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On contact problem for a plate having a crack

by

A.M. Khludnev

Lavrentyev Institute of Hydrodynamics,
Siberian Division of the Russian Academy of Sciences,
Prospekt Ak. Lavrentyeva 15,
Novosibirsk 630090 Russia

A contact problem for a plate having a vertical crack is considered. The solution satisfies two restrictions of the inequality type. The first restriction is imposed in the domain and represents the mutual nonpenetration condition in the system plate–punch, the second one is put on the crack faces and corresponds to the nonpenetration of these faces. The corresponding variational inequality describing an equilibrium of the plate has the fourth order along the normal to the plate and the second order in the horizontal directions. The regularity of the solution is under consideration. In particular, $H^2 \times H^2 \times H^3$-smoothness up to the interior crack points is established. Boundary conditions having a natural physical interpretation are found on the crack faces. The existence of extreme crack shapes is also investigated. Specifically, the cost functional is defined on the feasible set of functions describing the crack shapes. The functional characterizes the deviation of the displacement vector from a given function. The problem consists in maximizing this functional. The existence of solutions of the formulated problem is proved.

1. Introduction

The model of the plate considered in the paper actually corresponds to a shallow shell having zeroth curvatures. The gradient of the punch surface is assumed to be rather small, so that the nonpenetration condition imposed in the domain is the same as in the usual case for a plate. Meanwhile, the restriction imposed on the crack faces takes into account the dependence of horizontal displacements on the distance from a middle surface and, hence, contains three components of the displacements vector.

The internal regularity of solutions of unilateral contact problems for a plate was investigated in Caffarelly, Friedman (1979); Frese (1973); Schild (1984). In these papers the operator of the problem is biharmonic. The solution properties of the equilibrium problem for plates having a crack were studied in Morozov (1984). In this case the boundary conditions imposed on both crack faces are written as equalities. As for results on the solution regularity near angle vertices in a linear elasticity theory one may refer to Grisvard (1989). Extreme crack shapes for plates in simpler situations were analyzed in Khludnev (1994). In particular, the existence theorems were proved. The stability properties for solutions, when the crack shape is perturbed, are quite close to these investigations. See the book Sokołowski, Zolesio (1992) containing the results on the shape sensitivity analysis. An approximate method of finding crack shapes, using the minimization of a cost functional, may be found in Banichuk (1970). Other approaches for determining cracks, leading to inverse boundary problems, are suggested in Friedman, Vogelius (1989).

Let $\Omega \subset R^2$ be a bounded domain with a smooth boundary $\partial\Omega$ and the equation $y = \psi(x), x \in [0,1]$, describe a crack shape on the plane x, y. The graph of the function $y = \psi(x)$ is denoted by $\Gamma_\psi, \psi \in H_0^3(0,1), \Omega_\psi = \Omega \setminus \Gamma_\psi$. Denote next by $\chi = (W, w)$ a displacement vector of the middle-surface points, where $W = (w^1, w^2)$ is horizontal displacements and w is a vertical one. Let $\varepsilon_{ij} = \varepsilon_{ij}(W)$ be the strain tensor of the middle-surface points and $N_{ij} = N_{ij}(W)$ be the integrated stresses,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\right), \quad i,j = 1,2, \; x_1 = x, x_2 = y,$$

$$N_{11} = \varepsilon_{11} + \sigma\varepsilon_{22}, \; N_{22} = \varepsilon_{22} + \sigma\varepsilon_{11}, \; N_{12} = (1-\sigma)\varepsilon_{12},$$

$$0 < \sigma < \frac{1}{2}, \; \sigma = const.$$

Introduce the energy functional of the plate

$$\Pi_\psi(\chi) = \frac{1}{2}B_\psi(w,w) + \frac{1}{2}\langle N_{ij}(W), \varepsilon_{ij}(W)\rangle_\psi - \langle f, \chi\rangle_\psi.$$

Herein

$$f = (f_1, f_2, f_3) \in L^2(\Omega), \quad \langle p, q\rangle_\psi = \int\limits_{\Omega_\psi} pq d\Omega_\psi$$

and the bilinear form $B(.,.)$ is as follows

$$\begin{aligned} B_\psi(w,\bar{w}) \quad = & \int\limits_{\Omega_\psi} (w_{xx}\bar{w}_{xx} + w_{yy}\bar{w}_{yy} + \sigma w_{xx}\bar{w}_{yy} + \sigma w_{yy}\bar{w}_{xx} \\ & +2(1-\sigma)w_{xy}\bar{w}_{xy}) d\Omega_\psi. \end{aligned}$$

Assume that the equation $z = \Phi(x,y)$ describes a punch shape, $(x,y) \in \Omega$, $\Phi \in C^1(\bar{\Omega})$. A nonpenetration condition for the system plate–punch may be written as

$$w \geq \Phi \quad \text{in} \quad \Omega_\psi \tag{1}$$

provided $\nabla\Phi$ is small enough. Denote next by $\nu = \frac{(-\psi_x, 1)}{\sqrt{1+\psi_x^2}}$ the normal vector to the curve $y = \psi(x)$ and by $2h$ the thickness of the plate, $\nu = (\nu_1, \nu_2)$. Taking into account the linear dependence of the horizontal displacements $W(z) = (w^1(z), w^2(z))$ on the distance z from the middle plane (see Vol'mir, 1972)

$$w^i(z) = w^i - zw_{x_i}, \quad i = 1, 2, \quad |z| \leq h,$$

the nonpenetration condition of crack faces takes the form

$$[W - z\nabla w]\nu \geq 0 \quad \text{on} \quad \Gamma_\psi, \quad |z| \leq h, \tag{2}$$

where $[U] = U^+ - U^-$ is the jump of U on Γ_ψ and $U^\pm$ correspond to the positive and negative directions of ν. For simplicity we put $h = 1$. The following boundary conditions are assumed to be given at the external boundary

$$w = \frac{\partial w}{\partial n} = W = 0 \quad \text{on} \quad \partial\Omega.$$

Let the subspace $H^{1,0}(\Omega_\psi) \subset H^1(\Omega_\psi)$ consist of the functions equal to zero on $\partial\Omega$ and the subspace $H^{2,0}(\Omega_\psi) \subset H^2(\Omega_\psi)$ consist of functions equal to zero on $\partial\Omega$ with the first derivatives, $H(\Omega_\psi) = H^{1,0}(\Omega_\psi) \times H^{1,0}(\Omega_\psi) \times H^{2,0}(\Omega_\psi)$. Consider the convex and closed set

$$K_\psi(\Omega_\psi) = \left\{ (W,w) \in H(\Omega_\psi) \;\middle|\; (W,w) \quad \text{satisfy} \quad (1),(2) \right\}$$

assuming that the boundary value of Φ provides the nonemptiness of $K_\psi(\Omega_\psi)$. The equilibrium problem for the plate contacting with the punch $z = \Phi(x,y)$ and having the crack shape $y = \psi(x)$ may be formulated as variational

$$\inf_{\chi \in K_\psi(\Omega_\psi)} \Pi_\psi(\chi).$$

In view of the convexity and differentiability of Π_ψ this problem is equivalent to the next one: find the function $\chi = (W, w) \in K_\psi(\Omega_\psi)$ satisfying the inequality

$$B_\psi(w, \bar{w} - w) + \langle N_{ij}(W), \varepsilon_{ij}(\bar{W} - W)\rangle_\psi \geq \tag{3}$$

$$\langle f, \bar{\chi} - \chi\rangle_\psi \quad \forall\, \bar{\chi} \in K_\psi(\Omega_\psi).$$

It will be noted that the following inequalities hold true

$$B_\psi(w, w) \geq c\|w\|^2_{2,\Omega_\psi} \quad \forall\, w \in H^{2,0}(\Omega_\psi), \tag{4}$$

$$\langle N_{ij}(W), \varepsilon_{ij}(W)\rangle_\psi \geq c\|W\|^2_{1,\Omega_\psi} \quad \forall\, W \in H^{1,0}(\Omega_\psi) \tag{5}$$

with the constants independent of w and W, respectively. The relation (5) is precisely the first Korn's inequality. The inequalities (4),(5) provide the coercivity and weak lower semicontinuity of the functional Π_ψ on $H(\Omega_\psi)$, hence, the problem (3) has a unique solution.

2. Boundary conditions on Γ_ψ

Let us elucidate the boundary conditions on Γ_ψ for the solution (W, w) assuming that $w > \Phi$ in some neighbourhood $\mathcal{W}$ of the graph Γ_ψ. To this end, we first note that the equation

$$\Delta^2 w = f_3 \tag{6}$$

holds in $\mathcal{W} \setminus \Gamma_\psi$ in the sense of distributions. Indeed, to verify this equation, the test elements of the form $(W, w) + (0, \varepsilon\varphi)$ are substituted in (3), where φ is a smooth function having a compact support in $\mathcal{W} \setminus \Gamma_\psi$ and ε is a small parameter. Moreover, the following equations hold in Ω_ψ

$$-\frac{\partial N_{ij}}{\partial x_j} = f_i, \quad i = 1, 2, \tag{7}$$

in the sense of distributions. We next denote $F = (f_1, f_2)$ and assume that the solution (W, w) is quite regular. This assumption means that the arguments given below are formal. The restriction (2) may be written as

$$\left|\left[\frac{\partial w}{\partial \nu}\right]\right| \leq [W]\nu \quad \text{on} \quad \Gamma_\psi. \tag{8}$$

Let us put functions like $(\bar{W}, w)$ in the capacity of test ones in (3), where w is the third component of the solution (W, w). This yields

$$\langle N_{ij}(W), \varepsilon_{ij}(\bar{W} - W)\rangle_\psi \geq \langle F, \bar{W} - W\rangle_\psi. \tag{9}$$

In so doing, the test functions $\bar{W}$ should satisfy the inequality

$$\left|\left[\frac{\partial w}{\partial \nu}\right]\right| \leq [\bar{W}]\nu \quad \text{on} \quad \Gamma_\psi, \quad \bar{W} \in H^{1,0}(\Omega_\psi).$$

One can represent the vector $\{N_{ij}\nu_j\}$ on Γ_ψ^- as a sum of the normal and tangential components

$$\{N_{ij}\nu_j\} = N_\nu \nu + N_s s, \quad s = (-\nu_2, \nu_1).$$

A similar formula can be written on Γ_ψ^+. Choosing the functions $\tilde{W}$ having the property $\left[\tilde{W}\right]\nu \geq 0$ on Γ_ψ the test elements $\bar{W} = W + \tilde{W}$ may be substituted in (9). Since the boundary $\partial\Omega_\psi$ of domain Ω_ψ is a combination of the sets $\partial\Omega, \Gamma_\psi^+, \Gamma_\psi^-$ the integration by parts is easily carried out. This implies

$$N_s = 0,\ N_\nu \le 0,\ N_\nu\left(\left|\left[\frac{\partial w}{\partial \nu}\right]\right| - [W]\nu\right) = 0 \quad \text{on} \quad \Gamma_\psi. \tag{10}$$

On the other hand, let the functions $\bar{\chi} = (W, \bar{w})$ be chosen as the test ones in (3). This drives to the relation

$$B_\psi(w, \bar{w} - w) \ge \langle f_3, \bar{w} - w\rangle_\psi \tag{11}$$

satisfied for test functions $\bar{w}$ such that

$$\left|\left[\frac{\partial \bar{w}}{\partial \nu}\right]\right| \le [W]\nu \quad \text{on} \quad \Gamma_\psi, \quad \bar{w} \in H^{2,0}(\Omega_\psi). \tag{12}$$

Consider the boundary operators on $\partial\Omega_\psi$

$$M(u) = \sigma\Delta u + (1-\sigma)\frac{\partial^2 u}{\partial n^2},$$

$$T(u) = \frac{\partial}{\partial n}\Delta u + (1-\sigma)\frac{\partial^3 u}{\partial n \partial^2 s}, \quad s = (-n_2, n_1).$$

Making use of the Green's formula

$$B_\psi(u, v) = \langle M(u), \frac{\partial v}{\partial n}\rangle_{\partial\Omega_\psi} - \langle T(u), v\rangle_{\partial\Omega_\psi} + \langle \Delta^2 u, v\rangle_\psi$$

the relations (11),(12) imply

$$T(w) = 0, \quad M(w)\left[\frac{\partial w}{\partial \nu}\right] - |M(w)|[W]\nu = 0 \quad \text{on } \Gamma_\psi. \tag{13}$$

In particular, the strict inequality in (8) provides $M(w) = 0$. Otherwise, the second equality in (13) determines the sign of $M(w)$. We have to note at this point that the boundary conditions (10),(13) hold on $\Gamma_\psi^\pm$ and $[N_\nu] = 0, [M(w)] = 0$. Besides, (10) holds good irrespective of the inequality $w > \Phi$ in $\mathcal{W}$, i.e. this condition takes place in general case $w \ge \Phi$. At the same time, to derive (13), we make use of the equation (6) in $\mathcal{W} \setminus \Gamma_\psi$ which takes place provided that $w > \Phi$ in $\mathcal{W}$. Moreover, the inequality $w > \Phi$ in $\mathcal{W}$ provides two more relations

$$|M(w)| \le -N_\nu,\ M(w)\left[\frac{\partial w}{\partial \nu}\right] + N_\nu[W]\nu = 0 \quad \text{on} \quad \Gamma_\psi.$$

It is seen that all these conditions have clear physical interpretation.

3. Solution regularity

Let $x^0 \in \Gamma_\psi \backslash \partial\Gamma_\psi$ be any fixed point. Assume that $\Phi = c$ in some neighbourhood $\mathcal{O}(x^0)$ of the point $x^0, c = const$, and $\Gamma_\psi \cap \mathcal{O}(x^0)$ be a segment parallel to the $x-$axis. Denote next by $R_\delta(x^0)$ the ball of the radius δ centered at the point x^0. The following assertion holds.

Theorem 1. *Let the above hypotheses be fulfilled. Then the inclusions*

$$W \in H^2(R_\delta(x^0) \cap \Omega_\psi), \quad \frac{\partial w}{\partial x} \in H^2(R_\delta(x^0) \cap \Omega_\psi)$$

take place for δ small enough.

Proof. Choose a smooth function φ such that $\varphi \equiv 1$ in $R_\delta(x^0)$, $\varphi \equiv 0$ outside of $R_{\frac{3\delta}{2}}(x^0), 0 \leq \varphi \leq 1$ everywhere, $\frac{\partial \varphi}{\partial y} = 0$ on $\Gamma_\psi \cap \mathcal{O}(x^0)$. The inclusion $R_{2\delta}(x^0) \subset \mathcal{O}(x^0)$ is assumed to be valid. Introduce the notations

$$d_{\pm\tau} p(\bar{x}) = \tau^{-1}(p(\bar{x} \pm \tau e) - p(\bar{x})), \quad \Delta_\tau = -d_{-\tau} d_\tau,$$

where e is a unit vector of the axis x, $0 < |\tau| < \frac{\delta}{2}$. Now the functions

$$w_\tau^i = w^i + \frac{\tau^2}{2}\varphi^2 \Delta_\tau w^i, \ i = 1,2, \quad w_\tau = w + \frac{\tau^2}{2}\varphi^2 \Delta_\tau w$$

may be considered in Ω_ψ. By virtue of the assumptions the normal ν has the coordinates $(0,1)$ near x^0, hence the nonpenetration condition (8) on $\Gamma_\psi \cap \mathcal{O}(x^0)$ is of the form

$$\left| \left[\frac{\partial w}{\partial y} \right] \right| \leq [w^2]. \tag{14}$$

Let us notice the following. Assuming that a function p satisfies the inequality

$$p \geq 0 \quad \text{on} \quad \Gamma_\psi \cap \mathcal{O}(x^0)$$

it is easy to check that for the above function φ the relation

$$p + \frac{\tau^2}{2}\varphi^2 \Delta_\tau p \geq 0 \quad \text{on} \quad \Gamma_\psi \cap \mathcal{O}(x^0)$$

holds. In fact, one has for $\bar{x} \in \Gamma_\psi \cap \mathcal{O}(x^0)$

$$p(\bar{x}) + \frac{\tau^2}{2}\varphi^2(\bar{x}) \Delta_\tau p(\bar{x}) =$$

$$(1 - \varphi^2(\bar{x}))p(\bar{x}) + \frac{\varphi^2(\bar{x})}{2}[p(\bar{x} - \tau e) + p(\bar{x} + \tau e)] \geq 0.$$

Bearing in mind this fact the vector $\chi_\tau = (w_\tau^1, w_\tau^2, w_\tau)$ is easily proved to satisfy the restriction (14), that is

$$\left|\left[\frac{\partial w_\tau}{\partial y}\right]\right| \le [w_\tau^2] \quad \text{on} \quad \Gamma_\psi \cap \mathcal{O}(x^0).$$

Consequently

$$\left|\left[\frac{\partial w_\tau}{\partial \nu}\right]\right| \le [W_\tau]\nu \quad \text{on} \quad \Gamma_\psi.$$

Moreover, the function w_τ satisfies the inequality

$$w_\tau \ge \Phi \quad \text{in} \quad \Omega_\psi,$$

since $\Phi = c$ in $\mathcal{O}(x^0)$. To state this, we first notice that $w_\tau = w$ outside of $R_{2\delta}(x^0)$, so that $w_\tau \ge \Phi$ in $\Omega_\psi \setminus R_{2\delta}(x^0)$. On the other hand, one has in $R_{2\delta}(x^0)$

$$w - c + \frac{\tau^2}{2}\varphi^2 \Delta_\tau w = (w - c) + \frac{\tau^2}{2}\varphi^2 \Delta_\tau (w - c) \ge 0.$$

The aforesaid means that $\chi_\tau \in K_\psi(\Omega_\psi)$. Let us substitute χ_τ in (3) as a test function. In this case we easily arrive at the inequality

$$B_\psi(w, \varphi^2 \Delta_\tau w) + \langle N_{ij}(W), \varepsilon_{ij}(\varphi^2 \Delta_\tau W)\rangle_\psi \ge \tag{15}$$

$$2\tau^{-2}\langle f, \chi_\tau - \chi\rangle_\psi.$$

It can be verified that the difference between the terms

$$B_\psi(w, \varphi^2 \Delta_\tau w) \quad \text{and} \quad - B_\psi(d_\tau(\varphi w), d_\tau(\varphi w))$$

may be estimated from above by the value being in the right-hand side of the written below inequality (16). Analogously, the difference between the terms

$$\langle N_{ij}(W), \varepsilon_{ij}(\varphi^2 \Delta_\tau W)\rangle_\psi \quad \text{and} \quad - \langle N_{ij}(d_\tau \varphi W), \varepsilon_{ij}(d_\tau \varphi W)\rangle_\psi$$

may be estimated from above by the same quantity. Thus, the relation (15) implies

$$B_\psi(d_\tau(\varphi w), d_\tau(\varphi w)) + \langle N_{ij}(d_\tau(\varphi W)), \varepsilon_{ij}(d_\tau(\varphi W))\rangle_\psi \le \tag{16}$$

$$c\{\|\chi\|^2_{H(\Omega_\psi)} + \|d_\tau(\varphi\chi)\|_{H(\Omega_\psi)}(\|\chi\|_{H(\Omega_\psi)} + \|f\|_{0,\Omega_\psi})\}.$$

In view of (4),(5) the estimate

$$\|d_\tau(\varphi\chi)\|_{H(\Omega_\psi)} \le c$$

follows being uniform in τ. It clearly yields

$$\frac{\partial}{\partial x}(\varphi\chi) \in H(\Omega_\psi).$$

So, the assertion of Theorem 1 related to w is proved. Meanwhile, the equations (7) may be written down as

$$W_{yy} = G.$$

The function G depends on f_1, f_2, W_{xy}, W_{xx} linearly, so that in view of the above result, we have $G \in L^2(R_\delta(x^0) \cap \Omega_\psi)$. Hence, all derivatives of W up to the second order belong to $L^2(R_\delta(x^0) \cap \Omega_\psi)$. Theorem 1 has been completely proved. ■

In what follows we prove the solution regularity in a neighbourhood of points belonging to the crack faces and not having a contact with the punch. Let $x^0 \in \Gamma_\psi \setminus \partial\Gamma_\psi$ be any fixed point such that $w^\pm(x^0) > \Phi(x^0)$ and, moreover, a neighbourhood $\mathcal{O}(x^0)$ of the point x^0 is assumed to be chosen such that $\Gamma_\psi \cap \mathcal{O}(x^0)$ is a segment parallel to the axis x. The following statement is valid.

Theorem 2. *Let the above hypotheses be fulfilled. Then the inclusions*

$$W \in H^2(R_\delta(x^0) \cap \Omega_\psi), \quad w \in H^3(R_\delta(x^0) \cap \Omega_\psi)$$

hold provided δ is small enough.

Proof. The condition $w^\pm(x^0) > \Phi(x^0)$ implies the fulfilment of the equation

$$\Delta^2 w = f_3 \tag{17}$$

in $R_{2\delta}(x^0) \cap \Omega_\psi$ for small δ. Take a function φ and construct the vector $\chi_\tau = (w_\tau^1, w_\tau^2, w_\tau)$ as in Theorem 1, $0 < |\tau| < \frac{\delta}{2}$. The parameter δ is supposed to be fixed such that $R_{2\delta}(x^0) \subset \mathcal{O}(x^0)$ and $w_\tau \geq \Phi$ in $R_{\frac{3\delta}{2}}(x^0)$. In this case it is seen that $w_\tau \geq \Phi$ in Ω_ψ. Moreover, it has been proved that χ_τ satisfies the restriction (8). Hence, the inclusion $\chi_\tau \in K_\psi(\Omega_\psi)$ holds. Substituting χ_τ in (3) as a test function results in the relation like (15). The further arguments remind those of Theorem 1, so that

$$W \in H^2(R_\delta(x^0) \cap \Omega_\psi), \quad \frac{\partial w}{\partial x} \in H^2(R_\delta(x^0) \cap \Omega_\psi). \tag{18}$$

Meantime, the equation (17) may be written as

$$w_{yyyy} = Q.$$

According to (18) the inclusion $Q \in H^{-1}(R_\delta(x^0) \cap \Omega_\psi)$ holds. Whence, taking into account the relations $w_{yyy}, w_{yyyx} \in H^{-1}(R_\delta(x^0) \cap \Omega_\psi)$ and the results of Duvaut, Lions (1972) we arrive at the desired conclusion:

$$w_{yyy} \in L^2(R_\delta(x^0) \cap \Omega_\psi).$$

Theorem 2 has been proved. ■

Remark. Seemingly, the hypothesis relating to $\Gamma_\psi \cap \mathcal{O}(x^0)$ in Theorem 1 and Theorem 2 may be omitted, but it is not proved.

4. Extreme crack shapes

Suppose that the crack shape is described by the equation $y = \delta\psi(x)$ with a parameter δ. The space $H(\Omega_\delta)$ and the set $K_\delta(\Omega_\delta)$ are introduced analogously to $H(\Omega_\psi)$ and $K_\psi(\Omega_\psi)$, respectively. In the following we analyze the behaviour of the solution when $\delta \to 0$. It will enable us in the sequel to prove the existence of extreme crack shapes. The formulation of this problem will be given below. So, for every fixed δ there exists a solution $\chi^\delta = (W^\delta, w^\delta)$ of the problem

$$B_\delta(w^\delta, \bar{w} - w^\delta) + \langle N_{ij}(W^\delta), \varepsilon_{ij}(\bar{W} - W^\delta)\rangle_\delta \geq \tag{19}$$

$$\langle f, \bar{\chi} - \chi^\delta\rangle_\delta, \quad \chi^\delta \in K_\delta(\Omega_\delta), \quad \forall\, \bar{\chi} \in K_\delta(\Omega_\delta).$$

In order to study the solution convergence as $\delta \to 0$ we carry out the mapping of Ω_δ onto Ω_0. Of course, the graphs $y = \delta\psi(x)$ are assumed to belong to Ω for all $0 \leq \delta \leq \delta_0$. Extend the function ψ beyond $[0,1]$ by zero, then choose domains Ω_1, Ω_2 such that $\bar{\Omega}_1 \subset \Omega_2, \bar{\Omega}_2 \subset \Omega, \Gamma_\delta \subset \Omega_1$ for all δ small enough and a function ξ possessing the properties: $\xi \equiv 1$ in $\Omega_1, \xi \equiv 0$ in $\Omega \setminus \Omega_2$. The following transformation of the independent variables may be considered

$$\tilde{x} = x, \quad \tilde{y} = y - \delta\xi\psi. \tag{20}$$

It is clear that the Jacobian $q_\delta = 1 - \delta\psi\xi_y$ of this transformation converges uniformly to the unit on Ω as $\delta \to 0$. Introduce the notations

$$U^\delta(\tilde{x}, \tilde{y}) \equiv W^\delta(x, y), \;\; u^\delta(\tilde{x}, \tilde{y}) \equiv w^\delta(x, y), \;\; \omega^\delta = (U^\delta, u^\delta).$$

A substitution of a fixed test function $\bar{\chi}$ in (19) drives to the relation

$$B_\delta(w^\delta, w^\delta) + \langle N_{ij}(W^\delta), \varepsilon_{ij}(W^\delta)\rangle_\delta \leq$$

$$B_\delta(w^\delta, \bar{w}) + \langle N_{ij}(W^\delta), \varepsilon_{ij}(\bar{W})\rangle_\delta + \langle f, \chi^\delta - \bar{\chi}\rangle_\delta.$$

Omitting the sign δ in the functions it is easy to rewrite this inequality in the new variables

$$\int_{\Omega_0} \left\{ u_{\tilde{x}\tilde{x}}^2 + u_{\tilde{y}\tilde{y}}^2 + 2\sigma u_{\tilde{x}\tilde{x}} u_{\tilde{y}\tilde{y}} + 2(1-\sigma) u_{\tilde{x}\tilde{y}}^2 \right\} q_\delta^{-1} d\Omega_0 + \tag{21}$$

$$\langle N_{ij}(U), \varepsilon_{ij}(U) q_\delta^{-1}\rangle_0 - \langle f^\delta, (\omega - \bar{\omega}) q_\delta^{-1}\rangle_0 +$$

$$\delta \int_{\Omega_0} g(\tilde{x}, \tilde{y}, \delta, D^\alpha u, D^\alpha \bar{u}, D^\beta U, D^\beta \bar{U}) d\Omega_0 \leq 0.$$

Herein $f^\delta(\tilde{x}, \tilde{y}) \equiv f(x, y), |\alpha| \leq 2, |\beta| \leq 1$. A dependence of the function g on its arguments is fully determined by the transformation (20). It is of importance that this function has quadratic growth in the principal derivatives. In view of the inequality $q_\delta^{-1} > \frac{1}{2}$ holding for small δ we conclude from (21) that

$$\|\omega^\delta\|_{H(\Omega_0)} \le c$$

uniformly in $\delta \le \delta_0$. Choosing a subsequence, if necessary, one can assume that as $\delta \to 0$

$$\omega^\delta \to \omega \quad \text{weakly in} \quad H(\Omega_0). \tag{22}$$

The solution (U^δ, u^δ) satisfies the inequalities (1),(2) written in the new variables. To be more precise, we denote $\Phi^\delta(\tilde{x}, \tilde{y}) \equiv \Phi(x, y)$. Then the inequality (1) takes the form

$$u^\delta \ge \Phi^\delta \quad \text{in} \quad \Omega_0, \tag{23}$$

and the inequality (2) may be written as

$$\left[U^\delta - z(u^\delta_{\tilde{x}} - \delta\psi_x u^\delta_{\tilde{y}}, u^\delta_{\tilde{y}})\right](-\delta\psi_x, 1) \ge 0 \quad \text{on} \quad \Gamma_0, \quad |z| \le 1. \tag{24}$$

Let the set of all functions (U, u) from the space $H(\Omega_0)$ satisfying (23),(24) be denoted by $K_\delta(\Omega_0)$. The following statement is useful for further consideration.
Lemma. *For any fixed $(\bar{U}, \bar{u}) \in K_0(\Omega_0)$ there exists a sequence $(\bar{U}^\delta, \bar{u}^\delta) \in K_\delta(\Omega_0)$ such that as $\delta \to 0$*

$$(\bar{U}^\delta, \bar{u}^\delta) \to (\bar{U}, \bar{u}) \quad \textit{strongly in} \quad H(\Omega_0). \tag{25}$$

Proof. We make use of the following assertion proved in Khludnev (1994). For any fixed function $(\bar{U}, \bar{u}) \in H(\Omega_0)$ satisfying the inequality

$$\left[\bar{U} - z\nabla\bar{u}\right]\nu \ge 0 \quad \text{on} \quad \Gamma_0, \quad |z| \le 1, \quad \nu = (0, 1),$$

a sequence $(\bar{U}^\delta, \bar{u}^\delta) \in H(\Omega_0)$ may be constructed such that $(\bar{U}^\delta, \bar{u}^\delta) \to (\bar{U}, \bar{u})$ strongly in $H(\Omega_0)$ and, moreover, the restriction (24) holds good and $\bar{u}^\delta = \bar{u}$ for all δ. Let us take a fixed element $(\bar{U}, \bar{u}) \in K_0(\Omega_0)$ and bearing in mind the aforesaid construct a sequence $\bar{\chi}^\delta = (\bar{U}^\delta, \bar{u})$ having the above properties. We show that the appropriate changing the third component of $\bar{\chi}^\delta$ by $\bar{u}^\delta$ will imply the sequence $(\bar{U}^\delta, \bar{u}^\delta)$ to be needful, that is $(\bar{U}^\delta, \bar{u}^\delta) \in K_\delta(\Omega_0)$ and (25) takes place. Since $\Phi^\delta \to \Phi$ uniformly on Ω and $\Phi^\delta = \Phi$ near $\partial\Omega$ there exists a function θ^δ such that

$$\theta^\delta \ge |\Phi^\delta - \Phi| \quad \text{in} \quad \Omega, \qquad \theta^\delta \to 0 \quad \text{strongly in} \quad H^2(\Omega).$$

We should remark at this step that $\bar{u} \ge \Phi$ in Ω_0. Putting $\bar{u}^\delta = \bar{u} + \theta^\delta$ it is easily checked the sequence $(\bar{U}^\delta, \bar{u}^\delta)$ to satisfy all conditions. Indeed, the restriction (23) for $\bar{u}^\delta$ holds by construction of θ^δ. Since the jump $\left[(\theta^\delta_{\tilde{x}} - \delta\psi_x\theta^\delta_{\tilde{y}}, \theta^\delta_{\tilde{y}})\right]$ is equal to zero on Γ_0 the restriction (24) for $(\bar{U}^\delta, \bar{u}^\delta)$ aslo holds. The convergence (25) is evident. Lemma has been proved.

Let us now rewrite (19) in the new variables $\tilde{x}, \tilde{y}$. The convergence (22) and Lemma allow us to carry out the limiting procedure when $\delta \to 0$. Moreover, the limiting function $\omega = (U, u)$ is a solution of the variational inequality

$$B_0(u, \bar{u} - u) + \langle N_{ij}(U), \varepsilon_{ij}(\bar{U} - U)\rangle_0 \geq$$

$$\langle f, \bar{\omega} - \omega\rangle_0, \quad \omega \in K_0(\Omega_0), \quad \forall\, \bar{\omega} \in K_0(\Omega_0). \tag{26}$$

So, the following statement has been proved.

Theorem 3. *From the sequence $\chi^\delta = \omega^\delta$ of solutions of the problem (19) one may choose a subsequence, still denoted by ω^δ, such that as $\delta \to 0$ the convergence (22) takes place and, moreover, the limiting function satisfies (26).*

This result enables us to investigate the extreme crack shape problem. The formulation of the last one is as follows. Let $\Psi \subset H^3_0(0,1)$ be a convex, closed and bounded set. Assume that for every $\psi \in \Psi$ the graph $y = \psi(x)$ describes the crack shape. Consequently, for a given $\psi \in \Psi$ there exists a unique solution of the problem

$$B_\psi(w, \bar{w} - w) + \langle N_{ij}(W), \varepsilon_{ij}(\bar{W} - W)\rangle_\psi \geq \tag{27}$$

$$\langle f, \bar{\chi} - \chi\rangle_\psi, \quad \chi = (W, w) \in K_\psi(\Omega_\psi), \quad \forall\, \bar{\chi} \in K_\psi(\Omega_\psi).$$

Consider the cost functional

$$J(\psi) = \|\chi - \chi_0\|_{0,\Omega_\psi}$$

where $\chi_0 \in L^2(\Omega)$ is a prescribed element. We have to find a solution of the maximization problem

$$\sup_{\psi \in \Psi} J(\psi). \tag{28}$$

The following assertion holds.

Theorem 4. *Let the above hypotheses be fulfilled. Then, there exists a solution of the problem (28).*

We shall confine ourselves to short remarks. A maximizing sequence $\psi^n \in \Psi$ is evidently bounded in $H^3_0(0,1)$. Hence, without any loss, one may assume that as $n \to \infty$

$$\psi^n \to \psi \quad \text{weakly in} \quad H^3_0(0,1), \quad \psi \in \Psi, \tag{29}$$

$$|\psi^n_{xx} - \psi_{xx}| < \frac{1}{n} \quad \text{on} \quad [0,1].$$

For any n, there exists a solution (W^n, w^n) of the problem

$$B_{\psi^n}(w^n, \bar{w} - w^n) + \langle N_{ij}(W^n), \varepsilon_{ij}(\bar{W} - W^n)\rangle_{\psi^n} \geq \tag{30}$$

$$\langle f, \bar{\chi} - \chi^n\rangle_{\psi^n} \quad \forall\, \bar{\chi} \in K_{\psi^n}(\Omega_{\psi^n}).$$

The domains Ω_1, Ω_2 and the function ξ may be chosen as in the proof of Theorem 3. The transformation of the independent variables is of the form

$$\tilde{x} = x, \quad \tilde{y} = y + (\psi - \psi^n)\xi.$$

We prove that the solution $U^\delta(\tilde{x}, \tilde{y}) \equiv W^\delta(x, y)$, $u^\delta(\tilde{x}, \tilde{y}) \equiv w^\delta(x, y)$ satisfies the following estimate

$$\|U^n\|_{1,\Omega_\psi} + \|u^n\|_{2,\Omega_\psi} \leq c.$$

Without loss a generality, one may suppose that as $n \to \infty$

$$(U^n, u^n) \to (U, u) \quad \text{weakly in} \quad H(\Omega_0), \text{ strongly in} \quad L^2(\Omega_0). \tag{31}$$

For the passage to the limit in the relations obtained from (30) by a change of variables, we use the convergence (31) and the statement analogous to Lemma. The limiting function $\chi = (U, u)$ is a solution of the variational inequality (3) with the function ψ from (29), that is $\chi = \chi_\psi$. At last, it is easy to verify that

$$J(\psi) = \sup_{\bar{\psi} \in \Psi} J(\bar{\psi}).$$

This precisely means that the limiting function ψ is a solution of the extreme crack shape problem (28).

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