Control and Cybernetics

vol. **25** (1996) No. 3

Boundary control and dynamical inverse problem for nonselfadjoint Sturm–Liouville operator (BC–method)[1]

by

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Abstract. The paper develops an approach to inverse problems which is based on ideas of the boundary control theory, the so called BC-method (Belishev, 1987; Avdonin, Belishev and Ivanov, 1991). It also gives a new interpretation of the local approach (Blagoveshchenskii, 1971). Our main aim is an extension of the BC-method to the case of nonselfadjoint problems. The results obtained in this paper and the methods developed here will be useful in nonselfadjoint inverse problems for the hyperbolic type equations in multidimensional domains.

Keywords: inverse problems, boundary control, distributed parameter systems

1. Direct problem, boundary control problem

1.1. Problem I

Let $V(x)$ be a real $N \times N$ matrix-function with continuously differentiable elements for $x \in \mathbf{R}_+$. Consider the following initial boundary value problem (Problem I) for a vector-function $u(x,t)$ (here and below we assume that all functions are real)

$$u_{tt} - u_{xx} + V(x)u = 0 \ , \ \ (x,t) \in \mathbf{R}_+ \times (0,T) \ , \ T > 0 \ , \tag{1}$$

$$u(x,0) = u_t(x,0) = 0 \ , \tag{2}$$

[1]This work was supported by the Commission of the European Communities in the framework of the EC–Russia collaboration (contract ESPRIT P 9282 ACTCS), by the Russian Foundation of Fundamental Research (grant 95 01 00360a) and by the International Science Foundation (grant NSI300). The authors express their gratitude to these foundations.

$$u(0,t) = f(t) \ . \tag{3}$$

Solution to this problem is denoted by $u^f(x,t)$ to stress dependence on f in the boundary condition. Sometimes, following physical terminology, V, f and u^f are called *potential*, *source* or *control*, and *wave*, correspondingly.

Let matrix kernel $w(x,t)$ be a solution of the Goursat problem

$$\left\{ \begin{array}{ll} w_{tt} - w_{xx} + V(x)w = 0, & 0 < x < t < T, \\ w(0,t) = 0, & w(x,x) = -\frac{1}{2}\int_0^x V(s)ds \ ; \end{array} \right. \tag{4}$$

where $w(x,t)$ is a twice continuously differentiable function in the triangle $\{(x,t) : 0 \leq x \leq t \leq T\}$. The following assertions can be easily checked.

PROPOSITION 1.1 *(i) If $f \in C^2([0,T]; \mathbf{R}^N)$ and $f(0) = f'(0) = 0$ then Problem I has the unique classical solution $u = u^f(x,t)$.*

$$u^f(x,t) = \left\{ \begin{array}{ll} f(t-x) + \int_x^t w(x,s) f(t-s)ds & \text{if } x < t \\ 0 & \text{if } x \geq t \end{array} \right. \tag{5}$$

(ii) For $f \in L_2(0,T; \mathbf{R}^N)$ the function $u^f(x,t)$ defined by representation (5) satisfies equation (1) in the sense of the theory of distributions.

As follows from (5), at any fixed moment $t = \xi$

$$\operatorname{supp} u^f(\cdot,\xi) \subset \Omega^\xi, \ 0 \leq \xi \leq T, \tag{6}$$

where $\Omega^\xi := [0,\xi]$ is an interval on the x–axis. The same representation implies the inclusion

$$u^f(\cdot,T) \in L_2(\Omega^T; \mathbf{R}^N) \text{ for all } \ f \in L_2(0,T); \mathbf{R}^N). \tag{7}$$

Let $\mathcal{T}^{T,\xi}$ be a delay operator

$$(\mathcal{T}^{T,\xi})(t) := f_{T-\xi}(t) := \left\{ \begin{array}{ll} 0, & 0 \leq t < T-\xi, \\ f(t-(T-\xi)), & T-\xi \leq t \leq T, \end{array} \right. \tag{8}$$

where ξ is a parameter, $\xi \in (0,T)$;

$$\mathcal{T}^{T,T} f := f, \ \mathcal{T}^{T,0} f := 0. \tag{9}$$

Independence of the potential $V(x)$ of t implies the following well known property of the solution u^f:

$$u^{f_{T-\xi}}(\cdot,T) = u^f(\cdot,\xi). \tag{10}$$

While omitting details we will mention one more property of the solution u^f which follows from the hyperbolicity of system (1)–(3). It is the so–called "local principle". For any fixed $\xi \in (0,T/2)$ the values of $u^f(x,t)$ for $(x,t) : 0 \leq x \leq \xi, \ x \leq t \leq 2\xi - x$ are uniquely determined by the values of $V\,|_{x \leq \xi}$, being independent of the behaviour of $V\,|_{x > \xi}$.

1.2. Boundary control problem

We set up a boundary control problem (BCP) in the following way.

For a given $a \in L_2(\Omega^T; \mathbf{R}^N)$ find $f \in L_2(0,T; \mathbf{R}^N)$ such that

$$u^f(\cdot, T) = a\,. \tag{11}$$

This setting turns out to be natural in view of the properties (6), (7).

LEMMA 1.1 *For any $T > 0, a \in L_2(\Omega^T; \mathbf{R}^N)$ there exists the unique solution of the BCP.*

Proof. According to (5) the condition (11) in the BCP is equivalent to the equation

$$a(x) = f(T-x) + \int_x^T w(x,s) f(T-s)ds, \;\; x \in \Omega^T. \tag{12}$$

This is the Volterra equation of the second kind with respect to f. Its solvability implies solvability of the BCP. □

2. Operators of BC–method

2.1. Problem I as a dynamical system

In further considerations it is convenient to use some terms from the abstract theory of linear systems. The space of controls (inputs) $\mathcal{F}^T := L_2(0,T; \mathbf{R}^N)$ is called an outer space of dynamical system (1)–(3). The space $\mathcal{H}^T := L_2(\Omega^T; \mathbf{R}^N)$ is said to be an inner space; at any fixed moment $t = \xi$ the wave (state) $u^f(\cdot, \xi)$ belongs to $\mathcal{H}^T$ (see (6), (7)). The inner and outer spaces are connected by the (control) operator $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$,

$$W^T f = u^f(\cdot, T) \tag{13}$$

LEMMA 2.1 *For any $T > 0$ the operator W^T is bounded and boundedly invertible (i.e. W^T is an isomorphism).*

The proof follows directly from Lemma 1.1. The right hand side of (12) gives us the representation of the operator W^T:

$$(W^T f)(x) = f(T-x) + \int_x^T w(x,s) f(T-s)ds, \;\; x \in \Omega^T. \tag{14}$$

In the outer space $\mathcal{F}^T$ we introduce a family of extending subspaces $\mathcal{F}^{T,\xi}$ formed by delayed controls (see (8)–(10)),

$$\mathcal{F}^{T,\xi} := \mathcal{T}^{T,\xi} \mathcal{F}^T = \{\, f \in \mathcal{F}^T \; : \mathrm{supp} f \subset [T-\xi, T] \,\}, \; 0 \le \xi \le T. \tag{15}$$

The set

$$\mathcal{U}^{\xi} := W^T \mathcal{F}^{T,\xi} \tag{16}$$

is said to be a reachable set for the system (1)–(3) (in time ξ).

Lemma 2.1 shows that $\mathcal{U}^T = \mathcal{H}^T$. It is clear that an analogous equality is valid for any moment of time:

$$\mathcal{U}^{\xi} = \mathcal{H}^{\xi} \ , \ 0 \le \xi \le T, \tag{17}$$

where $\mathcal{H}^{\xi}$ are subspaces of $\mathcal{H}^T$,

$$\mathcal{H}^{\xi} = \left\{ a \in \mathcal{H}^T \ : \text{supp } a \subset \Omega^{\xi} \right\}.$$

2.2. Response operator

We consider a function $u_x^f(0,t) \in \mathcal{F}^T$ as an output of the dynamical system (1)–(3) corresponding to the input $f(t)$. An "input–output" correspondence is realized by the (response) operator R^T ,

$$R^T : \mathcal{F}^T \mapsto \mathcal{F}^T \ , \ \text{Dom } R^T = \left\{ f \in C^2([0,T];\mathbf{R}^N) : f(0) = f'(0) = 0 \right\},$$

$$\left(R^T f \right)(t) = u_x^f(0,t) \ , t \in [0,T].$$

It is defined correctly by virtue of Proposition 1.1. In the system theory this operator is called a dynamical transfer function. Proposition 1.1 also yields the following statement.

PROPOSITION 2.1 *For any $T > 0$ the following representation is valid:*

$$\left(R^T f \right)(t) = -f'(t) + \int_0^t r(t-s)f(s)ds \ , \ 0 < t < T \ , \tag{18}$$

where the matrix-function $r(t)$ is continuously differentiable for $t > 0$.

The response operator plays below a role of data of the inverse problem.

2.3. Duality relations

In what follows the upper index "$\#$ " means the matrix transposition.

Let us introduce the Problem $I_{\#}$:

$$u_{tt} - u_{xx} + V^{\#}(x)u = 0 \ , \ (x,t) \in \mathbf{R}_+ \times (0,T) \ , \ T > 0 \ ,$$

$$u(x,0) = u_t(x,0) = 0 \ , \ u(0,t) = g(t) \ .$$

Let $u_{\#}^g$ be the solution and $W_{\#}^T$, $W_{\#}^T g = u_{\#}^g(\cdot,T)$, be a corresponding control operator. Similarly as W^T, the operator $W_{\#}^T$ is an isomorphism.

Similarly to (18) the response operator $R_{\#}^T$, $(R_{\#}^T g)(t) = \left(u_{\#}^g\right)_x(0,t)$, may be represented in the form of

$$(R_{\#}^T g)(t) = -g'(t) + \int_0^t r_{\#}(t-s)g(s)ds \ . \tag{19}$$

LEMMA 2.2 *The matrix kernels $r(t)$ and $r_{\#}(t)$ in (18) and (19) are mutually transposed:*

$$r^{\#}(t) = r_{\#}(t) . \tag{20}$$

Proof. Let u^f be a solution of Problem I and $u^g_{\#}$ be a solution of Problem $I_{\#}$. Set $w^g(x,t) := u^g_{\#}(x, T-t)$. The function w^g satisfies the equation

$$w^g_{tt} - w^g_{xx} + V^{\#}(x)w^g = 0 \ , \ \ (x,t) \in \mathbf{R}_+ \times (0,T) \tag{21}$$

and the initial and boundary conditions

$$w^g(x,T) = w^g_t(x,T) = 0 \ , \ \ w^g(0,t) = g(T-t) . \tag{22}$$

From (19) we also have

$$w^g_x(0,t) = -g'(T-t) + \int_0^{T-t} r_{\#}(T-t-s)g(s)ds . \tag{23}$$

Using (1)–(3) and (21), (22), from integration by parts we obtain

$$\int_0^T \langle u^f_x(0,t), w^g(0,t)\rangle dt = \int_0^T \langle u^f(0,t), w^g_x(0,t)\rangle dt ,$$

where $\langle \cdot, \cdot \rangle dt$ is a scalar product in $\mathbf{R}^N$.

Taking into account (18) and (23) and changing the order of integration one can obtain (20). □

2.4. Connecting operator

Let us introduce an operator which plays a central role in our approach to inverse problems. For arbitrary functions $f,\ g \in \mathcal{F}^T$ and corresponding solutions $u^f,\ u^g_{\#}$ of Problems I and $I_{\#}$ we have:

$$\left(u^f(\cdot,T), u^g_{\#}(\cdot,T)\right)_{\mathcal{H}^T} = (W^T f, W^T_{\#} g)_{\mathcal{H}^T} = \left((W^T_{\#})^* W^T f, g\right)_{\mathcal{F}^T} =$$

$$= (C^T f, g)_{\mathcal{F}^T} \ ,$$

where

$$C^T : \mathcal{F}^T \ \mapsto \mathcal{F}^T \ \ , \ C^T := (W^T_{\#})^* W^T . \tag{24}$$

This operator is an isomorphism together with $W^T_{\#}$ and W^T .

A very important fact is that the operator C^T can be explicitly expressed via the response operator. To formulate the result we need to introduce the following operators:

the operator of odd continuation $S^T : \mathcal{F}^T \mapsto \mathcal{F}^{2T}$,

$$(S^T f)(t) = \begin{cases} f(t)\,, & 0 \le t \le T \\ -f(2T-t)\,, & T < t \le 2T; \end{cases}$$

the operator extracting an odd part of controls $Q^{2T} : \mathcal{F}^{2T} \mapsto \mathcal{F}^{2T}$,

$$(Q^{2T} f)(t) = \frac{1}{2}[f(t) - f(2T-t)]\ ;$$

the operator of restriction $N^T : \mathcal{F}^{2T} \mapsto \mathcal{F}^T$,

$$N^T f = f\mid_{[0,T]}\ .$$

It is easy to check that $(S^T)^* = 2N^T Q^{2T}$.

We shall also use the operators R^{2T} and J^{2T} — the response operator and the integration operator in the space $\mathcal{F}^{2T}$,

$$(J^{2T} f)(t) = \int_0^t f(s)ds\ , 0 \le t \le 2T\ .$$

THEOREM 2.1 *The following representations are valid*

$$C^T = -\frac{1}{2}(S^T)^* J^{2T} R^{2T} S^T\ , \tag{25}$$

$$(C^T f)(t) = f(t) + \int_0^T [p(2T-t-s) - p(\mid t-s\mid)]f(s)ds\ , \tag{26}$$

where

$$p(t) := \frac{1}{2}\int_0^t r(s)ds\ .$$

Proof. Let us choose arbitrary functions $f, g \in C_0^\infty([0,T]; \mathbf{R}^N)$ and set

$$f_- := S^T f,\ w^{fg}(s,t) := \left(u^{f_-}(\cdot,s), u_\#^g(\cdot,t)\right)_{\mathcal{H}^T}\ ;\ 0 \le s \le 2T,\ 0 \le t \le T\ .$$

It should be noted that $f_- \in \mathrm{Dom}\ R^{2T}$. Integrating by parts we have the equalities

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right] w^{fg}(s,t) =$$

$$\int_{\Omega^T} \left[\langle u^{f_-}(x,s), (u_\#^g)_{tt}(x,t)\rangle - \langle u_{ss}^{f_-}(x,s), (u_\#^g)(x,t)\rangle\right] dx =$$

$$\int_{\Omega^T} \Big[\langle u^{f_-}(x,s), (u_\#^g)_{xx} - V^\#(x) u_\#^g(x,t)\rangle -$$

$$\langle u_{ss}^{f_-}(x,s) - V(x)u^{f_-}(x,s), u_\#^g(x,t)\rangle\Big]\, dx =$$

$$\int_{\Omega^T}\Big[\langle u^{f_-}(x,s),(u^g_{\#})_{xx}(x,t)\rangle-\langle u^{f_-}_{xx}(x,s),(u^g_{\#})(x,t)\rangle\Big]\,dx=$$
$$\Big[\langle u^{f_-}(x,s),(u^g_{\#})_x(x,t)\rangle-\langle u^{f_-}_x(x,s),(u^g_{\#})(x,t)\rangle\Big]_{x=0}^{T}=$$
$$-\langle f_-(s),(R^T_{\#}g)(t)\rangle+\langle(R^{2T}f_-)(s),g(t)\rangle.$$

In the last equality we took into account that $u^{f_-}(T,s)=u^g_{\#}(T,t)=0$ for $f,g\in C_0^\infty([0,T];\mathbf{R}^N)$.

Thus, the function w^{fg} satisfies the equation

$$w^{fg}_{tt}-w^{fg}_{ss}=-\langle f_-(s),(R^T_{\#}g)(t)\rangle+\langle(R^{2T}f_-)(s),g(t)\rangle\,;\quad 0\le s\le 2T,\ 0\le t\le T \tag{27}$$

and the initial conditions (see (2))

$$w^{fg}(s,0)=w^{fg}_t(s,0)=0\,. \tag{28}$$

The values of w^{fg} for $0\le t\le s\le T$ may be found by means of the D'Alembert's formula. Using it for $t=s=T$ we get

$$w^{fg}(T,T)=-\frac{1}{2}\int_0^T d\eta\int_\eta^{2T-\eta}[\langle f_-(\xi),(R^T_{\#}g)(\eta)\rangle-\langle(R^{2T}f_-)(\xi),g(\eta)\rangle]d\xi\ .$$

Since $\int_\eta^{2T-\eta}f_-(\xi)d\xi=0$,

$$w^{fg}(T,T)=\frac{1}{2}\int_0^T d\eta\int_\eta^{2T-\eta}\langle(R^{2T}f_-)(\xi),g(\eta)\rangle d\xi\ . \tag{29}$$

On the other hand,

$$\int_\eta^{2T-\eta}(R^{2T}f_-)(\xi)d\xi$$
$$=(J^{2T}R^{2T}f_-)(2T-\eta)-(J^{2T}R^{2T}f_-)(\eta)=-2(Q^{2T}J^{2T}R^{2T}f_-)(\eta),$$

and equality (29) takes the form of

$$w^{fg}(T,T)=-\int_0^T\langle(N^TQ^{2T}J^{2T}R^{2T}S^Tf)(\eta),g(\eta)d\eta$$
$$=-\frac{1}{2}\Big((S^T)^*J^{2T}R^{2T}S^Tf,g\Big)_{\mathcal{F}^T}\ . \tag{30}$$

Recalling the definition of w^{fg} we have

$$w^{fg}(T,T)=\big(C^Tf,g\big)_{\mathcal{F}^T} \tag{31}$$

Comparing the right hand sides of (30) and (31) we get (25) because of arbitrariness of $f,g\in C_0^\infty([0,T];\mathbf{R}^N)$. The formula (26) can be easily obtained from (25) using (18). □

3. Inverse problem

3.1. Statement of the problem

The "local principle" (see Section 1.1) motivates the following statement of the inverse problem (IP): for a given response operator R^{2T} recover the potential $V(x)$ for $x \in \Omega^T$. It should be noted that to assign R^{2T} is the same as to assign its matrix kernel $r(t)$ for $t \in [0, 2T]$.

3.2. IP as a factorization problem

Let us introduce the operator $I^T : \mathcal{F}^T \mapsto \mathcal{H}^T, (I^T f)(x) := f(T-x), x \in \Omega^T$. Obviously,

$$I^T \mathcal{F}^{T,\xi} = \mathcal{H}^\xi. \tag{32}$$

Its adjoint $\left(I^T\right)^* : \mathcal{H}^T \mapsto \mathcal{F}^T$, is defined as $\left(\left(I^T\right)^* a\right)(t) := a(T-t), t \in [0,T]$. Along with it

$$\left(I^T\right)^* I^T = \mathbf{1}_{\mathcal{F}^T}, \quad I^T \left(I^T\right)^* = \mathbf{1}_{\mathcal{H}^T}. \tag{33}$$

The operators $\tilde{W}^T := \left(I^T\right)^* W^T$ and $\tilde{W}^T_\# := \left(I^T\right)^* W^T_\#$ acting in $\mathcal{F}^T$ possess the so-called triangular property (Gokhberg and Krein, 1967) : $\tilde{W}^T \mathcal{F}^{T,\xi} = \mathcal{F}^{T,\xi}$, $\tilde{W}^T_\# \mathcal{F}^{T,\xi} = \mathcal{F}^{T,\xi}$, $0 \le \xi \le T$, (see (16) and (17)).

On the other hand, using (24) and (33) we have the following representation for the connecting operator:

$$C^T = \left(W^T_\#\right)^* W^T = \left(\tilde{W}^T_\#\right)^* \tilde{W}^T_\#, \tag{34}$$

which gives us a triangular factorization of C^T along the family of subspaces $\{\mathcal{F}^{T,\xi}\}$ (see Gokhberg and Krein, 1967).

Suppose that we are able to realize the factorization of C^T in the form of (34) via inverse data R^{2T}. Then the IP can be solved in the following way. Knowing the operator $\tilde{W}^T$, we find the operator $W^T = I^T \tilde{W}^T$. Then, using the representation (14) we find the kernel w and recover the potential V from the formula

$$V(x) = -2 \frac{d}{dx}[w(x,x)],\ x \in \Omega^T. \tag{35}$$

(see (4)). Thus, the IP may be considered as a problem of a triangular factorization of the connecting operator C^T.

3.3. Operator integral

The factorization (34) can be realized with the help of the canonical procedure described by Gohkberg and Krein, (1967). We present here another method of factorization using a construction of an operator integral based on dynamical considerations.

Let us introduce two families of "cutting" operators depending on a parameter ξ, $0 \le \xi \le T$, :

$$X^{T,\xi} : \mathcal{F}^T \mapsto \mathcal{F}^T,$$
$$\left(X^{T,\xi} f\right)(t) = \begin{cases} 0, & 0 \le t < T - \xi \\ f(t), & T - \xi \le t \le T \end{cases}$$

and

$$P^{\xi} : \mathcal{H}^T \mapsto \mathcal{H}^T,$$
$$\left(P^{\xi} a\right)(x) = \begin{cases} a(x), & 0 \le x < \xi \\ 0, & \xi \le x \le T\,. \end{cases}$$

Let $\Xi = \{\xi_0, \xi_1, \ldots, \xi_n\}, 0 = \xi_0 < \xi_1 < \ldots < \xi_n = T$, be a partition of the interval Ω^T with the range $\lambda(\Xi) := \max_{1 \le i \le n}(\xi_i - \xi_{i-1})$. With the partition we associate the integral sum

$$S_{\Xi} : \mathcal{F}^T \mapsto \mathcal{H}^T, S_{\Xi} = \sum_{i=1}^{n} \Delta P^{\xi_i} W_{\#}^T \Delta X^{T,\xi},$$

where $\Delta P^{\xi_i} = P^{\xi_i} - P^{\xi_{i-1}}$ and $\Delta X^{\xi_i} = X^{\xi_i} - X^{\xi_{i-1}}$.

Lemma 3.1 *For any function $f \in \mathcal{F}^T$ the following representation is valid*

$$\left(\Delta P^{\xi_i} W_{\#}^T \Delta X^{T,\xi_i} f\right)(x) = \begin{cases} f(T-x) + y_i^f(x), & x \in [\xi_{i-1}, \xi_i] \\ 0, \quad x \in \Omega^T \setminus (\xi_{i-1}, \xi_i) \end{cases} \tag{36}$$

with y_i^f satisfying the estimate

$$\|y_i^f\|^2_{L^2([\xi_{i-1},\xi_i];\mathbf{R}^N)} \le c(\xi_i - \xi_{i-1})^2 \|f\|^2_{\mathcal{F}^T} \tag{37}$$

with the constant c not depending on f.

Proof. The representation (36) follows directly from (5), moreover,

$$y_i^f(x) = \int_x^{\xi_i} w(x,s) f(T-s) ds, \quad x \in [\xi_{i-1}, \xi_i].$$

From this equality the estimate (37) can be easily derived. □

The following theorem describes the limit of the integral sums S_{Ξ}.

Theorem 3.1 *For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that the inequality $\|S_{\Xi} - I^T\| < \epsilon$ is valid for any partition Ξ satisfying $\lambda(\Xi) < \delta$.*

Proof. For any $f \in \mathcal{F}^T$ by virtue of (36) and (37) we have the relations

$$\| (S_\Xi - I^T) f \|^2_{\mathcal{H}^T} = \sum_{i=1}^{n} \| y_i^f \|^2_{L^2([\xi_{i-1}, \xi_i]; \mathbf{R}^N)} \leq cT\lambda(\Xi) \|f\|^2_{\mathcal{F}^T}$$

proving the theorem. □

The limit of the sums S_Ξ is called an operator integral; it is denoted by

$$\lim_{\lambda(\Xi)\to 0} S_\Xi := \int_0^T dP^\xi W^T_\# dX^{T,\xi}.$$

Thus, we have a "dynamical" representation of the operator I^T:

$$I^T = \int_0^T dP^\xi W^T_\# dX^{T,\xi}.$$

As a consequence we obtain the representation

$$\left(I^T\right)^* := \int_0^T dX^{T,\xi} \left(W^T_\#\right)^* dP^\xi \tag{38}$$

Therefore, the operator $\tilde{W}^T = \left(I^T\right)^* W^T$ may be represented in the following form

$$\tilde{W}^T = \int_0^T dX^{T,\xi} d\Pi^{T,\xi} \tag{39}$$

where $\Pi^{T,\xi}$ is the operator–function

$$\Pi^{T,\xi} := \left(W^T_\#\right)^* P^\xi W^T = C^T \left(W^T\right)^{-1} P^\xi W^T. \tag{40}$$

Below we demonstrate how to express it via the inverse data.

3.4. Operator–function $\Pi^{T,\xi}$

Let us introduce in the space $\mathcal{F}^T$ the bilinear form

$$[f, g] := (C^T f, g)_{\mathcal{F}^T}.$$

For every subspace $\mathcal{F}^{T,\xi}$ let us introduce its "orthogonal" complement in $\mathcal{F}^T$:

$$\mathcal{F}^{T,\xi}{}_{[\perp]} := \left\{ f \in \mathcal{F}^T : [f, g]_{\mathcal{F}^T} = 0\ ,\ g \in \mathcal{F}^{T,\xi} \right\}$$

The equalities (16) and (17) imply that for any $\xi \in [0, T]$ the space $\mathcal{F}^T$ can be decomposed in the direct sum

$$\mathcal{F}^T = \mathcal{F}^{T,\xi} \dot{+} \mathcal{F}^{T,\xi}_{[\perp]}.$$

Along with it the operator $\mathcal{P}^\xi := \left(W^T\right)^{-1} P^\xi W^T$ turns out to be the skew projector $\mathcal{F}^T \mapsto \mathcal{F}^{T,\xi}$ parallel to $\mathcal{F}^{T,\xi}{}_{[\perp]}$. Indeed, $\mathcal{P}^\xi$ is idempotent, $\left(\mathcal{P}^\xi\right)^2 = \mathcal{P}^\xi$, and $\mathcal{P}^\xi \mathcal{F}^{T,\xi} = \mathcal{F}^{T,\xi}$, $\mathcal{P}^\xi \mathcal{F}^{T,\xi}{}_{[\perp]} = 0$.

Let us represent this operator via the operator C^T. Recall that, in turn, C^T can be expressed via R^{2T}.

Introduce the operators

$$\hat{X}^{T,\xi} : \mathcal{F}^T \mapsto \mathcal{F}^{T,\xi}, \ \left(\hat{X}^{T,\xi}\right)^* : \mathcal{F}^{T,\xi} \mapsto \mathcal{F}^T$$

$$\left(\hat{X}^{T,\xi} f\right)(t) = f(t), \ T-\xi \le t \le T,$$

$$\left(\left(\hat{X}^{T,\xi}\right)^* g\right)(t) = \begin{cases} 0, \ 0 \le t \le \xi \\ g(t), \ T-\xi \le t \le T \end{cases}$$

As one can easily check, the operator $C^{T,\xi} := \hat{X}^{T,\xi} C^T \left(\hat{X}^{T,\xi}\right)^*$ is an isomorphism in the space $\mathcal{F}^{T,\xi}$. Let us introduce the operator $\tilde{\mathcal{P}}^\xi : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$\tilde{\mathcal{P}}^\xi := \left(\hat{X}^{T,\xi}\right)^* \left[C^{T,\xi}\right]^{-1} \hat{X}^{T,\xi} C^T.$$

LEMMA 3.2 *The operator $\tilde{\mathcal{P}}^\xi$ coincides with the skew projector $\mathcal{P}^\xi$.*

Proof. The proposition of the lemma follows from the equalities $\left(\tilde{\mathcal{P}}^\xi\right)^2 = \tilde{\mathcal{P}}^\xi$, $\tilde{\mathcal{P}}^\xi \mathcal{F}^{T,\xi} = \mathcal{F}^{T,\xi}$, $\tilde{\mathcal{P}}^\xi \mathcal{F}^{T,\xi} = 0$.

Thus, we have shown that the operator–function $\Pi^{T,\xi}$ can be written in the form

$$\Pi^{T,\xi} = C^T \mathcal{P}^\xi = C^T \left(\hat{X}^{T,\xi}\right)^* \left[\hat{X}^{T,\xi} C^T \left(\hat{X}^{T,\xi}\right)^*\right]^{-1} \hat{X}^{T,\xi} C^T. \tag{41}$$

□

3.5. Procedure for solving the inverse problem

Representations (39) and (41) give a possibility to solve the IP in the following way:

a) construct the operator C^T via R^{2T} by means of (25);
b) recover the operator–function $\Pi^{T,\xi}$ using (41);
c) construct the operator $\tilde{W}^T$ by representation (39);
d) find the operator $W^T = I^T \tilde{W}^T$ and its kernel w (see (14)).
e) recover the potential V on Ω^T from the formula (35).

The procedure can be completed by characteristic description of inverse data.

THEOREM 3.2 *The operator $R : \mathcal{F}^{2T} \mapsto \mathcal{F}^{2T}$ is a response operator of the system (1)–(3) corresponding to a potential V continuously differentiable in Ω^T if and only if*

(i) *the operator R can be represented in the form of*

$$(Rf)(t) = -f'(t) + \int_0^t k(t-s)f(s)ds \ , \ 0 < t < 2T \ ,$$

with a matrix-valued function $k(t)$ continuously differentiable on $[0, 2T]$;

(ii) *for any $\xi \in [0, T]$ the operator C^ξ defined by the equalities*

$$(C^\xi f)(t) = f(t) + \int_0^\xi [q(2\xi - t - s) - q(|\, t - s\,|)]f(s)ds \ , 0 < t < \xi,$$

$$q(t) := \frac{1}{2}\int_0^t k(s)ds,$$

is an isomorphism in $\mathcal{F}^\xi$.

Under these conditions the matrix-function $V(x), x \in \Omega^T$, corresponding to R is unique.

The proof of the theorem will be presented in the forthcoming paper by the authors. Note only that a similar result was obtained in another form in Blagoveshchenskii (1971) for the scattering inverse problem.

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