Control and Cybernetics

vol. **24** (1995) No. 3

Fixed price equlibria in distribution economies

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We study some existence and efficiency properties of fixed-price equilibria in distribution economies, significant among them being the dual pricing equilibrium.

Subsequently, we turn to an economy with a produced public good and show that all voluntary and efficient allocations for such an economy must be ratio equilibrium allocation, thus establishing the inherent non-optimality of rationing schemes in mixed economies.

1. Introduction

Many justifications have been offered for the existence of price rigidities in an economy. A good exposition of the economic rationale behind price rigidities can be found in Silvestre (1986).

However, there is one justification for fix-price analysis which is rooted in the theory of linear economic models of production: the non-substitution theorem of von Weizsacker and Samuelson. Briefly stated, the non-substitution theorem tells us that the unit prices of produced commodities depend only on technological considerations under the conditions of constant returns to scale. Thus at any point of time, the consumption sector of the economy inherits from the production sector a vector of commodity prices and a consumption bundle, the latter being not necessarily uniquely determined by the prices of production. Hence the problem faced by the consumption sector is one of allocating this consumption bundle amongst the consumers.

As a by-product of the producer prices and the vector of commodities produced, we also get a distribution of income among the consumers. Thus, we are confronted with what Malinvaud (1985) describes as a distribution economy.

Now, the prices at which the produced commodities are to be distributed among the consumers, need not be prices at which they are produced. Distribution of resources can take place at any price. However, a departure from the producer prices would necessitate the existence of a clearing house which collects the payments made by the consumers, and reallocates it among the producers at producer prices, the value of eaach producer's productions. Hence, rigidity of prices in the consumption sector is not implied by the non-substitution theorem per se.

In the event that the producers have a significant role in determining prices, it is natural to expect that the price prevailing in the consumption sector is the same as the producer prices. It is then that we are left with fixed consumption prices in a distribution economy.

In this paper we focus on fixed price analysis in a distribution economy (as opposed to a trading economy, which has been studied by Silvestre, 1982). We extend some conventional fixed price equilibrium solutions to such an economy. They are the ones of Dreze (1975), Younnes (1975), Dreze and Müller (1980), Balasko (1979), since they seem more appropriate in the context under discussion. We modify the solution of Dreze and Müller (1980) and show that this modified solution is equivalent to the solution of Balasko (1979). This also offers a method of enforcing the solution that Balasko had proposed.

Amongst all the above solutions, the solution of Dreze (1975) and its adaptation to our context is the only solution with quantity constraints on the purchases of the consumer. However, its lack of efficiency is a disconcerning feature. We propose the concept of a dual pricing equilibrium (which in one form or the other has existed in the parlance of administrators and applied economists) as an institutional remedy for this non-optimality. We thereby do depart from orthodox fixed price analysis in order to accommodate this solution.

Silvestre (1985) shows that in a pure trade economy, all net trades which are voluntary and Pareto optimal must be Walrasian. In the context of a distribution economy consisting solely of private goods, the concept of voluntary net trades lack credibility since no one is actually trading, unless of course we consider a distribution economy to be "structurally isomorphic" to a pure trade economy with the initial endowments being allocated proportionately to income. We, however, choose to avoid such counter factual simulations and instead of considering an economy consisting solely of private goods, we consider a two good economy consisting of money and a public good. In such an economy, voluntary allocations have a meaning, and we show (under assumptions on preferences which are slightly more stringent than in the rest of the paper) that all voluntary and Pareto optimal allocations, must be ratio equilibrium allocations (a concept due to Kaneko (1977)).

It may be worthwhile to point out that the case for a distribution economy is much stronger than the case for the general validity of the non-substitution theorem. The concept of a distribution economy rests on the existence of a numeraire good in terms of which all value and costs can be measured, as for instance in Dierker and Lenninghaus (1986). The non-substitution theorem requires additional mathematical structure on the cost function for its general uniquely defined prices arising out of the production sector, then the fixed prices that we start out with can be considered to be a policy instrument available to a social planner in order to implement a desired distribution of resources in the consumption sector.

2. The model

To keep matters simple we will adopt the framework of a distribution economy as proposed by Blad and Keiding (1990).

There are n produced goods in the economy indexed by $j \in \{1, \ldots, n\}$ and m consumers indexed by $i \in \{1, \ldots, m\}$. The commodity space is the non-negative orthant of n-dimensional Euclidean space $\mathbf{R}^n_+$. Consumer i has preferences defined on consumption bundles in $\mathbf{R}^n_+$ which are represented by a continuous, strictly increasing, strictly quasi-concave utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}$ (i.e. $x, y \in \mathbf{R}^n_+,\ x \geq y,\ x \neq y \ \Rightarrow u_i(x) > u_i(y)$; $x, y \in \mathbf{R}^n_+,\ x \neq y \Rightarrow u_i(tx + (1-t)y) > \min\{u_i(x), u_i(y)\}\ \forall t \in (0,1)$). Each consumer i has in addition a strictly positive income $w_i > 0$. This income which may be in the form of paper money serves the purpose of a medium of exchange. The economy is endowed with a strictly positive vector of resources: $\omega \in \mathbf{R}^n_{++} \equiv \{x \in \mathbf{R}^n_+ | x_j > 0\ \forall j \in \{1, \ldots, n\}\}$.

A market equilibrium is a pair $(\bar{x}, \bar{p}) \in (\mathbf{R}^n_+)^m \times (\mathbf{R}^n_+ \setminus \{0\})$ where $\bar{x} \in (\mathbf{R}^n_+)^m$ is an allocation, $\bar{p} \in \mathbf{R}^n_+ \setminus \{0\}$ a price vector such that:

(i) $\sum_{i=1}^m \bar{x}^i = \omega (\bar{x}^i \in \mathbf{R}^n_+\ \forall i \in \{1, \ldots, m\})$

(ii) $\bar{x}^i$ maximizes $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ | \bar{p}, x \leq w_i\}$

(iii) $\bar{p}\omega = \sum_{i=1}^m w_i$

The following theorem due to Malinvaud (1985) will be stated for completeness:

THEOREM 2.1 *There exists a market equilibrium for the above economy.*

Let $\hat{p} \in \mathbf{R}^n_{++}$ be a vector of fixed prices such that $\hat{p}\omega = \sum_{i=1}^m w_i$.

A **Dreze-type equilibirum** (D-equilibrium) is a triple $(\hat{x}, \hat{L}, \hat{l}) \in (\mathbf{R}^n_+)^m \times \mathbf{R}^n_+ \times \mathbf{R}^n_+$ such that $\hat{x} \in (\mathbf{R}^n_+)^m$ is an allocation and $\hat{L} \in \mathbf{R}^n_+$, $\hat{l} \in \mathbf{R}^n_+$ are (quantity) constraints with $\hat{L}_j < \hat{l}_j\ \forall j = 1, \ldots, n$ satisfynig:

(i) $\sum_{i=1}^m \hat{x}^i = \omega$ and $\hat{p}\hat{x}^i \leq w_i\ \forall i = 1, \ldots, m$,

(ii) $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ | \hat{p}x \leq w_i, \hat{l}_j \leq x_j \leq \hat{L}_j\ \forall j = 1, \ldots, n\}$.

The following theorem and its proof establish the existence of a D-equilibrium.

THEOREM 2.2 *There exist a D-equilibrium for the above economy.*

Proof: Consider the set,

Let $x_i(L)$ maximize $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ | \hat{p}x \leq w_i, 0 \leq x \leq L\}$. Clearly $x^i : Q \to \mathbf{R}^n_+$ is a continuous function. Now consider the function $f : Q \to Q$ defined as follows:

$$f_j(L) = \min\{\max\{L_j - (\sum_{i=1}^{m} x^i_j(L) - \omega_j), 0\}, \omega_j\}.$$

Clearly, f is continuous. Thus by Brouwer's fixed point theorem, there exists $\hat{L} \in Q$ such that $f(\hat{L}) = \hat{L}$.

CASE 2.1 $\omega_j > L_j = f_j(\hat{L}) > 0$; *thus* $\hat{L}_j - (\sum_{i=1}^m x^i_j(\hat{L}) - \omega_j) = \hat{L}_j$, *and* $\sum_{i=1}^m x^i_j(\hat{L}) = \omega_j$.

CASE 2.2 $\hat{L}_j = f_j(\hat{L}) = \omega_j$; *thus* $\max\{\omega_j - (\sum_{i=1}^m x^i_j(\hat{L}) - \omega_j), 0\} \geq \omega_j > 0$, $\Rightarrow \sum_{i=1}^m x^i_j(\hat{L}) - \omega_j \leq 0$.

CASE 2.3 $\hat{L}_j = f_j(\hat{L}) = 0$; *thus* $0 - (\sum_{i=1}^m x^i_j(\hat{L}) - \omega_j) \leq 0 \Rightarrow \sum_{i=1}^m x^i_j(\hat{L}) - \omega_j \geq 0$.

which contradicts $\hat{L}_j = 0$ Hence by Cases 2.1 and 2.2, $\sum_{i=1}^m x^i_j(\hat{L}) - \omega_j \leq 0 \ \forall j = 1, \ldots, n$, with $\sum_{i=1}^m x^i_j(\hat{L}) = \omega_j$ if $\hat{L}_j < \omega_j$. Since $\hat{p}\omega = \sum_{i=1}^m w_i$, strict monotonicity of the preferences imply $\sum_{i=1}^m x^i_j(\hat{L}) = \omega_j \ \forall_j = 1, \ldots, n$. Let $\hat{x}^i = x^i(\hat{L})$ and $\hat{l} = 0$. This proves the theorem. ∎

We now define a **Younnes-type equilibrium** (Y-equilibrium). A **Y-equilibrium** is an allocation $\hat{x} \in (\mathbf{R}^n_+)^m$ such that

(i) $\sum_{i=1}^m \hat{x}^i = \omega$

(ii) $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ | \hat{p}, x \leq w_i, \min_{i'}\{\hat{x}^{i'}_j\} \leq x_j \leq \max_{i'}\{\hat{x}^{i'}_j\}$.

It is trivial to observe that an allocation is a Y-equilibrium if and only if it is a D-equilibrium.

We now proceed to the definition of a Dreze-Müller-type equilibrium (DM-equilibrium).

A **DM-equilibrium** is an allocation price pair $(\hat{x}, a^0) \in (\mathbf{R}^n_+)^m \times (\mathbf{R}^n_+ \setminus \{0\})$ such that:

(i) $\sum_{i=1}^m \hat{x}^i = \omega$, and $\hat{p} \cdot \hat{x}^i \leq w_i \ \forall i = 1, \ldots, m$.

(ii) If $x \in \mathbf{R}^n_+$ with $\hat{p} \cdot x \leq w_i$ and $u_i(x) > u_i(\hat{x}^i)$ then $a^0 \cdot x > a^0 \hat{x}^i \ \forall i = 1, \ldots, m$.

A DM-equilibrium allocation is enforced be allowing consumer i to purchase $\hat{x}^i$ at prices $\hat{p}$ and giving him coupons worth $a^0 \hat{x}^i$. In the event that he wishes to pursache any other consumption bundle x which is affordable at his original income, he has to obtain it by using the coupons he receives, the coupon-prices of the commodities being given by the vector a^0. The coupon-prices are such

3. Efficiency properties of equilibrium

It is easy to observe that a D-equilibrium allocation (or for that matter a Y-equilibrium allocation) is not Pareto optimal, where we define Pareto optimal allocations as follows:

An allocation $\bar{x} \in (\mathbf{R}^n_+)^m$ with $\sum_{i=1}^m \bar{x}^i = \omega$ is said to be **Pareto optimal** if there is no other allocation $x \in (\mathbf{R}^n_+)^m$ with $\sum_{i=1}^m x^i \leq \omega$ satisfying $u_i(x^i) \geq u_i(\bar{x}^i)\ \forall i = 1, \ldots, m$ with at least one strict inequality.

Given that with fixed prices $\hat{p}$, Pareto optimality may be difficult to achieve, one considers the alternative notion of restricted Pareto optimality:

An allocation $\bar{x} \in (\mathbf{R}^n_+)^m$ with $\sum_{i=1}^m \bar{x}^i = \omega$ and $\hat{p}\bar{x}^i \leq w_i\ \forall i = 1, \ldots, m$ is said to satisfy **restricted Pareto optimality** if there is no other allocation $x \in (\mathbf{R}^n_+)^m$ with $\sum_{i=1}^m x^i \leq \omega$ and $\hat{p}x^i \leq w_i\ \forall i = 1, \ldots, n$ satisfying $u_i(x^i) \geq u_i(\bar{x}^i)\ \forall i = 1, \ldots, m$ with at least one strict inequality.

It turns out that a D-equilibrium need not even satisfy **restricted Pareto optimality** as the following example reveals:

Example: $m = 2, n = 2$ (two person, two good case): $\omega = (w, w) \gg 0, w_1 = w_2 = w > 0$. Let $\hat{p} = (1, 1)$ and $u_1(x_1, x_2) = x_1^2 \cdot x_2$, $u_2(x_1, x_2) = x_1 \cdot x_2^2$, $(x_1, x_2) \in \mathbf{R}^n_+$.

Then, $\hat{x}^1 = \left(\frac{w}{2}, \frac{w}{2}\right)$, $\hat{x}^2 = \left(\frac{w}{2}, \frac{w}{2}\right)$, $L = \left(\frac{w}{2}, \frac{w}{2}\right)$, $l = (0, 0)$ is a D-equilibrium. However, let $x^1 = \left(\frac{w}{3}, \frac{2w}{3}\right)$, $x^2 = \left(\frac{2w}{3}, \frac{w}{3}\right) \cdot x^1 + x^2 = \omega$, $\hat{p}x^1 = \hat{p}x^2 = w$ and $u_1(x^1) > u_1(\hat{x}^1)$, $u_2(x^2) > u_2(\hat{x}^2)$.

On the other hand a DM-equilibrium allocation satisfies restricted Pareto optimality.

THEOREM 3.1 *Let $(\hat{x}, a^0) \in (\mathbf{R}^n_{++})^m \times (\mathbf{R}^n_+ \setminus \{0\})$ be a DM-equilibrium. Then x satisfies restricted Pareto optimality.*

Proof Suppose not. Then there exist $x \in (\mathbf{R}^n_+)^m$ with $\sum_{i=1}^m x^i \leq \omega, \hat{p}x^i \leq w_i \forall i = 1, \ldots, m$, $u_i(\bar{x}^i) \geq u_i(\hat{x}^i) \forall i = 1, \ldots, m$ with at least one strict inequality. Let $u_{i0}(x^{i0}) > u_{i0}(\hat{x}^{i0})$. Then $a^0 x^{i0} > a^0 \hat{x}^{i0}$.

Suppose $a_{0.}x^i < a_{0.}\hat{x}^i$, for some $i \neq i_0$. By strict quasi-concavity, $u_i(tx^i + (1-t)\hat{x}^i) > u_i(\hat{x}^i)\ \forall t \in (0, 1)$. Further, $a^0[tx^i + (1-t)\hat{x}^i] < a_0\hat{x}^{i0}\ \forall t \in (0, 1)$ and $p[tx^i + (1-t)\hat{x}^i] \leq w_i\ \forall t \in (0, 1)$ contracting that $(\hat{x}, a^0)$ is a DM-equilibrium. Hence $a_{0.}x^i \geq a_{0.}\hat{x}^i i$.

Then, $a^0\omega \geq \sum_{i=1}^m a^0 x^i > \sum_{i=1}^m a^0 \hat{x}^i = a^0\omega$ which again is a contradiction. Thus x satisfies restricted Pareto optimality. ■

The converse of Theorem 3.1 is easy to establish, i.e. that if $\hat{x}$ satisfies restricted Pareto optimality, then there exists $a^0 \in \mathbf{R}^n_+ \setminus \{0\}$ such that $(\hat{x}, a^0)$ is a DM-equilibrium.

THEOREM 3.2 *Let $\hat{x}$ satisfy restricted Pareto optimality. Then there exists $a^0 \in \mathbf{R}^n_+ \setminus \{0\}$ such that $(\hat{x}, a^0)$ is a DM-equilibrium.*

Proof Consider sets

$$U = \{\sum_{i=1}^{m} x^i | x^i \in \mathbf{R}^n_+, \hat{p}x^i \le w_i, u_i(x^i) \ge u_i(\hat{x}^i)\ \forall i\} + \mathbf{R}^n_+ \text{ and}$$
$$V = \omega - \mathbf{R}^n_+$$

Both U and V are non-empty since ω belongs to both. Since the utility functions are strictly quasi-concave, U is convex. Since x satisfies restricted Pareto optimality and the utility functions are strictly increasing, ω is a boundary point of U. By construction ω is a boundary point of V. Hence by the separating hyperplane theorem, there exists $a^0 \in \mathbf{R}^n_+ \backslash \{0\}$ such that $a^0 z \le a^0 z' \forall z \in V, z' \in U$.

Now suppose $x^i \in \mathbf{R}^n_+ \setminus \{0\}$ with $\hat{p}x^i \le w_i$ and $u_i(x^i) > u_i(\hat{x}^i)$. By strict monotonicity, continuity and restricted Pareto optimality of $\hat{x}$, $x^i + \sum_{k \ne i}^{n} \hat{x}^k \in$ interior of U.

Then $a^0(x^i + \sum_{k\ne i}^{n} \hat{x}^k) > a^0 \sum_{k=1}^{m} \hat{x}^k = a^0\omega$ i.e. $a^0 x^i > a^0 \hat{x}^i$, thus completing the proof. ∎

The set $\{\sum_{i=1}^{m} x^i \setminus x^i \in \mathbf{R}^n_+, \hat{p}x^i \le w_i, u_i(x^i) \ge u_i(\hat{x}^i)\ \forall i\}$ is convex with empty interior since $\hat{x}$ satisfies restricted Pareto optimality and $\hat{p}\cdot\omega = \sum_{i=1}^{m} w_i$. Hence it must be contained in an $n-1$ dimensional affine subspace of $\mathbf{R}^n_+$.

This brings us to the question of whether a DM-equilibrium exists. We now establish the existence of a DM-equilibrium. The proof is a slight modification of the proof of the existence of a market equilibrium due to Malinvaud (1985).

THEOREM 3.3 *For the distribution economy given in section 1, a DM-equilibrium exists.*

Proof Let $\{A_i\}_{i=1}^{m}$ be a set of m strictly positive real numbers.

Consider the set $A = \{a \in \mathbf{R}^n_+ | a_j \le \sum_{i=1}^{m} A_i/\omega_j\}$.

Let $x^i(a)$ maximize $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ | \hat{p}x \le w_i, ax \le A_i\}$ for $a \in A$ and $i = 1, \ldots, m$. Define $f : A \to A$ as follows:

$$f_j(a) = \min\{\max\{a_j + \sum_i x^i_j(a) - \omega_j, 0\}, \sum_i A_i/\omega_j\}$$

Clearly, f is continuous. Thus, by Brouwer's fixed point theorem, there exists $a^0 \in A$ such that $f(a^0) = a^0$.

Case 1: $0 < a^0_j < \sum_i A_i\omega_j$; then $\sum_i x^i_j(a^0) = \omega_j$.

Case 2: $a^0_j = \sum_i A_i/\omega_j$; then $\sum_i x^i_j(a^0) \ge \omega_j$.

If $\sum_i x^i_j(a^0) > \omega_j$, then $a^0_j \sum_i x^i_j(a^0) > \sum_i A_i$, which is impossible.

So, $\sum_i x^i_j(a^0) = \omega_j$.

Case 3: $a^0_j = 0$ then $\sum_i x^i_j(a^0) - \omega_j \le 0 < \sum_i A_i/\omega_j$.

If $\sum_i x^i_j(a^0) < \omega_j$, then considering the above two cases, $\hat{p}\sum_i x^i(a^0) < \hat{p}\omega$

Since preferences are strictly increasing, there exist some consumer $i \in \{1,\dots,m\}$, who could increase his consumption of good j, satify both his constraints and still do better than at $x^i(a^0)$ contradicting the definition of $x^i(a^0)$.

Thus we must have $\sum_i x^i_j(a^0) = \omega_j$.

Set $x^i(a^0) = \hat{x}^i \; \forall i = 1,\dots,m$.

Since $a^0\hat{x}^i \leq A_i \; \forall i = 1,\dots,m$, $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ | \hat{p}x \leq w_i, a^0x \leq a^0\hat{x}^i\} \; \forall i = 1,\dots,m$. This established the theorem. ■

What happens it we modify a DM-equilibrium?

A **modified DM-equilibrium** is an allocation price pair $(\hat{x}, a^0) \in (\mathbf{R}^n_+)^m \times (\mathbf{R}^n_+) \setminus \{0\}$ such that

(i) $\sum_{i=1}^m \hat{x}_i = \omega$, $\hat{p}\hat{x}_i = w_i \; \forall i = 1,\dots,m$

(ii) If $x \in \mathbf{R}^n_+$ with $u_i(x) > u_i(\hat{x}^i)$ then $a^0x > a_0\hat{x}^i$.

Such an allocation is easier to enforce. Consumer i is allowed to purchase $\hat{x}^i$ at prices $\hat{p}$ in lieu of which he is given coupons worth a^0, $\hat{x}^i$. Any further recontracting must take place using the coupons at coupon prices a^0. If x is a **modified DM-equilibrium** then no agent has any incentive to further recontact.

The following result is immediate.

THEOREM 3.4 *If $(\hat{x}, a^0)$ is a* **modified DM-equilibrium**, *then it is a market equilibrium for the distribution economy where the income of consumer i is $a^0\hat{x}^i$, the preference and initial endowment of resources being the same as before.*

Proof Immediate.

In addition we obtain the following result:

COROLLARY 3.1 *If $(\hat{x}, a^0)$ is a modified DM-equilibrium, then $\hat{x}$ is Pareto optimal.*

Proof Follows from the first fundamental theorem of welfare economics, which holds under our assumptions (see Blad and Keiding, 1990).

This brings us to the concept of a **budget constrained Pareto Efficient** (Optimal) (BCPE) allocation due to Balasko (1979).

An allocation $\hat{x} \in (\mathbf{R}^n_+)^m$ is said to be a BCPE allocation if:

(i) $\sum_{i=1}^m \hat{x}^i = \omega$

(ii) $\hat{p}\hat{x}^i \leq w_i \; \forall i = 1,\dots,m$

(iii) $\hat{x}$ is Pareto optimal.

The following result is of some interest, and follows almost directly from our previous analysis.

THEOREM 3.5 *x is a BCPE allocation if and only if there exists $a^0 \in \mathbf{R}^n_+ \setminus \{0\}$*

Proof If $(\hat{x}, a^0)$ is a modified DM-equilibrium, then it is a BCPE allocation as a consequence of Corollary 3.1.

Conversely if x is a BCPE allocation, then since it is Pareto optimal, it follows by the second fundamental theorem of welfare economies (as in Blad and Keiding, 1990), that there exists $a^0 \in \mathbf{R}^n_+ \setminus \{0\}$, such that $u_i(x) > u_i(\hat{x}^i)$ implies $a^0 x > a^0 \hat{x}^i$. Since $\hat{p}\hat{x}^i \leq w_i \; \forall i = 1, \ldots, m$, we have the proof. ∎

This establishes the equivalence of the concept of a BCPE allocation with the concept of a modified DM-equilibrium allocation, the latter suggesting a mechanism through which the solution can be enforced.

4. Dual pricing in a distribution economy

In order to remedy the possible ineffieciency of a D-equilibrium, we now propose the concept of a **dual pricing** (DP) equilibrium.

A DP equilibrium is a four-tuple $[\hat{x}, \hat{y}, \hat{L}, p^*] \in (\mathbf{R}^n_+)^m \times (\mathbf{R}^n_+)^m \times \mathbf{R}^n_+ \times \mathbf{R}^n_+ \setminus \{0\}$, where $\hat{x}, \hat{y}$ are allocations, $\hat{L}$ is a vector of quantity constraints on the market with fixed prices $\hat{p} \in \mathbf{R}^n_{++}$, and p^* is a price vector such that:

(i) $\hat{x}^i \leq \hat{L} \; \forall i = 1, \ldots, m$ and $\sum_{i=1}^m \hat{x}^i + \sum_{i=1}^m \hat{y}^i = \omega$

(ii) $\hat{p}\hat{x}^i + p^*\hat{y}^i = w_i$ and $(\hat{x}^i, \hat{y}^i)$ maximizes $u_i(x + y)$ on the set $\{(x, y) \in \mathbf{R}^n_+ \times \mathbf{R}^n_+ | x \leq \hat{L}, \hat{p}x + p^*y \leq w_i\} \; \forall i = 1, \ldots, m$.

Hence in a dual pricing equilibrium there are both fixed and flexible prices, with quantity constrainsts on the purchases of a consumer in the fixed price market.

THEOREM 4.1 *A DP-equilibrium exists for our distribution economy.*

Proof Let $\hat{L} = 0$, $\hat{x} = 0$ and $(\hat{y}, p^*)$ be a market equilibrium for our distribution economy, which exists by Therem 2.1. This yields a DP-equilibrium.

It is clear that the DP-equilibrium we obtain in our proof is Pareto optimal. It is significant however that all DP-equilibria are Pareto optimal, which is what we establish now.

THEOREM 4.2 *Let $[\hat{x}, \hat{y}, \hat{L}, p^*]$ be a DP-equilibrium. Then it is Pareto optimal.*

Proof Suppose not. Then because of our assumptions on preferences (i.e. that they are strictly increasing) we can find $z \in (\mathbf{R}^n_+)^m$ such that $\sum_{i=1}^m z^i = \omega$ and $u_i(z^i) > u_i(\hat{x}^i + \hat{y}^i) \forall i = 1, \ldots, m$. Thus clearly whatever $(x^i, y^i) \in \mathbf{R}^n_+ \times \mathbf{R}^n_+$ with $x^i \leq \hat{L}$ and $x^i + y^i = z^i$ we may choose we must have $\hat{p}x^i + p^*y^i > w_i$. Since $\sum_{i=1}^m z^i = \omega = \sum_{i=1}^m \hat{x}^i + \sum_{i=1}^m \hat{y}^i$, it is possible to choose (x^i, y^i), $i = 1, \ldots, m$ such that

(i) $x^i \leq \hat{L}, x^i + y^i = z^i$, $i = 1, \ldots, m$.

(ii) $\sum_{i=1}^m x^i = \sum_{i=1}^m \hat{x}^i$, $\sum_{i=1}^m y^i = \sum_{i=1}^m \hat{y}^i$.

But then we get using $w_i = \hat{p}\hat{x}^i + p^*\hat{y}^i$ that $\sum_{i=1}^m w_i = \hat{p}\sum_{i=1}^m \hat{x}^i + p^*\sum_{i=1}^m y^i = \hat{p}\sum_{i=1}^m x^i + p^*\sum_{i=1}^m y^i > \sum_{i=1}^m w_i$; which is a contradiction, Hence the remedy for the non-optimality of a D-equilibrium, is purely institutional: allow for both fixed and flexible prices with quanity rationing on the fixed price market. From the point of view of a social planner with redistributional objectives (which may or may not be embodied in an explicit Bergson-Samuelson social welfare function), a DP-equilibrium may well turn out to be supererior to a market equilibrium i.e. some $\hat{L}$ different form zero and a p' may be more equitable than a market equilibrium.

5. Fix price analysis in an economy with public good

In the context of a pure-exchange private ownership economy, Silvestre (1985) established that all Pareto optimal allocations which have voluntary net trades are Walrasian. This means:

(a) that a method of allocation which differs from the usual price mechanism leading to Walras equilibrium, and which is at the same time Pareto optimal must be involuntary;

(b) rationing is inherently non optimal.

We would like to pose a similar question now, in an economy with a public good. Resource allocation in an economy consisting of a public good is receiving growing attention. The solution concept which is usually offered as a means to decentralize resource allocation in a mixed economy (i.e. an economy with both private as well as public goods) is the concept of a ratio-equilibrium, due to Kanako (1977). The main conclusion of this section is that in such an economy all Pareto optimal allocations which are voluntary are ratio equilibrium allocations, provided the techno-economic considerations for the provision of the public good are summarized by a linear cost function.

As in Lahiri (1993), we consider an economy with one public good $y \geq 0$ producible from a single private good (money), $x \geq 0$. Taking the latter as numeraire, the cost function for the production of the public good is given by $c : \mathbf{R}_+ \to \mathbf{R}_+$ where $c(y) = cy \; \forall y \geq 0$ and c is a strictly positive real number.

There are N consumers, each endowed with a strictly positive amount of numeraire: let $w_i > 0$ be agent i's endowment of the private good. The preferences of consumer i are described by a utility function $u_i : \mathbf{R}^2_+ \to \mathbf{R}$ which is assumed to be strictly increasing, quasi-concave and differentiable.

A **state** of the economy is a vector $(x, y) \in \mathbf{R}^n_+ \times \mathbf{R}_+$ A state (x, y) is **feasible** if $cy \leq \sum_i (w_i - x_i)$. A feasible state (x, y) is **Pareto Optimal** if there is no other feasible state (x^1, y^1) such that $u_i(x^1_i, y^1) \geq u_i(x_i, y)$ for all i, with strict inequality for at least one i.

A **feasible state** (x, y) is said to be **voluntary** if $u_i(x_i, y) \geq u_i \; (w_i + \lambda(x_i - w_i), \lambda y) \; \forall \lambda \in [0, 1]$. A state which is voluntary is individually rational in the sense that $u_i(x_i, y) \geq u_i(w_i, 0)$ (put $\lambda = 0$ above) $\forall i$; however it conveys more information that just that. It says, for instance, that given the option of contracting both his payment as well as his choice of the public good by the

A feasible state (x, y) is said to be a **ratio equilibrium allocation** if there exists a vector $(t_1, \ldots, t_N) \in \mathbf{R}_+^N$ such that

(i) $\sum_{i=1}^N t_i = 1$
(ii) $\forall i = 1, \ldots, N,\ (x, y)$ solves
$\max u_i(x_i^1, y^1)$
s.t. $x_i^1 + t_i c y^1 \leq w_i$
$x_i^1 \geq 0,\ y^1 \geq 0$

We now prove that essentially any allocation which is voluntary and is at the same time Pareto optimal, must be a ratio-equilibrium allocation.

THEOREM 5.1 *Let (x, y) be a feasible state with $x_i > 0\, \forall i = 1, \ldots, N$, $y > 0$. If (x, y) is Pareto optimal and λ voluntary, then it is ratio equilibrium allocation.*

Proof Under our hypothesis (which also implies semi-strict quassi-concavity of preferences i.e. $u_i(x_i^1, y^1) > u_i(x_i^{11}, y^{11}), (x_i^1, y^1), (x_i^{11}, y^{11}) \in \mathbf{R}_+^2 \Rightarrow u_i(tx_i^1 + (1-t)x_i^{11}, ty^1 + (1-t)y^{11}) > u_i(x_i^{11} y^{11})\ \forall t \in (0, 1)$) we get by Proposition 3 in Lahiri (1993) that there exists a vector $(t_1, \ldots, t_N) \in \mathbf{R}_+^N$ such that

(i) $\sum_{i=1}^N t_i = 1$
(ii) $\forall i = 1, \ldots, N,\ (x_i, y)$ solves
$\max u_i(x_i^1, y^1)$
s.t. $x_i^1 + t_i c y^1 \leq x_i + t_i c y$
$x_i^1 \geq 0,\ y^1 \geq 0$

Since u_i is strictly increasing for each i, we must have $t_i > 0$.

Suppose that $x_i + t_i c y < w_i$ for some i. Then there exists $1 > E_0 > 0$ such that for $\forall E, E_0 < E < 1$ we have

$$u_i(w_i + E(x_i - w_i), Ey) > u_i(x_i, y),$$

contradicting that the state is voluntary. Thus, $x_i + t_i c y \geq w_i$, for all i. However, $\sum_{i=1}^N x_i + cy \leq \sum_{i=1}^N w_i$ and $\sum_{i=1}^N t_i = 1$ implies $x_i + t_i c y = w_i\ \forall i$,establishing our result.

It is worthwhile to comment on a restriction without which the above result is not valid. This restriction is the assumption of linear cost funtions. Assuming differentialbility of the cost functions, we would require $c'(y) \leq \frac{c(y)}{y}$ in order to obtain the contradiction in the proof above. However, this would in the case of non-linearity imply a non-covex cost function, and then the existence of the equilibrium prices in the first part of the proof cannot be established. Thus a linear cost function is a non-trivial requirement for the above analysis to hold. In any event, linearity of the cost function is not a serious handicap in equilibrium analysis with or without public goods.

The differentiability assumption on preferences is required for the existence of the equilibrium prices. This has been discussed in Lahiri (1993). Even with differentiability, our set of admissible economies is sufficiently large, to make

6. Conclusion

In this section we clarify some as yet unanswered questions that this paper may have posed.

First of all, in most of our analysis quasi-concavity of preferences would suffice. In Theorem 2.1, for instance, we would then have to appeal to the Kakutani fixed point theorem with a slightly modified correspondence to which it would apply. If preferences are strictly increasing and continuous, then a slight modification of the proof of Theorem 3.1 would validate the result under weaker assumptions on preferences. Thus, technically our assumptions are not very restrictive.

Turning now to the underlying economics, the question may be asked as to how a distribution economy takes care of the labor-leisure choice made by the individual consumer. To answer this question we must take note of the fact that, if we assume the wage rate of "standard" labor to be one, i.e. labor as the numeraire good in terms of which all costs are measured, w_i for a wage-earner would be the units of "standard" labor he is endowed with. In our framework, aggregate leisure (measured in "standard" units) is an output of the production process as well. In order to be within his/her budget set, each consumer i has to consume an amount of leisure which is less than or equal to w_i. Hence there is no conflict between the construct of a distribution economy and actual problems of resource allocation, with or without fixed prices for the consumption sector.

Acknowledgement

This paper grew out of discussions that I have had over a sustained period of time with Prof. Dipankar Dasgupta. I would like to thank him for all the time and encouragement he has spend on me.

My interest in public goods owes its genesis to Prof. Parkash Chander, who has in various ways tried to make me realize the wider applicability of the concept. I would also like to thank two anonymous referees who pointed out to me several lapses in the paper.

This paper was written while I was on visiting assigment to McMaster University (School of Business). I would like to thank them for their kind hospitality and to Prof. Prakash Abad of the Management Sciences and Information Systems area for making the visit possible.

Finally, I would like to thank the Economics Department seminar participants of the University of Waterloo, for their kind comments on sections of this paper.

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Appendix

In this appendix we prove the existence of some equilibria discussed in the paper, when the agents' preferences are semi-strictly quasi-concave. We therefore assume a distribution economy with aggregate endowment being $\omega \in \mathbf{R}^n_{++}$. There are m consumers with initial income $w_i > 0$ for the i-th consumer and the preferences of the i-th consumer being represented by a utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}$ which is

(i) continuous and strictly increasing;

(ii) semi-strictly quasi-concave: $x, y \in \mathbf{R}^n_+, u_i(x) \neq u_i(y) \Rightarrow u_i(tx+(1-t)y) > \min\{u_i(x), u_i(y)\}\ \forall t \in (0,1)$.

Let $C = \{x \in \mathbf{R}^n_+ | \hat{x}_j \leq \omega_j + 1\ \forall j = 1, \ldots, n\}$

LEMMA 6.1 *If $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in C : px \leq w_i\}$ where $p \in \mathbf{R}^n_+ \setminus \{0\}$ and $\hat{x}_i < \omega$, then $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in \mathbf{R}^n_+ : px \leq w_i\}$.*

Proof Suppose there exists $y_i \in \mathbf{R}^n_+ : py^i \leq w_i$ and $u_i(y^i) > u_i(\hat{x}^i)$. Then $y^i \in \mathbf{R}^n_+ \setminus C$. Thus there exists $t \in (0,1)$ such that $ty^i + (1-t)\hat{x}^i \in C$, since $\hat{x}^i < \omega$. By semi-strict quasi-concavity $u_i(ty^i + (1-t)\hat{x}^i) > u_i(\hat{x}^i)$ and further $p(ty^i + (1-t)\hat{x}^i) \leq w_i$. this contradicts the definition of x^i and proves the

Let $mC = \{mx | x \in C\}$, where m is the number of consumers. mC is a compact, convex subset of $\mathbf{R}^n_+$.

Let $P = \{p \in \mathbf{R}^n_+ | p\frac{\omega}{\sum_{i=1}^m w_i} = 1\}$ P is a compact convex subset of $\mathbf{R}^n_+$.

THEOREM 6.1 *There exists a market equilibrium for the distribution economy defined above.*

Proof For each $p \in P$, let $x^i(p) = \{x \in C | x$ maximizes $u_i(y)$ subject to $py \leq w_i,\ y \in C\}$.

The correspondence $\sum_{i=1}^m x^i(.) : P \to\to mC$ is convex valued, non-empty valued and has and has a closed graph. Consider the correspondence, $f : P \times mC \to\to P \times mC$, defined as follows:

$$f(p,z) = \left(\frac{p_j + \max\{z_j - \omega_j, 0\}}{1 + \left(\sum_{j=1}^n \max\{z_j - \omega_j, 0\} \frac{\omega_j}{\sum_{i=1}^m w_i} \right)} \right)_{j=1}^n \times \sum_{i=1}^m x^i(p)$$

Clearly f is well defined, non-empty valued, convex-valued and has a closed graph. Thus by Kakutani's fixed point theorem there exists $(p^*, z^*) \in P \times mC$ such that $(p^*, z^*) \in f(p^*, z^*)$. Thus, $p_j^* = \frac{p_j^* + \max\{z_j^* - \omega_j, 0\}}{1 + \left(\sum_{j=1}^n \max\{z_j^* - \omega_j, 0\} \frac{\omega_j}{\sum_{i=1}^m w_i} \right)}$ $\forall j = 1, \ldots, n$, or $p_j^* + \sum_{j=1}^n \max\{z_j^* - \omega_j, 0\} \frac{\omega_j}{\sum_{i=1}^m w_i} = \max\{z_j^* - \omega_j, 0\}$.

Since $z^* \in \sum_{i=1}^m x^i(p^*)$, $p^* z^* \leq \sum_{i=1}^m .w_i = p\omega$. Then $0 \geq \sum_{j=1}^n p_j^*(z_j^* - \omega_j) \sum_{k=1}^n \max\{z_k^* - \omega_k, 0\} \frac{\omega_k}{\sum_{i=1}^m w_i} = \sum_{j=1}^n (z_j^* - \omega_j) \max\{z_j^* - \omega_j, 0\}$. So $z_j^* \leq \omega_j\ \forall j = 1, \ldots, n$.

Given that the preferences are strictly increasing and $z^* \in \sum_{i=1}^m x_i(p^*)$, we get $z^* = \omega$, thus proving the theorem. ■

Let $\hat{p} \in \mathbf{R}^n_{++}$ be a vector of fixed prices such that $\hat{p}\omega = \sum_{i=1}^m w_i$.

THEOREM 6.2 *There exists a D-equilibrium for the above economy.*

Proof Consider the set Q as defined in Theorem 2.2 and let $x^i(L) = \{x \in \mathbf{R}^n_+ | px \leq w_i,\ 0 \leq x \leq L$ and x maximizes $u_i(y)$subject to $py \leq w_i, 0 \leq y \leq L\}$.

The corresopondence $\sum_{i=1}^m x^i(.) : Q \to \mathbf{R}^n_+$ is non-empty valued, convex valued and has a closed graph. Consider the correspondence $f : Q \times mC \to\to Q \times mC$ defined thus:

$$f(L,z) = (\min\{\max\{L_j - (z_j - \omega_j), 0\}, \omega_j\})_{j=1}^n \times \sum_{i=1}^m x_i(L)$$

f is well defined, non-empty valued, convex valued and has a closed graph. Hence there exists $(\hat{L}, \hat{z}) \in Q \times mC$ such that $(\hat{L}, \hat{z}) \in f(\hat{L}, \hat{z})$. The proof now

THEOREM 6.3 *For the distribution economy defined above there exists a DM-equilibrium.*

Proof Let $\{A_i\}_{i=1}^m$ and A be defined as in the proof of Theorem 3.3.

For each $a \in A$, let $x_i(a) = \{x \in \mathbf{R}_+^n | \hat{p}x \leq w_i,\ ax \leq A_i$ and x maximizes $u_i(y)$ subject to $\hat{p}y \leq w_i,\ ay \leq A_i,\ y \in \mathbf{R}_+^n\}$.

The correspondence $\sum_{i=1}^m x_i(.) : A \rightarrow \mathbf{R}_+^n$ is non-empty valued, convex-valued and has a closed graph. Further, $\forall a \in A$, $\sum_{i=1}^m x_i(a) \in mC$.

Define the correspondence $f : A \times mC \rightarrow A \times mC$ as follows:

$$t(a, z) = (\min\{\max\{a_j + z_j - \omega_j, 0\}, \sum_i A_i/\omega_j\})_{j=1}^n \times \sum_{i=1}^m x_i(a).$$

The correspondence f is non-empty valued, convex-valued, and has a closed graph. Thus by Kakutani's fixed point theorem there exists $(a^0, z^0) \in A \times mC$ such that $(a^0, z^0) \in f(a^0, z^0)$. Note $z^0 \in \sum_{i=1}^m x_i(a^0)$. Now the proof proceeds exactly as in the proof of Theorem 3.3, establishing the results. ∎