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An analytical method for solving a type of fuzzy optimization problems

by

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A new method for solving a type of fuzzy optimization problems is proposed. It concerns fuzzy optimization problems with crisp parameters and *soft* (*flexible*) constraints and objective(s) under certain additional conditions. For this special case of fuzzy mathematical programming (FMP) the necessary and sufficient conditions for analytical solving the problem are formulated. A comparison between the results derived by this method and by the widely used Zimmermann's method is presented.

Keywords: Bellman-Zadeh's approach, fuzzy linear programming, Zimmermann's method

1. Introduction

In formulating real optimization problems, it is usually difficult to state constraints and criteria exactly Dumitru, Luban (1986); Kacprzyk, Orlovski (1987); Verdegay (1984). The main reasons for this are impreciseness and vagueness of the data, subjectivity of criteria formulation etc. Negoita, Ralescu (1977); Negoita, Sularia (1976); Orlovski (1978); Rommelfanger (1993). One of the possible tools for coming to grips with such problems is fuzzy sets theory Yingjun Feng (1978); Zimmermann (1978;1985). Fuzzy mathematical programming (FMP) was originally introduced by Tanaka, Okuda and Asai; Verdegay (1982) as a tool dealing with *soft* optimization problems when the objective(s) are determined as fuzzy sets and small violations of constraints are possible. This concept was extended to the case of multiobjective programming Chanas (1989); Hannan (1981); Kacprzyk (1983); Verdegay (1984); Yazenin (1987), linear programming Deldago, Verdegay, Vila (1989); Streng (1975); Werners (1987), optimal control Angelov, Tzonkov (1993); Baldwin, Pilsworth (1982); Filev, Angelov (1992); Negoita, Minoiu, Stan (1976).

The constraints (equalities and non-equalities) and criteria in FMP problems are represented by their membership functions. A number of mathematical expressions are used: piece-wise linear Carlson, Korhonen (1986); Chanas (1992); Maleki, Mashinchi (1993), concave fractional Nakamura (1984); Rommelfanger, Hanusheck, Wolf (1989), exponential functions Sakawa, Zimmermann (1983), logistic functions Zimmermann (1979) etc.

There exist various formulations and techniques for solving flexible programming problems Luhandjula (1988); Negoita (1984); Oheigeartaigh (1982); Ramik (1987), nearly all of them are based on Bellman-Zadeh's concept Bellman, Zadeh (1970). In some particular cases mathematical programming problems with fuzzy parameters are also solved via Bellman-Zadeh's approach Ramik (1986); Rommelfanger, Hanusheck, Wolf (1989); Tanaka, Asai (1984). Most of the authors treating FMP problems propose to transform them into non-fuzzy optimization problems which can be solved numerically (by simplex or gradient techniques). Recently, a special property of the solution of FMP problems under certain conditions has been discussed Angelov (1993). The paper contains a comprehensive study of this property and some more detailed conclusions are made for fuzzy linear programming (FLP) problems. A simple illustrative example is presented.

2. Problem statement

FLP problem is formulated in the sense of flexible programming as follows Zimmermann (1983):

$$J = Cx \Rightarrow \tilde{\max}\ x \in R^n;\ J \in R^p,\ C \in R^{p\times n} \tag{1}$$

$$Ax \tilde{\leq} b\ \ A \in R^{q\times n};\ b \in R^{q\times 1}; \tag{2}$$

where

x vector of unknowns,
p number of objectives,
q number of inequalities,
A, b, C crisp parameters,
$\tilde{\max}$ fuzzy maximum,
$\tilde{\leq}$ fuzzy inequality.

Fuzzy sets expressing the objective(s) and the constraints are represented by their membership functions uniquely: $\mu_i(x)$, $i = 1, \ldots, p$ for objectives; and $\mu_j(x)$, $j = 1, \ldots, q$ for constraints. Membership functions represent the acceptance of given values of unknowns. In FLP membership functions are piece wise linear.

The problem is considered symmetrically as consisting of $(p+q)$ – fuzzy conditions represented by their membership functions. Fuzzy solution is determined as an intersection of all fuzzy sets (fuzzy objective(s) and fuzzy constraints):

$$\mu_D(x) = \cap_{r=1}^{p+q}\{\mu_r(x)\} \tag{3}$$

where

$\mu_D(x)$ denotes decision membership functions

In Bellman-Zadeh's approach the so called "min-operator" is applied as intersection:

$$\mu_D(x) = \min_{r=1}^{p+q}\{m_r(x)\} \tag{4}$$

Deterministic solution (x^0) which is the final result of the problem solving is found in Bellman-Zadeh's approach by maximization of $\mu_D(x)$ on x:

$$x^0 = \max_x \mu_D(x) \tag{5}$$

Negoita and Sularia (1976) and Zimmermann (1983) have proposed to transform the problem (4)-(5) to a crisp MP problem:

$$\max \lambda \tag{6}$$

$$\lambda \leq \mu_k(x) \; k = 1, \ldots, p+q \tag{7}$$

$$\lambda \leq 1 \tag{8}$$

$$\lambda \geq 0 \tag{9}$$

where

$\lambda = \min_{k=1}^{p+q} \mu_k(x)$

This problem can be solved by simplex method in the case of piece-wise linear membership functions $\mu_k(x)$, $k = 1, \ldots, p+q$ or by a gradient method otherwise. However, the extremum may be a local one Zimmermann (1985) in the non-linear case.

In general, crisp (deterministic) solution is determined by defuzzification Angelov (1994). In Bellman-Zadeh's approach "mean of maximum" method of defuzzification (5) is used. Interesting results have been reached Angelov (1993) when BADD defuzzification method Filev, Yager (1991) is applied for small scale problems. For large scale problems these results are not so effective because of the discrete character of the BADD defuzzification method.

The method proposed here applies "min-operator" and "mean of maximum" defuzzification method. It can be considered as a detailization of the Bellman-Zadeh's approach for FLP under certain conditions. It treats the problem from a different point of view and allows to find solution directly (without transformation to a crisp optimization problem) and determines necessary and sufficient conditions for determination of the global extremum.

3. A new method for analytical solving of FLP problems in certain cases

Solution of FMP problems possesses an important property which was described recently, Angelov, Kuncheva (1993). It is valid for all types FMP problems for which membership functions of fuzzy sets are such that for x^1 and x^2 ($x^1 \in R^n$; $x^2 \in R^n$; $\mu_r(x^2) > \mu_r(x^1)$) it follows that $\cos\psi > 0$; $\psi = (\vec{x^1, x^2}; \vec{\nabla\mu_r(x^1)})$, where

$\vec{x^1, x^2}$ vector between points x^1 and x^2,

$\vec{\nabla\mu_r(x^1)}$ gradient vector of $\mu_r(x)$ in point x^1.

This restriction is quite loose and, practically, almost all of the usually used types of membership functions (including piece wise linear) satisfy it.

The following theorem gives both necessary and sufficient conditions for solving of this problem in certain cases:

THEOREM 3.1 *Let an FLP problem of the type (1)-(2) be given which fuzzy sets are defined by piece wise linear membership functions. Let the decision membership function be defined as* $\mu_D(x) = \min_{r=1}^{p+q}\{\mu_r(x)\}$. *Suppose that there exists* x^0 *such that:*

1. $\mu_{r_s}(x^0) = \mu_D(x^0) = constant\ r_s \in]1; p+q[\ s = 1, \ldots, n-1$

2. $\exists$ *a convex combination such that* $\sum_{r_s=1}^{n-1} K_{r_s} \vec{\nabla\mu_{rs}(x^1)} = 0\ s = 1, \ldots, n-1$ *where* K_{r_s} *- coefficients, such that* $K_{r_s} \geq 0$ *for* $r_s \in]1; p+q[;\ s = 1, \ldots, n-1$ *with at least one* $K_{r_s} > 0$

Then $\mu_D(x^0) = \sup \mu_D(X)$.

Proof Let us assume that there exists x^*:

$$\mu_D(x^*) > \mu_D(x^0) \tag{10}$$

Then from the definition of the decision membership function it directly follows that $\mu_{r_s}(x^*) > \mu_D(x^*)$; $s = 1, \ldots, n-1$.

From statement 1) for x^0 it follows that $\mu_D(x^0) = \mu_{r_s}(x^0)$; $s = 1, \ldots, n-1$.

Combining both conditions the following inequality holds:

$$\mu_{r_s}(x^*) > \mu_D(x^*) > \mu_D(x^0) = \mu_{r_s}(x^0)\ s = 1, \ldots, n-1 \tag{11}$$

or

$$\mu_{r_s}(x^*) > \mu_{r_s}(x^0)\ s = 1, \ldots, n-1 \tag{12}$$

From the definition of the membership functions it follows that $\cos(\vec{x^0, x^*}; \vec{\nabla\mu_{rs}(x^0)}) > 0$; $s = 1, \ldots, n-1$.

The second condition of the theorem can be projected:

$$\sum_{r_s=1}^{n-1} K_{rs} \|\nabla\mu_{rs}(x^0))\| \cos\psi = 0 \tag{13}$$

where $\|\cdot\|$ – denotes the norm of a vector

The value of the norm of the gradient vector of $\mu_{r_s}(x)$ in the point x^0 is non-negative. It is equal to zero only in the case when $x^0 = \max_x \mu_{rs}(x)$. However, this case directly rejects the assumption of existing of x^* because of (12). Therefore the norm of the gradient vector is positive.

From the second condition of the theorem (as a convex combination) $K_{r_s} \geq 0$; $s = 1, \ldots, n-1$ with at least one $K_{r_s} \neq 0$.

Hence, the second condition of the theorem never holds for x^* which satisfy (10). ■

Analyzing the theorem for special cases of FLP problems (for different dimensions of the vector of unknowns (n), the numbers of inequalities (q) and objectives (p)) more detail and practically useful results can be formulated:

COROLLARY 3.1 *If a FLP problem is given and the following equality holds:*

$$n = p + q - 1 \tag{14}$$

Then the deterministic result (x^0) is found as the solution of a system of linear equations.

It gives the global extremum because the solution of a system of linear equations is unique Streng (1975). Membership functions are considered as linear and a normalization ($0 \leq \mu_s(X) \leq 1$) is applied after reaching the solution of the system of equations. An example for the case of $n = 1$ is presented in Fig. 1. The coefficients of the membership functions should satisfy the second condition of the theorem.

COROLLARY 3.2 *If a FLP problem is given and the following inequality holds:*

$$n < p + q - 1 \tag{15}$$

Then the deterministic result (x^0) is found as the solution of a system of linear equations.

The first condition of the theorem leads to the following system of linear equations:

$$\begin{aligned} \mu_1(x^0) - \mu_2(x^0) &= 0 && (16)\\ \mu_2(x^0) - \mu_3(x^0) &= 0 && (17)\\ &\cdots \\ \mu_{p+q-1}(x^0) - \mu_{p+q}(x^0) &= 0 \end{aligned}$$

Taking various combinations of n equations out of $(p+q-1-n)$ equations a number of systems of linear equations are derived. They give the unique solution

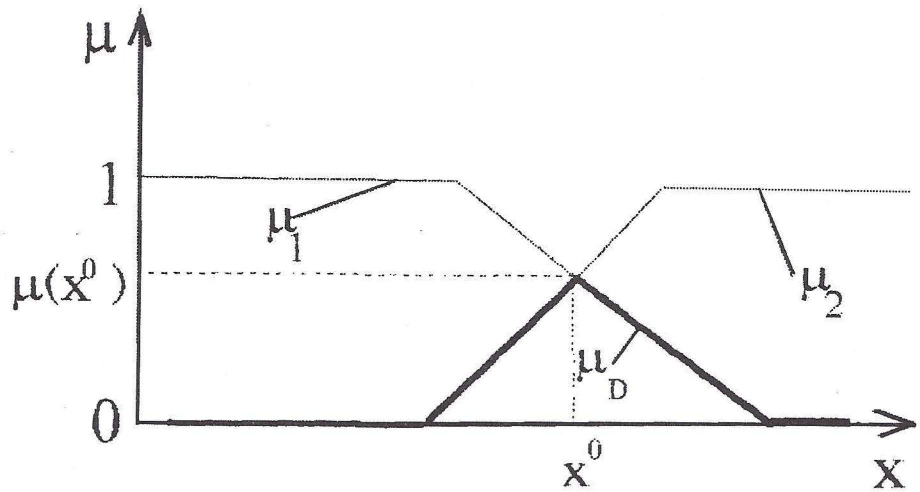

Figure 1. Graphical representation of the deterministic result (x^0) at a FLP for $n = 1$ (Corollary 1).

of every system. The final result is obtained after maximization of some partial results (the solutions of systems of linear equations):

$$x^0 = \max_{v=1}^{M} x_v^0 \; v = 1, \ldots, M \; M = C_{p+q-1}^n \tag{18}$$

where C_{p+q-1}^n – denotes permutations of n between $(p + q - 1)$ elements.

4. Example

Let us consider FLP problem of the type (1)-(2) where:

$$A = \begin{vmatrix} 2 & 0 & 0 & 5 & 4 & 0 \\ 1 & 0 & 0 & 4 & 0 & 4 \\ 0 & 2 & 4 & 0 & 0 & 1 \\ 5 & 0 & 0 & 6 & 5 & 3 \\ 2 & 0 & 0 & 10 & 8 & 0 \\ 0 & 0 & 0 & 10 & 8 & 0 \end{vmatrix} \quad b = \begin{vmatrix} 27 \\ 24 \\ 33 \\ 26 \\ 0 \\ 1 \end{vmatrix} \quad C = \begin{vmatrix} 3 & 0 & 6 & 4 & 0 & 5 \end{vmatrix}$$

$n = 6;\ p = 1;\ q = 6$

Fuzzy sets are represented by the membership functions μ_l and μ_2 for the objectives and $\mu_3, \ldots, \mu_8$ for the constraints respectively:

$$\mu_i = \begin{cases} 0; & \sum_{j=1}^n c_{ij}x_j < 0 \\ \sum_{j=1}^n c_{ij}x_j/\gamma_i; & 0 \le \sum_{j=1}^n c_{ij}x_j \le \gamma_i \quad \gamma_i = 100;\ i = 1, 2 \\ 1; & \sum_{j=1}^n c_{ij}x_j > \gamma_i \end{cases}$$

$$\mu_k = \begin{cases} 1; & \sum_{j=1}^n a_{kj}x_j < \alpha_k \\ 1 - \frac{\sum_{j=1}^n a_{kj}x_j - \alpha_k}{\beta_k}; & \alpha_k \le \sum_{j=1}^n a_{kj}x_j \le \beta_k \quad k = 2, \ldots, 5 \\ 0; & \sum_{j=1}^n a_{kj}x_j > \beta_k \end{cases}$$

$$\alpha = \begin{vmatrix} 0 \\ 37 \\ 26 \\ 24 \\ 0.5 \\ 0 \end{vmatrix} \quad \beta = \begin{vmatrix} 4 \\ 3 \\ 5 \\ 7 \\ 0.5 \\ 1 \end{vmatrix}$$

$$\mu_r = \begin{cases} 0; & \sum_{j=1}^n a_{rj}x_j < 0 \\ \left(\sum_{j=1}^n a_{rj}x_j - \alpha_j\right)/\beta_r; & \alpha_r \le \sum_{j=1}^n a_{rj}x_j \le \beta_r \quad r = 6, 7; \\ 1; & \sum_{j=1}^n a_{rj}x_j > \beta_r \end{cases}$$

Analogical crisp (non-fuzzy) problem is given as an example in Steuer (1986).

This FLP problem can be transformed by Zimmermann's approach to the following crisp LP problem:

$$\max \lambda \tag{19}$$

$$\lambda \le \mu_i \quad i = 1, \ldots, 7$$

$$0 \le \lambda \le 1$$

Then this problem can be solved numerically by the simplex method. The result is:

$$x^* = \begin{vmatrix} -0.426471 \\ -4.395221 \\ 8.178309 \\ 6.779412 \\ -8.143382 \\ 2.665441 \end{vmatrix}, \ \lambda = 0.882353$$

The initial problem can be solved easily using the theorem and especially Corollary 3.1. Then the following linear system of equations is derived:

$$\lambda = \mu_1 \ \lambda = \mu_2 \ \lambda = \mu_3 \ \lambda = \mu_4 \ \lambda = \mu_5 \ \lambda = \mu_6 \ \lambda = \mu_7 \tag{20}$$

It becomes, after transformations:

$$\lambda = \sum_{j=1}^n c_{ij}x_j/\gamma_i$$

$$\lambda = 1 - \frac{\sum_{j=1}^{n} a_{kj}x_j - \alpha_k}{\beta_k} \quad k = 2, \ldots, 5$$

$$\lambda = \left(\sum_{j=1}^{n} a_{rj}x_j - \alpha_j \right) / \beta_r \quad r = 6, 7$$

$$D\xi^0 = d \tag{21}$$

where

$$D = \begin{vmatrix} -\gamma & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ \beta_2 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \beta_3 & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ \beta_4 & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \beta_5 & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ -\beta_6 & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \\ -\beta_7 & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} \end{vmatrix}; \; d = \begin{vmatrix} 0 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \alpha_4 + \beta_4 \\ \alpha_5 + \beta_5 \\ \alpha_6 \\ \alpha_7 \end{vmatrix}; \; \xi = \begin{vmatrix} \lambda \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{vmatrix}$$

The system of linear equations (22) can be solved directly and solution ξ^0 is determined from:

$$\xi^0 = D^{-1}d \tag{22}$$

The solution is the same:

$$x^0 = \begin{vmatrix} -0.4264706 \\ -4.395221 \\ 8.178309 \\ 6.779412 \\ -8.143382 \\ 2.665441 \end{vmatrix}, \; \lambda = 0.882353$$

This example illustrates the possibility to solve easily FLP problems which satisfy the conditions of the Theorem 3.1 and of Corollary 3.1. The result reached under these conditions is the same as this reached by other techniques. The main advantage is the analytical way for its determination. In certain cases (in linear fuzzy dynamic programming problems) the result (optimal control and optimal trajectory) can be determined analytically using the results presented here. It allows to design algorithms for on-line optimal control in fuzzy environment for linear systems. This case is under consideration.

5. Conclusion

An approach for analytical solving of certain cases of FMP problems is proposed and discussed. Both necessary and sufficient conditions for solving this problems have been derived. The initial problem is converted to a system of equations and the global extremum is found for FLP under considered conditions. An illustrative example is presented. An extension of the result to the case of LP problems with fuzzy parameters and to on-line fuzzy optimal control are under consideration.

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