respect to S_i, $i = 1, 2$, by*

$$v_S(T) = \begin{cases} 1 & \textit{if } S \subseteq T \textit{ and } v(T) = 1 \\ 0 & \textit{otherwise} \end{cases} \quad \textit{for any } T \subseteq N.$$

Then the following statements are true:

(1) v_1 and V_2 are mergeable,

(2) $v = v_1 \oplus v_2$,

(3) $M(v_1 \oplus v_2) = \{S_1, S_2\}$.

Proof: 1) is obvious. Let us prove 2). Given any $T \subseteq N$, if $v(T) = 0$, then $v_1(T) = 0$, and $v_2(T) = 0$ so $v_1 \oplus v_2(T) = 0$. If $v(T) = 1$, from a) of Remark 2.1, we know that either the MWC S_1 or S_2 must be contained in T, so $v_1(T) = 1$ or $v_2(T) = 1$, and thus $v_1 \oplus v_2(T) = 1$. Therefore we obtain $v = v_1 \oplus v_2$.

We now prove 3). First, $S_1, S_2 \in M(v_1 \oplus v_2)$ is obviously true. For any $T \subseteq N$ with $v_1 \oplus v_2(T) = 1$, we know that $v_1(T) = 1$ or $v_2(T) = 1$, and then $S_1 \subseteq T$ or $S_2 \subseteq T$. So if $T \neq S_1$ and $T \neq S_2$, then T cannot be an MWC in $v_1 \oplus v_2$. Thus $M(v_1 \oplus v_2) = \{S_1, S_2\}$. ■

REMARK 2.3 *Assume that there are only three MWCs S_1, S_2, and S_3 in a simple game v. Similarly, the games v_1, v_2, and v_3 corresponding to S_1, S_2, and S_3, respectively, can also be well defined. It is easy to see that $v_1 \oplus v_2$ and v_3 are mergeable.*

The next important result we should show is $(v_1 \oplus v_2) \oplus v_3 = v_1 \oplus (v_2 \oplus v_3)$. In fact,

$$(v_1 \oplus v_2) \oplus v_3(T) = \begin{cases} 1 & \text{if} \quad [v_1 \oplus v_2](T) = 1 \quad \text{or} \quad v_3(T) = 1 \\ 0 & \text{if} \quad [v_1 \oplus v_2](T) = 0 \quad \text{and} \quad v_3(T) = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if} \quad v_1(T) = 1 \quad \text{or} \quad v_2(T) = 1 \quad \text{or} \quad v_3(T) = 1 \\ 0 & \text{if} \quad v_1(T) = 0 \quad \text{and} \quad v_2(T) = 0 \quad \text{and} \quad v_3(T) = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if} \quad [v_2 \oplus v_3](T) = 1 \quad \text{or} \quad v_1(T) = 1 \\ 0 & \text{if} \quad [v_2 \oplus v_3](T) = 0 \quad \text{and} \quad v_1(T) = 0 \end{cases}.$$

This proves the result. Moreover, given k games $v_1, \ldots, v_k$, then $v_1 \oplus v_2 \oplus \cdots \oplus v_k$ can be well defined by

$$v_1 \oplus v_2 \oplus \cdots \oplus v_k(T)$$

$$= \begin{cases} 1 & \text{if} \quad v_1(T) = 1 \quad \text{or} \quad v_2(T) = 1 \quad \text{or} \quad \cdots \quad v_k(T) = 1 \\ 0 & \text{if} \quad v_1(T) = 0 \quad \text{and} \quad v_2(T) = 0 \quad \text{and} \quad \cdots \quad v_k(T) = 0 \end{cases}.$$

Then, the following result can be proved

$$v_1 \oplus v_2 \oplus \cdots \oplus v_k = ((\cdots (v_1 \oplus v_2) \cdots) \oplus v_{k-1}) \oplus v_k.$$

Furthermore, through the induction procedure, we can extend the results in Lemma 2.2 to the case of a simple game v with k MWCs, $S_1, S_2, \ldots, S_k$. This leads to the following lemma.

LEMMA 2.3 *Assume that a simple game v has only k MWCs $S_1, \ldots, S_k$. We define k games $v_1, \ldots, v_k$ with respect to $S_1, \ldots, S_k$, respectively, by*

$$v_i(T) = \begin{cases} 1 & \textit{if } S_i \subseteq T \textit{ and } v(T) = 1 \\ 0 & \textit{otherwise} \end{cases} \qquad \textit{for any } T \subseteq N.$$

Then 1) $v_1, \ldots, v_k$ are mergeable, 2) $v = v_1 \oplus \cdots \oplus v_k$ and 3) $M(v_1 \oplus \cdots \oplus v_k) = M(v_1) \cup M(v_2) \cup \cdots \cup M(v_k)$.

The proof of this lemma is similar to that of the last one.

We now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1: We first denote all the MWCs of v by $\{S_1, S_2, \ldots, S_k\}$. According to Lemma 2.1 and Remark 2.3, the game v can be represented by $v = v_1 \oplus v_2 \oplus \cdots \oplus v_k$. Applying L2 to v sequentially, we get $p_i(v) = p_i[((\cdots (v_1 \oplus v_2) \cdots) \oplus v_{k-1}) \oplus v_k] = p_i((\cdots (v_1 \oplus v_2) \cdots) \oplus v_{k-1}) + p_i(v_k) = \cdots = p_i(v_1) + p_i(v_2) + \cdots + p_i(v_k)$.

According to Lemma 2.1, we know that if $i \in S_j$, then $p_i(v_j) = v_j(S_j) = 1$; if $i \notin S_m$, $p_i(v_m) = 0$. Therefore, we have

$$p_i(v) = \sum_{i \in S_j,\, S_j \in M(v)} v(s_j) = c_i.$$

■

In order to see briefly the relationship between the p-value and the PGI, let p_i be divided by the sum of powers of all players, $\sum_{i=1}^{n} p_i(v)$. Thus we have

$$p_i(v)/\sum_{i=1}^{n} p_i(v) = c_i/\sum_{i=1}^{n} c_i(v) = \hbar_i(v).$$

3. Generalization of p-value from simple games to general ones

Holler and Packel (1983) mentioned that the public good index (PGI) may be extended from simple games to general games of a characteristic function form. However, we feel that generalizing the PGI by means of the HP-axioms will cause a problem because of the complexity of the axiom HP4. This difficulty now can be avoided by generalizing the p-value (rather than the PGI) through the alternative axiomatic approach proposed in the last section. In this section we will generalize the concept of the public value to general games. To start with, we need to extend some preliminary concepts from simple games to general ones, which is essential to the definition of the p-value and the axiomatic approach to it.

First of all, for the concept of the minimum winning coalition in a simple game, we should define an equivalent concept in a general game.

DEFINITION 3.1 *For an n-person general game (N, v), a subset $S \subseteq N$ is called a real gaining coalition (RGC) if for any $T \subset S$, $v(S) - v(T) > 0$. The collection of all the RGCs in the game v is denoted by $R(v)$. A player i is called a dummy if i does not belong to any RGC.*

We can immediatly see that for a simple game v, $R(v) = M(v)$.

REMARK 3.1 *(a) Recall the carrier concept of a game v. $F \subseteq N$ is called a carrier if and only if for any $T \subseteq N$, $v(T) = v(T \cap F)$. The intersection set C of all carriers is also a carrier which is called the smallest carrier. Any $S \in R(v)$ must be a subset of the smallest carrier C. To show this, assume, contrary to our claim, there is $i \in S$ such that $i \notin C$. Then we will find the following contradiction: $v(S/\{i\}) = v[(S/\{i\}) \cap C] = v(S \cap C) = v(S) > v(S/\{i\})$. Thus $S \subseteq C$ must be true.*

(b) For any subset $T \subseteq N$ with $v(T) > 0$, there must exist an $S \in R(v)$ such that $S \subseteq T$.

The following will prove b). If $T \in R(v)$, the proof is completed. If T is not an RGC, there exist an $S_1 \subset T$ such that $v(S_1) \geq v(T)$. If $S_1 \in R(v)$, the proof ends. If S_1 is not an RGC, there exists an $S_2 \subset S_1$ such that $v(S_2) \geq v(S_1)$. Continuing in this fashion, we will discover that 1) we find some $S_k \subset T$ such that $S_k \in R(v)$ and $v(S_k) \geq v(T)$, or 2) eventually we can find a singleton $\{j\} \subset T$ with $v(j) \geq v(T)$. Obviously $\{j\} \in R(v)$. The proof is completed. This means that we can actually contact the set T to an RGC S with $v(S) \geq v(T)$.

The concepts of the sum and the mergeability of two games are also important for our axiomatic approach, since they will be used to decompose a game into several component games, and further axioms will be applied to this game with the decomposed form. The two concepts will be generalized by the following two definitions.

DEFINITION 3.2 *For two general games v_1 and v_2, we define the sum $v_1 \oplus v_2$ of v_1 and v_2 by*

$$v_1 \oplus v_2(S) = \max\{v_1(S),\ v_2(S)\}.$$

DEFINITION 3.3 *We say that two general games v_1, v_2 are mergeable if $S_1 \in R(v_1)$ and $S_2 \in R(v_2)$ implies $S_1 \neq S_2$, i.e., $R(v_1) \cap R(v_2) = \emptyset$.*

REMARK 3.2 $(v_1 \oplus v_2) \oplus v_3 = v_1 \oplus (v_2 \oplus v_3)$.

In fact, $(v_1 \oplus v_2) \oplus v_3 = \max\{(v_1 \oplus v_2), v_3\} = \max\{\max\{v_1, v_2\}, v_3\} = \max\{v_1, v_2, v_3\} = \max\{v_1, \max\{v_2, v_3\}\} = \max\{v_1, (v_2 \oplus v_3)\} = v_1 \oplus (v_2 \oplus v_3)$.

In general, for k games $v_1, v_2, \ldots, v_k$, we can define $v_1 \oplus v_2 \oplus \cdots \oplus v_k$ by $v_1 \oplus v_2 \oplus \cdots \oplus v_k = \max\{v_1, v_2, \ldots, v_k\}$. It is easy to show that $v_1 \oplus v_2 \oplus \cdots \oplus v_k = ((\cdots(v_1 \oplus v_2)\cdots) \oplus v_{k-1}) \oplus v_k$.

In the following we will propose a method of decomposition of a general game v, which can be regarded as a generalization of the decomposition method for a simple game.

LEMMA 3.1 *Assume that a ganearl game v has only two RGCs S_1 and S_2. We now define two games v_1 and v_2 with respect to S_1 and S_2, respectively, by*

$$v_i(T) = \begin{cases} v(S_i) & \text{if } T \supseteq S_i \text{ and } v(T) \geq v(S_i) \\ v(T) & \text{if } T \supseteq S_i \text{ and } v(T) < v(S_i) \\ 0 & \text{otherwise} \end{cases} \quad \text{for any } T \subset N,\ i = 1, 2.$$

Then 1) v_1 and v_2 are mergeable, 2) $v = v_1 \oplus v_2$, and 3) $R(v_1 \oplus v_2) = R(v_1) \cup R(v_2)$.

Proof: 1) We first prove $R(v_1) = \{S_1\}$. From $v_1(S_1) = v(S_1) > 0$ (for $S_1 \in R(v)$) and $v_1(S) = 0$, for any $S \subset S_1$, we have $S_1 \in R(v_1)$. For any $T \subseteq N$, $T \neq S_1$, if $S_1 \not\subset T$, $v_1(T) = 0$. Then T is not an RGC of v_1. If $S_1 \subset T$, we have

$$v_1(T) = \begin{cases} v(S_1) & \text{if} \quad v(T) \geq v(S_1) \\ v(T) & \text{if} \quad v(T) < v(S_1) \end{cases}.$$

Then $v_1(T) \leq v_1(S_1)$ for this case. Thus T is not an RGC of v_1. Therefore $R(v_1) = \{S_1\}$. In a similar way, we can also prove $R(v_2) = \{S_2\}$. So we get $R(v_1) \cap R(v_2) = \emptyset$. This leads to the conclusion that v_1 and v_2 are mergeable.

2) To prove $v = v_1 \oplus v_2$, consider any $T \subseteq N$. If $v(T) = 0$, we know that $v_1(T) = v_2(T) = 0$, and hence $v = v_1 \oplus v_2$. If $v(T) > 0$, from b) of Remark 3.1, we can find at least one $S \in R(v)$ such that $S \subseteq T$ and $v(S) \geq v(T)$. Take S_i satisfying $v(S_i) = \max\{v(S_j) | S_j \in R(v) \text{ and } S_j \subset T \text{ and } v(S_j) \geq v(T)\}$, where $j = 1$ or 2. Then $v_i(T) = v(T)$. For other $k \neq i$, $k = 1$ or 2, if $S_k \not\subseteq T$, then $v_k(T) = 0$. If $S_k \subseteq T$, we discuss further the following two cases: i) $v(S_k) \leq v(T)$, we can have $v_k(T) = v(S_k) \leq v(S_i) = v(T)$; ii) $v(S_k) > v(T)$, we can know $v_k(T) = v(T)$. Therefore we can get $v_1 \oplus v_2(T) = \max_k\{v_i(T), v_k(T)\} = v_i(T) = v(T)$. So, $v = v_1 \oplus v_2$.

3) $R(v_1 \oplus v_2) = R(v) = \{S_1, S_2\}) = R(v_1) \cup R(v_2)$. ■

We can extend Lemma 3.1 to the general case in which a game v has m RGCs $S_1, S_2, \ldots, S_m$. So we obtain the following lemma.

LEMMA 3.2 *Assume that a game v has only m RGCs $S_1, \ldots, S_m$. If we define m games $v_1, \ldots, v_m$ with respect to $S_1, \ldots, S_m$, respectively, by*

$$v_i(T) = \begin{cases} v(S_i) & \text{if } T \supseteq S_i \text{ and } v(T) \geq v(S_i) \\ v(T) & \text{if } T \supseteq S_i \text{ and } v(T) < v(S_i) \\ 0 & \text{otherwise} \end{cases}$$

for any $T \subset N$, $i = 1, 2, \ldots, m$,

then 1) $v_1, \ldots, v_m$ are mergeable, 2) $v = v_1 \oplus \cdots \oplus v_m$ and 3) $R(v_1 \oplus \cdots \oplus v_m) = R(v_1) \cup R(v_2) \cup \ldots \cup R(v_m)$.

This lemma can be proved by following the exact method of proving Lemma 3.1. For convenience of speaking, we call each v_k a component game of the game v. The concept of the p-value can now be formally extended to general games as follows.

DEFINITION 3.4 *Let p be a map from the collection of all n-person general games to the n-dimensional Euclidean space $\mathbf{R}^n$. That is, for any game (N, v), we have $p(v) = (p_1(v), \ldots, p_n(v)) \in \mathbf{R}^n$. The map p is called a p-map if p satisfies the axioms HP1, HP3, L1 and L2, where all the above axioms are applied to general games, the term MWC should be replaced by the RGC, and the sum and the mergeability of two games should follow Definition 3.2 and Definition 3.3, respectively. The vector value $p(v) = (p_1(v), \ldots, p_n(v))$ is called a public value (p-value) for the game v.*

Similar to the pattern of dealing with other values (or indices), we now want to obtain a specific formula for the p-value. the following theorem, as the main result of this paper, meets this demand.

THEOREM 3.1 *Let (N, v) be a general game. Then an n-vector $p(v) = (p_1(v), \ldots, p_n(v))$ is the p-value for v if and only if for each player $i \in N$, $p_i(v) = \sum_{S_j \ni i,\, S_j \in R(v)} v(S_j)$.*

The following lemma will be helpful for the proof of the above theorem.

LEMMA 3.3 *Consider each component game v_i, with respect to the RGC S_i, of the game v. Applying HP1, HP3, and L1 to it, we can get*

$$p_j(v_i) = \begin{cases} v(S_i) & \text{if } \ i \in S_i \\ 0 & \text{if } \ i \in N \setminus S_i \end{cases}$$

for each player j.

The proof of this lemma is similar to that of Lemma 2.1.

We will now prove Theorem 3.1.

Proof of Theorem 3.1: It is easy to prove the sufficient condition. We need to prove the necessary condition only. Suppose that the game v has only m RGCs $S_1, \ldots, S_m$. The game v can then be decomposed into m games, that is, $v = v_1 \oplus \cdots \oplus v_m$, where each v_j, $j = 1, \ldots, m$, is a component game of v. Since $v_1, \ldots, v_m$ are mergeable, and from Remark 3.2, $v_1 \oplus v_2 \oplus \cdots \oplus v_k = ((\cdots(v_1 \oplus v_2)\cdots) \oplus v_{k-1}) \oplus v_k$, we can apply L2 to v sequentially as follows: $p_i(v) = p_i[((\cdots(v_1 \oplus v_2)\cdots) \oplus v_{k-1}) \oplus v_k] = p_i((\cdots(v_1 \oplus v_2)\cdots) \oplus v_{k-1}) + p_i(v_k) = p_i((\cdots(v_1 \oplus v_2)\cdots) + p_i(v_{k-1}) + p_i(v_k) = \cdots = p_i(v_1) + p_i(v_2) + \ldots + p_i(v_{k-1}) + p_i(v_k)$. Then we finally have $p_i(v) = p_i(\oplus_{j=1}^{m} v_j) = \sum_{j=1}^{m} p_i(v_j)$. From Lemma 3.3, we know that

$$\sum_{j=1}^{m} p_i(v_j) = \sum_{S_j \ni i} v(S_j) + \sum_{S_k \not\ni i} v(S_k) = \sum_{S_j \ni i} v(S_j) + \sum_{S_k \not\ni i} 0$$

where S_j, $S_k \in R(v)$. Therefore we get $p_i(v) = \sum_{S_j \notin i,\, S_j \in R(v)} v(S_j)$. ■

4. Discussion

To illustrate the application of the public value to the power distribution among players in a game, we will study the following two examples.

Example 1. Consider a committee of a city, which consists of seven persons, labelled 1, 2, ..., and 7. they will vote on a proposal of dumping waste near the city. The players have been assigned voting weights as follows:

Player:	1	2	3	4	5	6	7
Weights:	0.3	0.1	0.1	0.1	0.2	0.1	0.1

Both players 1 and 2 are definitely supposed to support that proposal, say the proposal is given by their party, while players 5, 6, and 7 will certainly reject it, say they belong to Greenpeace. Players 3 and 4 can vote for either side. The rule of the voting is that passing the proposal needs at least 60% of the voting weights, for the reason of concern over the damage of environment, while 50% of the weights can negate the proposal. Of course, this story describes a situation of voting on a public good (or bad).

Our interest is to analyse the distribution of the voting power among players. the voting problem first can be modelled as a simple game v. It is important to note that any group of players containing the players from the two opposite sides will be a false coalition in the sense that the members of this coalition cannot agree on an unanimous decision. So the outcome of such a false coalition must be zero. Write out all the winning coalitions as below.

For the proposal: $\{1, 2, 3, 4, \}$.

Against the proposal: $\{3, 4, 5, 6\}$, $\{3, 4, 5, 7\}$, $\{3, 5, 6, 7\}$, $\{4, 5, 6, 7\}$ and $\{3, 4, 5, 6, 7\}$.

Therefore the characteristic function of the game v is $v(1, 2, 3, 4) = v(3, 4, 5, 6) = v(3, 4, 5, 7) = v(3, 5, 6, 7) = v(4, 5, 6, 7) = v(3, 4, 5, 6, 7,) = 1$ and for any other coalition T, $v(T) = 0$. It is a nonmonotonic (simple) voting game. [By monotonicity we mean that for any S, $T \subseteq N$, if $S \subset T$, then $v(T) \geq v(S)$].

The collection $M(v)$ of the minimum winning coalitions then is $M(v) = \{\{1, 2, 3, 4\}, \{3, 4, 5, 6\}, \{3, 4, 5, 7\}, \{3, 5, 6, 7\}, \{4, 5, 6, 7\}\}$.

Take $\{3, 5, 6, 7\}$ for example to analyse the power distribution among the members of an MWC. Suppose this coalition forms, then it wins the game. After winning the voting, the waste will not be dumped near their city. The preservatoin of the environment as the joint payoff of $\{3, 5, 6, 7\}$ is a public good. Every member of $\{3, 5, 6, 7\}$ will consume the whole good, and no one will or is able to retain a part of it and go away. Therefore, the value one, which indicates the winning of a coalition, should be assigned to each member of $\{3, 5, 6, 7\}$ to measure his/her power on deciding the winning of this coalition.

To measure the entire voting power of a player in this game, we should take into account all the MWCs containing him/her and assign the value one, with respect to each of the MWCs, to him/her.

Thus the p-value is an appropriate measure of the voting power of a player in this game. We now calculate the p-value for each player. Player 1 is decisive for only one coalition, $\{1, 2, 3, 4\}$, so his/her public value is one. Similarly, the public value of every player can be worked out as follows:

$$p_1 = p_2 = 1,\ p_3 = p_4 = p_5 = 4,\ p_6 = p_7 = 3.$$

Example 2: A small community of ten people, indexed by $\{1, 2, \ldots, 10\}$ will hold a celebration party. Player 1 can cook a nice big cake and a pot of good coffee. Player 2 can perform a wonderful puppet show. Player 3 can make a small firework show. Players 1 and 2 working together can present a musical show. Unfortunately, player 3 is a bad tempered man. If he works with someone else, they will argue with each other so that nothing can be done. For the celebration, other players can do nothing but enjoy the party, as they are old people or little kids. Since the party is a public celebration, nobody will get a wage for his work, but the enjoyment from organising and taking part in the party. Assume further that the whole community has an unanimous utility valuation over all the celebrating events as follows:

Events:	Cake and Coffee	Puppet Show	Fireworks	Music Show
Utility:	1	3	3	4

Clearly, some people contribute to the celebration more while others contribute little or nothing. An interesting question is how to measure the individual contribution of a person to this party. To answer this question, the whole story should first be formally represented as the ten-person game with the following characteristic function:

$v(1) = 1$, $v(2) = 3$, $v(3) = 2$, $v(1,2) = 4$, $v(1,3) = v(2,3) = v(1,2,3) = 0$; for any T with $\{1,3\} \subseteq T$ or $\{2,3\} \subseteq T$, $v(T) = 0$; for any $S \subseteq N/\{3\}$, $v(\{1\} \cup S) = 1$, $v(\{2\} \cup S) = 2$, $v(\{1,2\} \cup S) = 4$; for any $A \subseteq N/\{1,2\}$, $v(\{3\} \cup A) = 3$; and for any $S \subseteq \{4,5,7,8,9,10\}$, $v(S) = 0$. It is a nonmonotonic game.

To distinguish between the players with the capability to contribute to the party and those without it, we first should note that if a coalition T with $v(T) > 0$ is not a real gaining coalition, there must exist an RGC $S \subset T$ with $v(S) \geq v(T)$. This has two implications: 1) the coalition T is not really worthy forming, for the smaller subcoalition S can gain more than or at least the same as T can do, there is no incentive for T to form; 2) all the players in T/S are redundant for achieving the value $v(S)$. Moreover, each player in S is essential for the gaining of S.

Thus only the players in some RGCs are important for making the party, while the players belonging to no RGCs should be regarded as redundant, and this is why we call them dummies in general. The individual contribution of a player to the party is naturally connected with the values of the RGCs to which he/she belongs. In order to measure the contributions of players, we now list the collection of all real gaining coalitions as follows: $R(v) = \{\{1\}, \{2\}, \{3\}, \{1,2\}\}$.

The outcome resulting from cooparation of each RGC, e.g., a firework show or a music show, obviously is a public good. Then, to measure the individual contribution (or power) for producing such public goods, we should assign to each player the whole value of the real gaining coalition containg him/her, for he/she will not divide the value with other players. So the public value is available to do the job. For instance, player 1 is the member of two RGCs $\{1\}$ and $\{1,2\}$, then both the entire value $v(1)$ and $v(1,2)$ should be used to express the power of player 1 for producing the public goods: the cake and coffee, and the music show, respectively. Thus the whole power of player 1 is $p_1 = v(1) + v(1,2) = 1 + 4 = 5$.

Similarly we can get $p_2 = v(2) + v(1,2) = 2 + 4 = 6$, $p_3 = v(3) = 3$, and

$p_i = 0$ for $i = 4, 5, 6, 7, 8, 9$, and 10.

The two examples above show how the public value works in analysing power distribution in both simple and general games.

For a general game v, the public value to a player can be interpreted as a measure of his absolute power to produce or to decide on public goods, for he/she is decisive for every RGC S of which he/she is a member and he enjoys the consumption of the public good $v(S)$ with his/her partners in S.

To measure the relative power of player i to produce or to decide on public goods, we can divide $p_i(v)$ by the sum of all the public values of all players in the game v. So we obtain

$$\frac{\sum_{S_j \ni i,\, S_j \in R(v)} v(S_j)}{\sum_{i \in N} \sum_{S_j \ni i,\, S_j \in R(v)} v(S_j)} \cdots (\#)$$

For simple games, this formula is reduced to the public good index defined by Holler (1982) and Holler and Packel (1983). Therefore, we can naturally regard formula (#) as the definition of the generalized public good index for general games.

The public value of a player measures the aggregate amount of the coalition values for which this player is decisive whereas the public good index expresses this amount as a percentage of the total value produced and consumed by all the players in all RGCs. The relationship between the public value and the public good index is thus clear; the former measures the absolute power of a player while the latter measures the relative power.

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