Control and Cybernetics

vol. **25** (1996) No. 3

On identification of a diffusion coefficient

by

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Abstract. We describe some recent results on recovery of the principal coefficient of a second order partial differential equation of parabolic type, given one or all possible sets of the lateral Cauchy data of its solution. We outline ideas of proofs referring for details to other publications. The results are expected to be of importance in the inverse heat conduction and the inverse hydraulics problem.

Keywords: inverse problems in parabolic equations, numerical solution of ill-posed problems, heat conduction, underground hydraulics.

1. Formulation of the inverse problem

We are interested in finding the diffusion coefficient a of the following parabolic equation

$$u_t - \mathrm{div}(a\nabla u) = f \qquad \text{in} \quad Q = \Omega \times (0,T), \tag{1.1}$$

where Ω is a domain in R^n (a bounded one, a half-space or the whole space). We assume the zero initial conditions

$$u = 0 \qquad \text{on} \quad \Omega \times \{0\}, \tag{1.2}$$

the lateral boundary condition

$$u = g \qquad \text{on} \quad \partial\Omega \times (0,T) \tag{1.3}$$

and the boundedness condition on u if Ω is unbounded. If $a \in L^\infty(Q)$ and is strictly positive, the known (Friedman, 1964; Ladyzenskaja et al., 1968) results guarantee uniqueness, existence and stability of a (generalized) solution u to the direct problem (1.1)-(1.3) in a natural classical (Hölder or Sobolev) functional space. Our object is the following:

INVERSE PROBLEM *Find the coefficient a from the additional lateral boundary data*

$$a\partial_\nu u = h \quad \text{on} \quad \Gamma \times (0,T),\ \Gamma \subset \partial\Omega \tag{1.4}$$

or from u given outside of a bounded domain Ω_0 if $\Omega = \mathbf{R}^n$.

[1]The work of the second author was in part supported by the NSF grant DMS 9501410.

2. Single boundary measurements

The single boundary measurements mean that we are given h in (1.4) for only one (or, maybe) few g. If the initial data are zero there is actually only one uniqueness result in the one-dimensional case.

THEOREM 2.1 *Let $\Omega = (0,1)$ in $\mathbf{R}^1$. Let $g(t) = t^{1/2}$ and $\Gamma = \{0\}$. Then the diffusion coefficient $a \in C^1[0,1]$ is uniquely determined by the data (1.4) of the parabolic problem (1.1)-(1.3).*

We will give an idea of a proof which is based on a reduction to a one-dimensional inverse hyperbolic problem whose theory is rather elementary and relatively well understood.

Consider the transform

$$u(x,t) = (\pi t)^{-1/2} \int_0^\infty \exp(-\theta^2/(4t)) u^*(x,\theta) d\theta \tag{2.1}$$

where u^* solves the hyperbolic problem

$$\begin{aligned} &\partial_\theta^2 u^* - \partial_x(a(x)\partial_x u^*) = 0 \quad \text{in} \quad Q^* = \Omega \times (0,T^*), \\ &u^* = \partial_\theta u^* = 0 \quad \text{on} \quad \Omega \times \{0\}, \\ &u^* = g^* \quad \text{on} \quad \partial\Omega \times (0,T^*). \end{aligned} \tag{2.2}$$

One can check that in our case $g^*(\theta) = \sqrt{\pi}\theta/2$ if $x = 0$ and is zero if $x = 1$. Integrating (2.1) by parts and using fact that the function on the right-hand side satisfies the heat equation in t, θ one can conclude that

$$\partial_t u(x,t) = (\pi t)^{-1/2}(\partial_\theta u^*(x,0) + \int_0^\infty \exp(-\theta^2/(4t))\partial_\theta^2 u^*(x,\theta)d\theta).$$

Hence if u^* solves (2.2), then the corresponding u will solve the parabolic problem (1.1)–(1.3).

The additional data

$$\partial_x u^* = h^* \quad \text{on} \quad \partial\Omega \times (0,T^*) \tag{2.3}$$

(where T^* is any (large) number) can be obtained by inverting the relation (2.1) where h is the Neumann data of the parabolic problem. It is clear that this inversion is unique, but not stable in the Hadamard sense. Originally, the parabolic problem is considered on the interval $(0,T)$, but it can be solved with the same g on the interval $(0,\infty)$. Since the coefficient of the parabolic equation is time independent and the boundary data are analytic, the solution $u(x,t)$ is analytic with respect to $t \in (0,\infty)$, so it is uniquely determined by its values on $(0,T)$. This analytic continuation is a conditionally stable operation which is discussed in more detail in Isakov (1995).

Using standard arguments in one-dimensional inverse hyperbolic problems (Isakov, 1990) one can show uniqueness of a.

Observe that all the instability in the inverse parabolic problem is isolated in the inversion of the transform (2.1), and the inverse hyperbolic problem is stable. However, it is strongly nonlinear.

Similarly, one can treat the boundary data g generated via (2.1) by any function $g^* \in C^k([0,\infty))$, growing at infinity not more rapidly than $\exp(C\tau)$ and whose k-th order derivative is not zero at the origin.

At present there are no uniqueness results in the multidimensional case when the initial data are zero.

3. Many boundary measurements: uniqueness

In this case h is given for all regular g (say, in $C_0^2(\partial\Omega \times (0,T))$) which are zero outside $\Gamma \times (0,T)$, or in other words we are given the so-called lateral Dirichlet-to-Neumann map Λ. In Theorems 3.1 and 3.2 we assume that $f = 0$ and that Ω is a bounded domain with the C^2-boundary. We emphasize that Γ can be any arbitrarily small (but open and nonempty) part of $\partial\Omega$.

THEOREM 3.1 *Λ uniquely determines a t-independent $a \in C^1(\bar{\Omega})$.*

We will give only an idea of the proof which will be published elsewhere.

As above, we can associate with the parabolic problem (1.1)–(1.3) the following hyperbolic one

$$\partial_\theta^2 u^* - div(a\nabla u^*) = 0 \qquad \text{on} \quad \Omega \times (0,T^*)$$

with the zero initial conditions at $\theta = 0$ and the lateral boundary Dirichlet condition

$$u^* = g^* \qquad \text{on} \quad \partial\Omega \times (0,T^*).$$

We claim that the lateral Dirichlet-to-Neumann map $g \to h$ for the parabolic equation uniquely determines the Dirichlet-to-Neumann map Λ^* (with any finite T^*) for the hyperbolic equation. Indeed, let g^* be any function in $C^2(\partial\Omega \times (0,T^*)$ (which is zero outside $\Gamma \times (0,T^*)$. We extend this function as zero if $T^* < \theta$. Consider the parabolic problem (1.1)–(1.3) with the data g. Since g^* is compactly supported, the definition (2.1) shows that $g(x,t)$ is analytic with respect to $t \in (0,\infty)$. The known properties of the parabolic problems with time-independent coefficients guarantee that the solution $u(x,t)$ and its first order derivatives are analytic with respect to $t \in (0,\infty)$, therefore the Neumann data h which are originally given on $\Gamma \times (0,T)$ are determined uniquely (by the analytic continuation) on $\Gamma \times (0,\infty)$. Since the inversion of (2.1) is unique, h^* is uniquely determined on $\Gamma \times (0,T^*)$ for any finite T^*. For $a \in C^2(\bar{\Omega})$ and $\Gamma = \partial\Omega$ Theorem 3.1 can be derived from the results of Nachman ($n = 2$) (see Nachman, 1995) and of Sylvester and Uhlmann ($3 \le n$) (see Sylvester and Uhlmann, 1987) on the inverse conductivity problem.

In the theory of inverse hyperbolic problems it has been already proved that Λ^* given for large T^* uniquely determines a. It was done by Belishev (1987) who first used methods of optimal control in inverse problems. In the mentioned paper he considered the case $\Gamma = \partial\Omega$ and a slightly different hyperbolic equation. Recently (AMS-SIAM Summer Conference in Seattle, 1995) he announced also uniqueness for local data (Γ is not the whole $\partial\Omega$).

In very important practical problems (inverse hydraulics) the coefficient a is discontinuous. To present the result for such coefficients we define the lateral boundary $\partial_x Q^*$ of an open subset Q^* of the layer $R^n \times (0,T)$ as the closure of $\partial Q^* \cap \{0 < t < T\}$. We say that Q^* is x-Lipschitz, if its lateral boundary is locally the graph of a Lipschitz function $x_j = x_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n, t)$.

In the next theorem we assume that a_0 is a given positive $C^2(\bar{\Omega})$–function.

THEOREM 3.2 *Let $n \geq 2$. Suppose that Q^* is an open Lipschitz subset of Q and $\partial_x Q^* \cap \partial_x Q$ is empty. Assume that*

$$\text{the sets} \quad (Q \setminus \bar{Q}^*) \cap \{t = \tau\} \quad \text{are connected when } 0 < \tau < T \tag{3.1}$$

Then Λ uniquely determines $a = a_0 + k\chi(Q^*)$, $k = k(x) \in C^2(\bar{\Omega})$, $k \neq 0$ on $\partial_x Q^*$.

Proof. We will outline the proof of this result referring for details to the forthcoming paper (Elayyan and Isakov, to appear).

Assume that $Q_1 \neq Q_2$. Then we can assume that there is a point $(x_0, t_0) \in \partial_x Q_1 \setminus \bar{Q}_2$ which is contained also in $\bar{Q}_3$, where Q_3 is the union (over $\theta \in (0,T)$) of $Q_{3\theta}$. Here $Q_{3\theta}$ is the connected component of $\Omega \setminus (\bar{Q}_{1\theta} \cup \bar{Q}_{2\theta})$ whose boundary contains Γ. $Q_{j\theta}$ are defined as $Q_j \cap \{t = \theta\}$. Taking $g = 0$ for $t < t_0$ and using the translation we can also assume that $t_0 = 0$, $x_0 = 0$. Choose a ball B in $\mathbf{R}^n$ centered at the origin and a cylinder $Z = B \times (-\tau, \tau)$ so that $\bar{B} \subset \Omega, \bar{Z}$ is disjoint with $\bar{Q}_2$ and the lateral boundary of Q_1 in Z is a Lipschitz surface. By the Whitney Extension Theorem there is a $C^2(\bar{Q}_1 \cup \bar{Z})$–function a_3 coinciding with a_1 on Q_1. Extend a_3 onto $Q \setminus (Q_1 \cup Z)$ as a_0.

First we establish the so-called orthogonality relations

$$\int_{Q_1} k_1 \nabla u_3 \cdot \nabla u_2^* = \int_{Q_2} k_2 \nabla u_3 \cdot \nabla u_2^* \tag{3.2}$$

for any solution u_3 to the differential equation

$$\partial_t u_3 - \operatorname{div}(a_3 \nabla u_3) = 0 \quad \text{near} \quad \bar{Q}_4, \qquad u_3 = 0 \quad \text{if} \quad t < 0$$

and for any solution u_2^* to the (adjoint) equation

$$-\partial_t u_3^* - \operatorname{div}(a_2 \nabla u_2^*) = 0 \quad \text{near} \quad \bar{Q}_4, \qquad u_2^* = 0 \quad \text{if} \quad T < t.$$

Here Q_4 is $Q \setminus \bar{Q}_3$.

To explain why the relation (3.2) holds we subtract two equations (1.1) with $a = a_2$ and $a = a_1$ to obtain the following equation for the difference u of their solutions u_2 and u_1

$$\partial_t u - \mathrm{div}(a_2 \nabla u) = \mathrm{div}(k_2 \chi(Q_2) - k_1 \chi(Q_1)).$$

Since u_1, u_2 have the same lateral Cauchy data on $\Gamma \times (0, T)$, we derive from known uniqueness of the continuation results in parabolic equations (Isakov, 1990) that $u = 0$ on Q_4. Then "multiplying" the equation by u_2^* and "integrating by parts" (or, more precisely, using the definition of a weak solution) we obtain the relation (3.2) with u_1 instead of u_3 because the integral on the left-hand side is zero. To obtain (3.2) for u_3 we can use the Runge approximation theorem and to approximate a_3 by a sequence of coefficients which are equal to a_1 in neighborhoods of $\bar{Q}_1$.

To obtain a contradiction with the initial assumption we take as u_3 and u_2^* the singular (fundamental) solutions K^+ and K^- to the forward and backward diffusion equations with the coefficients a_3 and a_2 with the poles placed outside of $\bar{Q}_3$ which converge to the origin. Using special coordinates we can assume that $\partial_x Q_1$ contains the points $(0, t) : 0 < t < T$ and that the direction of the x_n-axis coincides with the exterior normal to $\partial_x Q_1$ at the origin. The fundamental solutions mentioned above have the following structure

$$K^+ = K_1^+ + K_0^+, \qquad K^- = K_1^- + K_0^-,$$

where the first terms are the so-called parametrices

$$K_1^+(x,t;y,\tau) = C(t-\tau)^{-n/2} a_3^{-1}(y) \exp(-|x-y|^2/(a_3(y)(t-\tau)),$$
$$K_1^-(x,t:y,\tau) = C(\tau-t)^{-n/2} a_3^{-1}(y) \exp(-|x-y|^2/(a_0(y)(\tau-t))$$

and K_0^+, K_0^+ are remainders, with weaker singularities (Friedman, 1964).

Letting $u_3 = K^+(;0,0), u_2^* = K(;0,\tau)$ in the relations (3.2) (which we can do first by taking poles at $(y,0),(y,\tau)$ with $y = (0,...,0,-\delta)$ and then letting δ converge to zero), splitting the fundamental solutions into parametrices and less singular parts, and breaking the integration domain Q_1 into $Q_1 \cap Z$ and its complement we obtain

$$I_1 = I_2 + I_3, \tag{3.3}$$

where

$$I_1 = \int_{Q_1 \cap Z} k_1 \nabla_x K_1^+ \cdot \nabla_x K_1^-$$

$$I_2 = -\int_{Q_1 \setminus Z} k_1 \nabla_x K^+ \cdot \nabla_x K^- + \int_{Q_2} k_2 \nabla_x K^+ \cdot \nabla_x K^-$$

and

$$I_3 = \int_{Q_1 \cap Z} k_1(\nabla_x K_1^+ \cdot \nabla_x K_0^- + \nabla_x K_0^+ \cdot \nabla_x K_1^- + \nabla_x K_0^+ \cdot \nabla_x K_0^-).$$

The basic analytic step of the proof is to show that

$$\tau^{-n/2} \le C|I_1|, |I_2| \le C\tau^{-n/2+1}\epsilon^{-2}\exp(-\epsilon^2/(M\tau)),$$
$$|I_3| \le C\tau^{-n/2+3/2}\epsilon^{-2}(1 - \exp(-\epsilon^2/(M\tau)), \tag{3.4}$$

where M depends only on an upper bound of a_3, a_0 over Q.

With this preparation the end of the proof follows very quickly. We let

$$\epsilon^2 = E\tau,$$

where (large) E is to be chosen later. From (3.3) and (3.4) we have

$$\tau^{-n/2} \le C^2(\tau^{-n/2+1}\epsilon^{-2}\exp(-E/M) + \tau^{-n/2+3/2}\epsilon{-2}(1 - \exp(-E/M)).$$

Using the relation between ϵ and τ and multiplying both parts of the last inequality by $\tau^{n/2}$ we obtain

$$1 \le C^2E^{-1}\exp(-E/M) + C^2E^{-1}\tau^{1/2}(1 - \exp(-E/M)).$$

Choose E so large that $E^{-1}e^{-E/M} \le 1/(2C)^2$, then the first term on the right-hand side is smaller than 1/4. Then choose τ so small that the last term is smaller than 1/4 as well. We arrive at a contradiction which shows that our original assumption was wrong, so $Q_1 = Q_2$.

If the unknown domains are equal it is relatively easy to show that $k_1 = k_2$ by modifying slightly the argument in Isakov (1989). □

4. Numerical results

The considered inverse problem is strongly nonlinear (with respect to (u, a) that both are to be determined) and ill-posed, so it must be quite difficult (if at all possible) to obtain its numerical solution. As the first step we decided to linearize the inverse problem and to test it numerically.

First we observe that our original inverse problem with several lateral boundary measurements for the equation (1.1) with $f = 0$ in a bounded domain can be reduced to a problem in the whole space with $f(x,t) = \delta(x - x^*)\delta(t)$.

Indeed, let u_ϕ be the (bounded at infinity) solution to the Cauchy problem (1.1), (1.2) with $a = 1 + \phi, \operatorname{supp}\phi$ is in a bounded domain Ω_0, and with f equal to the delta-function. Then, u_0 is a known function (the fundamental solution to the Cauchy problem for the heat equation), and the difference $w = u_\phi - u_0$ solves the following Cauchy problem

$$w_t - \Delta w = \operatorname{div}(\phi\nabla u_\phi) \text{ in } R^n \times (0,T), \qquad w = 0 \text{ for } t = 0.$$

Since $\operatorname{supp}\phi$ is in Ω_0, the function w satisfies the homogeneous heat equation outside Ω_0. Let Ω_1 be any bounded domain with smooth boundary containing closure of Ω_0. Outside Ω_1 the function w can be represented as the simple layer potential

$$w(x,t) = S\rho(x,t) = \int_0^t \int_{\partial\Omega_1} \rho(y,\tau)\Gamma(x-y,t-\tau)dS(y)d\tau,$$

where Γ is the well-known fundamental solution to the Cauchy problem for the heat equation.

Now, let Λ be the lateral Dirichlet-to-Neumann operator for the equation (1.1) corresponding to the domain Ω_1. Denoting by w^- the function w outside $\Omega_1 \times (0,T)$ we obtain

$$\partial_\nu w^- = \partial_\nu u_\phi - \partial_\nu u_0 = \Lambda(u_0 + w) - \partial_\nu u_0 = \Lambda S\rho + (\Lambda - \partial_\nu)u_0.$$

The well-known jump relations for the normal derivatives of simple layer potentials (Ladyzenskaya et al., 1968) give

$$\partial_\nu w^-(x,t) = -2^{-1}\rho(x,t) + W\rho(x,t),$$

where W is the normal derivative of the simple layer on $\partial\Omega_1$. From these two relations we have the following integral equation for ρ

$$(I + (-2^{-1}I - W + \Lambda S))\rho = U_0,$$

where I is the identity operator and $U_0 = (\partial_\nu - \Lambda)u_0$. It is proved in Elayyan and Isakov (1995) that the operator $-2^{-1}I - W + \Lambda S$ is a contraction in $L^\infty(\partial\Omega_1 \times (0,T))$ if T is small, so for ρ we have an integral equation which can be solved in a stable and efficient way. Summing up, given any point x^* outside the closure of Ω_1 we can efficiently find ρ, w and u_ϕ outside Ω_1.

Now we will study the equation (1.1) with $f = \delta(x - x^*), \Omega = R^n$ assuming that $a = 1 + \phi$ does not depend on t, $\operatorname{supp}\phi \subset \Omega_0$ (a bounded domain), and either $\|\phi\|_\infty(\Omega_0)$ is small or $\phi = k\chi(\Omega_2)$, where k has the bounded C^1-norm and $\operatorname{vol}\Omega_2$ is small. The solution u of the problem (1.1),(1.2) (the Cauchy Problem for the parabolic equation) is $u(x,t;x^*)$. Linearizing the original problem around $a = 1$ we arrive at the following equations

$$v_t - \Delta v = \operatorname{div}(\phi\nabla u_0) \qquad \text{on} \quad \mathbf{R}^n \times (0,T) \tag{4.1}$$

with the zero initial condition. Here u_0 is the solution to the unperturbed (heat) equation with the delta-function as the source term. The solution of the equation (4.1) with the zero initial data is $v(x,t;x^*)$. Now, motivated by the inverse backscattering, we let $x = x^*$ and fix T to reduce the overdeterminancy of our inverse problem by posing the following:

LINEARIZED INVERSE PROBLEM *Find* $\phi \in L^2(\Omega_0)$ *given the solution* $v(x,T;x)$ *of the linearized equation (4.1), where* $x \in \Omega^*$

Using the well-known representation of a solution to the Cauchy problem for the heat equation via potentials we arrive at the integral equation

$$A\phi(x) = \Phi(x), \qquad x \in \Omega^*, \tag{4.2}$$

where $A\phi(x) = \int_{\Omega_0} k(x-y)\phi(y)dy$ and

$$k(x) = -4^{-1}(4\pi)^{-n} \int_0^T |x|^2(\theta(T-\theta))^{-\frac{n}{2}-1}\exp(-\frac{|x|^2 T}{4\theta(T-\theta)}d\theta.$$

THEOREM 4.1 *A solution $\phi \in L^2(\Omega_0)$ to the equation (4.2) is unique.*

Proof. We will give an almost complete proof of this theorem. Since the integral equation is linear, to prove uniqueness it suffices to consider $\Phi = 0$ on Ω^* and to show that $\phi = 0$. Assume that ϕ is not zero.

Applying the Fourier transform to the both parts of (4.2) and using the fact that the Fourier transform of the convolution is the product of the Fourier transforms we will have

$$\hat{\Phi}(\xi) = \hat{k}(\xi)\hat{\phi}(\xi). \tag{4.3}$$

As can be seen from its integral representation, the function Φ is analytic outside Ω_0. Since it is zero on the open set Ω^*, it is zero outside some compact set by uniqueness of the analytic continuation. Using also our conditions we conclude that both Φ and ϕ have compact supports, hence by the Paley-Wiener theorem their Fourier transforms are entire analytic functions of order 1. So they are defined for all complex vectors ζ and

$$|\hat{\phi}(\zeta)| + |\hat{\Phi}(\zeta)| \leq Ce^{C|\zeta|}, \quad \zeta \in \mathbf{C}^n \tag{4.4}$$

On the other hand, using the explicit formulae for the Fourier transform of quadratic functions one can show that

$$\hat{k}(\xi) = c_n \int_0^T (2\theta(1-\frac{\theta}{T})\xi\cdot\xi - n)exp(-\xi\cdot\xi\theta(1-\frac{\theta}{T}))d\theta$$

Substituting $\xi = \zeta = iR\xi_0$, where ξ_0 is a non-zero real vector, and $R \geq 1$, using the fact that the two terms in the integrand have the same sign and dropping the first term we obtain

$$|\hat{k}(\zeta)| \geq c_n n \int_0^T \exp(|\zeta|^2\theta(1-\frac{\theta}{T}))d\theta \geq c_n n \int_{T/4}^{T/2} \exp(|\zeta|^2 3T/16)d\theta,$$

where we shrank the integration interval in the first integral and have used that on the smaller interval $3T/16 \leq \theta(1-\theta/T)$. Summing up we conclude that for the chosen ζ there is a positive ϵ such that

$$|\hat{k}(\zeta)| \geq \epsilon e^{\epsilon|\zeta|^2}. \tag{4.5}$$

Now we are ready to complete the proof of Theorem 4.1. If $\phi \neq 0$ then $\hat{\phi}(\xi_0) \neq 0$ for some nonzero $\xi_0 \in \mathbf{R^n}$ and therefore the entire function $\phi_0(z) = \phi(z\xi_0)$ of one complex variable is of order 1 and not identically zero. By the Littlewood theorem there are a constant C and a sequence r_j convergent to $+\infty$ such that $\min|\phi_0(z)| > e^{-Cr_j}$ where the minimum is taken over $|z| = r_j$. From (4.3), (4.4) and (4.5) it follows now that

$$|\hat{k}(\zeta)| \leq Ce^{2Cr_j}, |\zeta| = r_j|\xi_0|$$

which contradicts (4.4).

The contradiction shows that the initial assumption was wrong and that $\phi = 0$. The proof of Theorem 4.1 is complete. □

We anticipate logarithmic (conditional) stability of solution of the integral equation (4.2) which is not proven yet. It is quite clear, however, that this equation represents a severely ill-posed problem which must be regularized to be solved numerically. Observe that the operator A is a convolution operator, but since the function Φ is not defined on the whole $\mathbf{R}^n$, the equation (4.2) is not a convolution equation, so the Fourier transform will not be of much help.

In our numerical investigations (Elayyan and Isakov, 1995) which gave surprisingly good results, we replaced the equation (4.2) by its standard regularization

$$(\alpha + A^*A)\phi_\alpha = A^*\Phi,$$

where A^* is the operator adjoint to A acting from $L^2(\Omega_0)$ into $L^2(\Omega^*)$. We discretized the integrals by the trapezoid method to obtain functions of a discrete argument defined on the rectangular uniform N or $N \times N$ grid, and solved the resulting system of linear equations by the conjugate gradient method on our IBM mainframe.

We have considered the one-dimensional case: $\Omega_0 = (0,2), \Omega^* = (3,5)$. We chose $N = 30$, $\alpha = 10^{-6}$, $T = 4$ and recovered the function $\phi(x) = \sin(\pi x/2)$ with high precision which slightly deteriorated when we added a random noise of 0.01 (in maximum norms) to the data of our problem. We had to generate the data numerically because it was impossible to calculate them analytically. We did it using a different method that the one used to solve the inverse problem to create an additional noise.

An important test was a two-dimensional one, where we took $\Omega_0 = (0,2) \times (0,2)$, $\Omega^* = (3,5) \times (0,2)$. We put $\alpha = 10^{-14}$ and recovered the function ϕ from the previous example even better. The same value of the parameters was used to find the discontinuous function ϕ defined as 1 on $(0,1) \times (0,2)$ and as 0 otherwise. Now the discontinuity surface was not recovered sharply, but its location was easy to be distinguished.

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