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On unique continuation of solutions of the parabolic equation from a curve[1]

by

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Abstract. The unique continuation property of a second-order parabolic operator $\mathbf{L}$ with time-varying coefficients is studied. Given $T > \varepsilon > 0$, we ask if there is a curve $\hat{x}(t) \in \bar{\Omega}, t \in (\varepsilon, T)$ such that, if $\mathbf{L}u = 0$ in $Q = \Omega \times (0, T)$ and u vanishes along $\hat{x}(\cdot)$, then $u \equiv 0$ in Q. A generalization of this problem, based on the notion of a set-valued map, and applications to approximate and exact null-controllability are also discussed.

Keywords: second-order parabolic equations, unique continuation, controllability

1. Introduction

Let Ω be a bounded, open, connected set in R^n with boundary $\partial\Omega$. In Ω we consider the following homogeneous Dirichlet problem for the parabolic equation:

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) - \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} - a(x,t)u$$

$$\text{in} \quad Q = (0,T) \times \Omega, \tag{1.1a}$$

$$u\,|_{\Sigma} = 0 \quad \text{in} \quad \Sigma = \partial\Omega \times (0,T), \qquad u\,|_{t=0} = u_0 \quad \text{in} \quad \Omega$$

under the condition of uniform ellipticity, namely,

$$\mu \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\,(x,t)\xi_i\xi_j \quad \forall \xi_i \in R \quad \text{a.e. in} \quad Q, \quad \mu > 0, \tag{1.1b}$$

where $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(Q)$, $i,j = 1,\ldots,n$. The following two questions constitute the problems with which we deal in this paper.

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PROBLEM 1.1 Let $n \leq 3$. Given $T > \varepsilon > 0$, we ask whether there exists a curve $(\varepsilon, T) \ni t \to \hat{x}(t) \in \bar{\Omega}$ such that every solution $u \in H_0^{2,1}(Q)$ to (1.1) which vanishes along $\hat{x}(\cdot)$ vanishes in Q.

PROBLEM 1.2 Given $T > \varepsilon > 0$, we ask whether there exists a set-valued map $(\varepsilon, T) \ni t \to S(t) \subset \Omega$, $\text{mes}\{\text{S(t)}\} > 0$ such that every solution $u \in C([0,T]; L^2(\Omega)) \bigcap H_0^{1,0}(Q)$ to (1.1) which satisfies the equality $\int_{S(t)} u dx = 0$ on (ε, T) vanishes in Q.

In this paper we give a positive answer to both of these questions. Our proofs are based on the derivation of estimates (3.1) and (4.2) (which are also of interest in themselves), followed by application of a suitable backward uniqueness result.

REMARK 1.1 *In the above and elsewhere in the paper, the standard notations for the Sobolev spaces are used. In particular,* $H_0^{1,0}(Q) = \{\phi \mid \phi,\ \phi_{x_i} \in L^2(Q),\ i = 1, \ldots, n, \phi \mid_\Sigma = 0\}$ *and* $H_0^{2,1}(Q) = \{\phi \mid \phi,\ \phi_{x_i x_j}, \phi_{x_i}, \phi_t \in L^2(Q),\ i,j = 1, \ldots, n,\ \phi \mid_\Sigma = 0\}$.

The problem of unique continuation from an arbitrary open subset Q^* of Q to its horizontal component $\{(x,t) \in Q \mid \exists x^* : (x^*, t) \in Q^*\}$ for the second order parabolic operator is well known in the literature (see, e.g., Saut and Scheurer, 1987, and the bibliography therein). To our knowledge, the problem addressed in this paper is open. Motivation for its consideration comes from the following observations.

It is well known that the unique continuation property is closely related to the issue of controllability, see, e.g., Lions (1988). It is quite common in applications (e.g., in mechanical engineering or environmental problems) that only pointwise sensors/actuators can act upon the system. In this regard, a curve $\hat{x}(\cdot)$ ensuring (3.1) ((4.2) is its a "space-averaged" version) can be seen as that for the allocation of a scanning point sensor, whose outputs are stable with respect to measurement errors from $L^\infty(\varepsilon, T)$. We refer to Lasiecka and Triggiani (1991), Lions (1992) (and to the bibliography therein) for the discussion focusing on static point controls for different types of partial differential equations.

In the time-invariant framework (allowing one to use the techniques of harmonic analysis) and in *one* space dimension the estimates analogous to (3.1) were given, e.g., by Dolecki (1973). On the other hand, Müntz-Szász type theorems also imply that even for the constant elliptic operator the estimate proposition (3.1) does *not* hold for $n > 1$ if its left-hand side is static, that is, $\hat{x}(t) \equiv \bar{x}$, $t \in (0,T)$. On the contrary, in the dynamic setting the estimate (3.1) was established for the heat equation in an arbitrary space dimension on the set of its continuous solutions in the recent paper by Khapalov (1994). This was achieved by using the time-invariancy of the Laplacian, combined with the separability of $C(\bar{\Omega})$). An analogous result for the wave equation is given with respect to the energy norm in Khapalov (1995). Sharp correspondence between the internal regularity of solutions of a second-order hyperbolic equation with time-varying coefficients and a type of a lumped expression in the left-hand side of the corresponding estimate of type (3.1)/(4.2) was discussed in Khapalov (1995).

The paper is organized as follows. In the next section a number of assumptions on (1.1) providing suitable regularity of its solutions are given. Sections 3 and 4 deal, accordingly, with the derivation of estimates (3.1) and (4.2). In Section 5 the main uniqueness results are formulated. Their control theoretic consequences are discussed in Section 6.

2. Preliminaries

The following three sets of the assumptions on (1.1) are considered below:

$$u_0 \in L^2(\Omega), \quad \| \sum_{i=1}^{n} b_i^2, a\|_{q,r,Q} \leq \mu, \quad \frac{1}{r} + \frac{n}{2q} = 1 - \kappa, \tag{2.1a}$$

$$\begin{cases} q \in [\frac{n}{2(1-\kappa)}, \infty], \quad r \in [\frac{1}{1-\kappa}, \infty], \quad 0 < \kappa < 1, \quad \text{for} \quad n \geq 2, \\ q \in [1, \infty], \quad r \in [\frac{1}{1-\kappa}, \frac{2}{1-2\kappa}], \quad 0 < \kappa < \frac{1}{2}, \quad \text{for} \quad n = 1, \end{cases} \tag{2.1b}$$

where $\|z\|_{q,r,Q} = \|z\|_{L_{q,r}(Q)} = (\int_0^T (\int_\Omega |z|^q \, dx)^{\frac{r}{q}} dt)^{\frac{1}{r}}$;

$$\frac{\partial a_{ij}}{\partial t} \in L^1(0,T; L^\infty(\Omega)), \quad b_i, a \in L^\infty(Q), \quad i,j,k = 1, \ldots, n; \tag{2.2}$$

$$u_0 \in H_0^1(\Omega), \quad \partial\Omega \in C^2, \quad \frac{\partial a_{ij}}{\partial x_k}, b_i, a \in L^\infty(Q), \quad i,j,k = 1, \ldots, n. \tag{2.3}$$

Conditions (2.1) ensure the existence and uniqueness of a solution to (1.1) from the space $C([0,T]; L^2(\Omega)) \bigcap H_0^{1,0}(Q)$ (see, e.g., Ladyzhenskaya, Solonnikov and Ural'ceva, 1968, pp. 160, 181), which satisfies the energy estimate:

$$\|u\|_{C([0,T];L^2(\Omega))} + \|u\|_{H^{1,0}(Q)} \leq c\|u_0\|_{L^2(\Omega)}. \tag{2.4}$$

Here c depends on T and the parameters in (1.1b), (2.1). Under the assumptions (2.3) this solution lies in $H_0^{2,1}(Q)$, see, e.g., Ladyzhenskaya, Solonnikov and Ural'ceva (1968), pp. 178, 180-181 (where other suitable conditions are also discussed). Assumptions (2.2) allow one to use the backward uniqueness result due to Bardos and Tartar (1973) for the homogeneous boundary problem (1.1), (2.1).

The following generalized *maximum principle* (see, e.g., Ladyzhenskaya, Solonnikov and Ural'ceva, 1968, pp. 181, 192-193) for the solutions of (1.1), (2.1) plays an important role in the further discussion:

$$\|u\|_{L^\infty(Q)} \le c_1 \|u(\cdot,0)\|_{L^\infty(\Omega)}, \qquad \text{provided} \quad u(\cdot,0) \in L^\infty(\Omega), \tag{2.5a}$$

otherwise, for any $\varepsilon > 0$,

$$\|u\|_{L^\infty((\varepsilon,T)\times\Omega)} \le c_2 \|u\|_{L^2(Q)}, \tag{2.5b}$$

where c_1 depends on T, c_2 depends on ε, T and the parameters in (1.1b), (2.1).

Given $\varepsilon \in (0,T)$, let $S(\cdot)$ be such that the set $\{(x,t) \mid x \in S(t), t \in (\varepsilon, T)\}$ is measurable with respect to Lebesgue measure on Q and $\text{mes}\{S(t)\} > 0$ for almost all $t \in (\varepsilon, T)$. Then (2.4) and (2.5b) imply the following estimates:

$$\|(\text{mes}\{S(\cdot)\})^{-1} \int\limits_{S(\cdot)} u(x,\cdot)dx\|_{L^\infty(\varepsilon,T)} \le \|u\|_{L^\infty((\varepsilon,T)\times\Omega)}$$

$$\le cc_2(T-\varepsilon)^{1/2} \, \|u(\cdot,0)\|_{L^2(\Omega)}. \tag{2.6}$$

For $n \le 3$, by the embedding theorems, $H_0^2(\Omega) \subset C(\bar{\Omega})$ and

$$\|z\|_{C(\bar{\Omega})} \le c_3 \|z\|_{H^2(\Omega)} \qquad \forall z \in H_0^2(\Omega), \quad c_3 > 0. \tag{2.7}$$

Hence, due to (2.5b) and (2.7), any solution of (1.1), (2.3) is an element of both $L^2(0,T;C(\bar{\Omega}))$ and $L^\infty(\varepsilon,T;C(\bar{\Omega}))$ $\forall \varepsilon \in (0,T)$. Given measurable $\hat{x}(t), t \in (\varepsilon, T)$, it follows from (2.6) that:

$$\|u(\hat{x}(\cdot),\cdot)\|_{L^\infty(\varepsilon,T)} \le cc_2(T-\varepsilon)^{1/2}\|u(\cdot,0)\|_{L^2(\Omega)}, \tag{2.8}$$

provided $u \in H_0^{2,1}(Q)$.

3. Dynamic pointwise a priori estimate

The main result of this section is the following.

THEOREM 3.1 *Let $n \le 3$. Given $T > \varepsilon > 0$, there exists a measurable curve $\hat{x}(\cdot)$ for which the estimate*

$$\|u(\hat{x}(\cdot),\cdot)\|_{L^\infty(\varepsilon,T)} \ge \gamma \|u(\cdot,T)\|_{L^2(\Omega)}, \qquad \gamma = \gamma(\hat{x}(\cdot)) > 0 \tag{3.1}$$

holds for any solution of (1.1), (2.3) from $H_0^{2,1}(Q)$.

Proof. The scheme of the proof is as follows. Selecting a parameter $\nu \in (0,1)$, we employ Galerkin's approach to specify certain countable ν-net in the set of all the solutions of (1.1), (2.3) from $H_0^{2,1}(Q)$. We look then for a suitable $\hat{x}(\cdot)$ as a spline-curve consisting of a countable number of pieces each of which is associated with a pre-assigned net-element for which (3.1) is fulfilled on the corresponding part of $(0,T)$. To do this, we apply a finite-dimensional optimization technique, which makes use of the maximum principle (2.5). The conclusion of Theorem 3.1 follows then by the density argument.

Step 1: A curve for a single solution. Fix any $\varepsilon \in (0,T)$, $\nu \in (0,1)$, $\beta > 0$ and a (nontrivial) solution u to (1.1), (2.3). Select in (ε, T) an arbitrary subinterval τ. Let

$$e = \{(x,t) \in \bar{\Omega} \times \tau \mid u^2(x,t) \geq \operatorname*{ess\,max}_{(x,t)\in\Omega\times\tau} u^2(x,t) - \beta\}.$$

By (2.5b), the set e can always be defined and $\mathrm{mes}\{e\} > 0$.

Let $F(t)$, $t \in r$ be the following set-valued map: $r \ni t \to F(t) = \{x \mid (x,t) \in e\}$, where $r = \mathrm{dom}\ F(t) = \{t \in \tau \mid \mathrm{mes}\{x|(x,t) \in e\} > 0\}$. Since $H_0^2(\Omega) \subset C(\bar{\Omega})$, the sets $F(t)$, $t \in r$ are closed for almost all $t \in r$. Applying the theorem on measurable selection (see, e.g., Aubin and Frankowska, 1990), yields the existence of a measurable $\hat{x}(t)$, $t \in r$ such that $\hat{x}(t) \in F(t)$ a.e. on r. Combining this with (2.4), applied on $[t,T]$, $t \in \bar{r}$ (without loss of generality, with the same constant c), yields:

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq c\|u\|_{C(\bar{r};L^2(\Omega))} \leq c(\mathrm{mes}\{\Omega\})^{1/2}(\|u(\hat{x}(\cdot),\cdot)\|_{L^\infty(\tau)} + \sqrt{\beta}).$$

Extend arbitrarily the $\hat{x}(\cdot)$ found to the whole interval $(0,T)$ assuming only that its values lie in Ω. By linearity, for $\beta = (\nu/c(\mathrm{mes}\{\Omega\})^{-1/2}\|u(\cdot,T)\|_{L^2(\Omega)})^2$ the last estimate gives (3.1) on the set $\{u^* = \alpha u,\ \alpha \in R\}$ with $\gamma = c^{-1}(\mathrm{mes}\{\Omega\})^{-1/2}$ $(1-\nu)$:

$$\|u^*(\hat{x}(\cdot),\cdot)\|_{L^\infty(\varepsilon,T)} \geq c^{-1}(\mathrm{mes}\ \{\Omega\})^{-1/2}(1-\nu)\|u^*(\cdot,T)\|_{L^2(\Omega)}. \tag{3.2}$$

Note that the γ found does not depend on the choice of u and that the description of the curve $\hat{x}(\cdot)$ used in (3.2) is essential only on r. Hence, selecting an arbitrary sequence of non-overlapping intervals $\tau_k \subset (\varepsilon, T)$, $k = 1, \ldots,$ one can extend (3.2) to any countable set of the solutions of (1.1), (2.3).

Step 2: Galerkin's basis. Let $\{\omega_i\}_{i=1}^\infty \subset H_0^2(\Omega)$ be an arbitrary basis in $H_0^1(\Omega)$ which has been orthonormalized in $L^2(\Omega)$. (An example of such a basis is the eigenvalues of the homogeneous Dirichlet problem for the Laplacian.) Denote by $W_k(\Omega)$ the subspace of $L^2(\Omega)$ spanned by $\{\omega_i\}_{i=1}^k$. Let

$$\Phi_k = \{\phi \mid \|\phi\|_{L^2(\Omega)} = 1, \quad \phi \in W_k(\Omega)\};$$

$$\delta_k^* = (\mathrm{mes}\{\Omega\})^{-1/2} \min\{\|u(\cdot,T)\|_{L^2(\Omega)} \mid u(\cdot,0) \in \Phi_k\}; \tag{3.3a}$$

$$C = \min\{c^{-1}, (c_1c_3)^{-1}\}, \tag{3.3b}$$

where c, c_1, c_3 are from (2.4), (2.5a) and (2.7). Note that, because of a finite dimension of Φ_k, the minimum in (3.3a) is achieved. Without loss of generality, one can assume that it is positive. Otherwise, we would deal only with the maximal subspace of W_k for which it is so (on its orthogonal supplement estimate (3.1) is trivial).

REMARK 3.1 *The value δ_k^* is positive if (1.1) possesses the backward uniqueness property, which is so, e.g., under the assumptions (2.2).*

Take any positive parameter δ_k such that

$$0 < \delta_k < \frac{1}{3c}\nu\delta_k^*. \tag{3.3c}$$

Specify next (in an arbitrary way) in Φ_k a finite $C\delta_k$-net $\Phi_k^{\delta_k} = \{\phi_k^j\}_{j=1}^{J_k}$, $\phi_k^j \in \Phi_k$ (where J_k depends upon δ_k) with respect to $H^2(\Omega)$-norm. This means that for any element $\phi \in \Phi_k$ there exists a positive integer $j_* \leq J_k$ such that $\|\phi - \phi_k^{j_*}\|_{H^2(\Omega)} \leq C\delta_k \leq \delta_k$. Hence, in view of (3.3b), (2.4), (2.5a) and (2.7):

$$\|u - u_k^{j_*}\|_{C([0,T];L^2(\Omega))} \leq \delta_k, \quad \|u - u_k^{j_*}\|_{L^\infty(\varepsilon,T;C(\bar{\Omega}))} \leq \delta_k, \tag{3.4}$$

where u and u_k^{j*} are the solution of (1.1) in accordance with $u(\cdot,0) = \phi$ and $u_k^{j*}(\cdot,0) = \phi_k^{j*}$. In other words, the set $\Phi_k^{\delta_k}(\cdot) = \{u_k^j(x,t), u_k^j(x,0) = \phi_k^j(x)\}_{j=1}^{J_k}$ is a δ_k-net with respect to both $C([0,T];L^2(\Omega))$- and $L^\infty(\varepsilon,T;C(\bar{\Omega}))$-norms in the set of all those solutions to (1.1) whose initial conditions lie in Φ_k.

REMARK 3.2 *The requirement $\{\omega_k\}_{k=1}^\infty \subset H_0^2(\Omega)$ can be omitted, in which case one has to take in Φ_k a net with respect to $L^2(\Omega)$-norm, making use of estimate (2.5b) instead of (2.5a).*

Step 3. Take any δ_k satisfying (3.3c). Select in (ε,T) an arbitrary monotone sequence $\varepsilon = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$, $t_k \to T$ as $k \to \infty$ and denote $\tau_k = (t_{k-1}, t_k)$. Let $t_{k-1} = t_k^0 < t_k^1 < \ldots < t_k^{J_k} = t_k$ be any monotone sequence in τ_k. Denote $\tau_{kj} = (t_k^{j-1}, t_k^j)$, $j = 1, \ldots J_k$.

Making use of the argument of Step 1, namely, by setting $\beta = \delta_k^2$ in the corresponding estimate, one can conclude that for every τ_{kj} there exists a measurable function $\bar{x}_{kj}(t) \in \Omega$, $t \in \tau_{kj}$ such that

$$\|u_k^j\|_{L^\infty(\tau_{kj};C(\bar{\Omega}))} \leq \|u_k^j(\bar{x}_{kj}(\cdot),\cdot)\|_{L^\infty(\tau_{kj})} + \delta_k. \tag{3.5}$$

We show now that the curve

$$\hat{x}(t) = \bar{x}_{kj}(t), \quad t \in \tau_{kj}, \qquad k = 1,\ldots, \quad j = 1,\ldots,J_k \tag{3.6}$$

satisfies the requirements of Theorem 3.1.

Step 4: Verification of (3.1) for $W_k(\Omega)$. Take any solution u to (1.1), (2.3) such that $u(\cdot,0) \in W_k(\Omega)$. From (2.4), applied on $[t,T]$, $t \in \tau_{kj}$, it follows that:

$$\|u(\cdot,T)\|_{L^2(\Omega)} \le c \sup_{t\in\tau_{kj}} \|u(\cdot,t)\|_{L^2(\Omega)} \le c(\text{mes}\{\Omega\})^{1/2} \|u\|_{L^\infty(\tau_{kj};C(\bar\Omega))}. \quad (3.7)$$

Denote $\alpha = \|u(\cdot,0)\|_{L^2(\Omega)}$. Without loss of generality, we can assume that $\alpha \ne 0$ (otherwise, (3.1) is trivial). Select an element $u_k^{j*} \in \Phi_k^{\delta_k}(\cdot)$ such that inequalities (3.4) hold with $\alpha^{-1}u$ substituted for u. Then, combining (3.7) and the second estimate in (3.4) we get:

$$(\text{mes}\{\Omega\})^{-1/2} \|u(\cdot,T)\|_{L^2(\Omega)} \le \alpha c \|u_k^{j*}\|_{L^\infty(\tau_{kj};C(\bar\Omega))} + \alpha c\delta_k.$$

From here and (3.5) we deduce:

$$(\text{mes}\{\Omega\})^{-1/2} \|u(\cdot,T)\|_{L^2(\Omega)} \le \alpha c \|u_k^{j*}(\hat x(\cdot),\cdot)\|_{L^\infty(\tau_{kj})} + 2\alpha c\delta_k.$$

Applying again the second estimate in (3.4) to evaluate $\|u_k^{j*}(\hat x(\cdot),\cdot)\|_{L^\infty(\tau_{kj})}$, we obtain:

$$(\text{mes}\{\Omega\})^{-1/2} \|u(\cdot,T)\|_{L^2(\Omega)} \le c \|u(\hat x(\cdot),\cdot)\|_{L^\infty(\tau_{kj})} + 3\alpha c\delta_k. \quad (3.8)$$

Since, by our choice of δ_k (see (3.3a)-(3.3c)),

$$3\alpha c\delta_k < \nu\alpha\delta_k^* \le (\text{mes}\,\{\Omega\})^{-1/2}\nu\|u(\cdot,T)\|_{L^2(\Omega)},$$

combining the latter and (3.8), we arrive at the required estimate (3.1) on $W_k(\Omega)$:

$$\|u(\cdot,T)\|_{L^2(\Omega)} \le (\text{mes}\{\Omega\})^{1/2} \frac{c}{1-\nu} \|u(\hat x(\cdot),\cdot)\|_{L^\infty(\varepsilon,T)}. \quad (3.9)$$

Step 5. Since the system $\{\omega_i\}_{i=1}^\infty$ is fundamental in $H_0^1(\Omega)$, the set of the solutions to (1.1) starting from $\bigcup_{k=1}^\infty W_k(\Omega)$ is dense in the set of all the solutions to (1.1), (2.3) with respect to both $C([0,T];L^2(\Omega))$- and $L^\infty(\varepsilon,T;C(\bar\Omega))$-norms (and, hence, to $\|u(\hat x(\cdot,\cdot)\|_{L^\infty(\varepsilon,T)})$, see (2.4), (2.5) (and (2.8)). The conclusion of Theorem 3.1 with γ as in (3.9) now follows by density. □

4. Nonsmooth case

Let us recall that $u \in C([0,T];L^2(\Omega))$ and, hence, for any $t \in [0,T]$ the set of Lebesgue points of the function $u(x,t)$, namely,

$$\Omega(u(\cdot,t)) = \{x \in \Omega \mid \lim_{h\to 0} \mid (\text{mes}\{S_h(x)\})^{-1} \int_{S_h(x)} u(s,t)ds - u(x,t) \mid= 0\},$$

where $S_h(x) = \{\xi \in R^n \mid \|\xi - x\|_{R^n} \le h\}$, has a full measure in Ω:

$$\mathrm{mes}\{\Omega\backslash\Omega(u(\cdot,t))\} = 0 \qquad \forall t \in [0,T]. \tag{4.1}$$

We shall say that the map $F(t)$ from $[0,T]$ into the set of all the measurable subsets of Ω is continuous with respect to Lebesgue measure at $t = t^*$ if $\mathrm{mes}\{F(t^*)\Delta F(t^*+s)\} \to 0$ as $s \to 0$, where Δ denotes the symmetric difference: $A\Delta B = (A\backslash B)\bigcup(B\backslash A)$. (For various definitions of the continuity of set-valued maps see, e.g., Aubin and Frankowska (1990) and the bibliography therein.)

THEOREM 4.1 *Given $T > \varepsilon > 0$, there exists a set-valued map $S(\cdot)$, continuous with respect to Lebesgue measure on (ε, T) and whose values are of positive measure, for which the estimate*

$$\begin{aligned}&\|(\mathrm{mes}\{S(\cdot)\})^{-1} \int\limits_{S(\cdot)} u(x,\cdot)dx\|_{L^\infty(\varepsilon,T)} \geq \gamma\|u(\cdot,T)\|_{L^2(\Omega)},\\ &\gamma = \gamma(S(\cdot)) > 0\end{aligned} \tag{4.2}$$

holds for any solution to (1.1), (2.1) from $C([0,T];L^2(\Omega))\bigcap H_0^{1,0}(Q)$.

Proof. We treat the estimate (4.2) as a generalization of (3.1) to the case of discontinuous solutions. The latter is important in applications. The following argument follows that described in Section 3 in Steps 1-5 with a few modifications as given below.

Step 1. Fix any $\varepsilon \in (0,T)$, $\nu \in (0,1)$, $\beta > 0$. Take any solution u to (1.1), (2.1) and interval $\tau \subset (\varepsilon, T)$. Specify next the set e of positive measure as given in Step 1 of Section 3. In view of (4.1), there exists a triplet $\{(x^*,t^*) \in e, h^* > 0\}$ such that:

$$\begin{aligned}&(\mathrm{mes}\{S_{h^*}(x^*)\})^{-1} \int\limits_{S_{h^*}(x^*)} u(x,t^*)dx)^2 \geq \operatorname*{ess\ max}_{(x,t)\in\Omega\times\tau} u^2(x,t) - \beta,\\ &S_{h^*}(x^*) \subset \Omega.\end{aligned}$$

Since all the solutions of (1.1), (2.1) are continuous in time in $L^2(\Omega)$-norm, an arbitrary continuous set-valued map $S(t)$, $t \in [0,T]$ (whose values are of positive measure) such that $S(t^*) = S_{h^*}(x^*)$ provides the following estimate (compare with the corresponding estimate in Section 3):

$$\begin{aligned}&\|u(\cdot,T)\|^2_{L^2(\Omega)} \leq c^2\|u\|^2_{C(\bar{\tau};L^2(\Omega))}\\ &\leq c^2\mathrm{mes}\,\{\Omega\}(\|(\mathrm{mes}\{S(t)\})^{-1} \int\limits_{S(t)} u(x,t)dx\|^2_{L^\infty(\tau)} + \beta).\end{aligned}$$

Step 2. Let $\{\omega_i\}_{i=1}^\infty$ be an arbitrary orthonormalized basis in $L^2(\Omega)$. We preserve the previous notations for $W_k(\Omega)$. The argument of this step follows the corresponding one of Section 3 with the only exceptions that we select a $C\delta_k$-net in Φ_k with respect to $L^2(\Omega)$-norm and that we replace (3.4) by the following inequalities:

$$\|u - u_k^{j_*}\|_{C([0,T];L^2(\Omega))} + \|u - u_k^{j_*}\|_{H_0^{1,0}(Q)} \leq \delta_k, \tag{4.3}$$

$$\|u - u_k^{j_*}\|_{L^\infty((\varepsilon,T)\times\Omega)} \leq \delta_k.$$

REMARK 4.1 *The choice of δ_k for the system (1.1), (2.1) requires, in general, a modification of the constant C in (3.3b). One can set, e.g.:*

$$C = \min\{c^{-1}, (cc_2)^{-1}(T-\varepsilon)^{-1/2}\},$$

where c, c_2 are defined in (2.4), (2.5b).

Step 3. Analogously to the lines (3.5), (3.6), one can obtain, making use of Steps 1-2 of this section, a sequence of triplets $\{x_{kj}, t_k^j, h_{kj}\}$, $k = 1, \ldots$, $j = 1, \ldots, J_k$ such that a suitable map can be selected to be continuous on $(0, T)$, while:

$$S(t_k^j) = S_{h_{kj}}(x_{kj}), \qquad k = 1, \ldots, \quad j = 1, \ldots, J_k. \tag{4.4}$$

Steps 4-5, follow the lines (3.6)-(3.9) by using the space $L^\infty(\Omega \times \tau_{kj})$ instead of $L^\infty(\tau_{kj}; C(\bar{\Omega}))$. It is readily seen that the above constructions ensure the estimate (4.2) with the same constant γ as in (3.9). This ends the proof of Theorem 4.1. □

REMARK 4.2 *The arguments of Theorems 3.1 and 4.1 can be summarized in an algorithmic procedure for the construction of $\hat{x}(\cdot)$ and $S(\cdot)$ satisfying (3.1) and (4.2).*

5. Unique continuation

Combining Theorems 3.1 and 4.1 and Theorem II.1 by Bardos and Tartar (1973), p. 13 on the backward uniqueness of problem (1.1) under assumption (2.2), we obtain the following result.

THEOREM 5.1 (UNIQUE CONTINUATION) *Let (2.2) hold. Then under the assumptions of Theorems 3.1 and 4.1 (see (2.3) and (2.1)) the curves and set-valued maps constructed along the arguments of Theorems 3.1 and 4.1 solve Problems 1.1 and 1.2.*

6. Approximate controllability

Let $B(\cdot)$ be a linear operator defined on a linear manifold $V \subseteq L^2(0,T)$ by one of the following formulae:

$$B(T-t)v(t) = v(t)\delta(x - \hat{x}(T-t)), \qquad \hat{x}(t) \in \bar{\Omega} \quad \text{a.e. in} \quad [0,T], \tag{6.1}$$

where $\delta(x)$ is Dirac's function, or

$$B(T-t)v(t) = v(t) \times \begin{cases} 1, & \text{if } x \in S(T-t), \\ 0, & \text{if } x \notin S(T-t), \end{cases} \quad S(t) \subset \Omega \text{ a.e. in } [0,T]. \tag{6.2}$$

Consider the system:

$$\frac{\partial \varphi}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,T-t)\frac{\partial \varphi}{\partial x_j}) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x,T-t)\varphi)$$
$$-a(x,T-t)\varphi + B(T-t)v(t) \quad \text{in} \quad Q, \tag{6.3}$$
$$\varphi = 0 \quad \text{in} \quad \Sigma, \qquad \varphi\mid_{t=0} = 0 \quad \text{in} \quad \Omega,$$

where $v \in V$ is a control function.

We are now concerned with the global approximate controllability properties of the dynamics of (6.3). Assume that (6.3) admits a unique solution from the space $C([0,T];H)$, where H is a Hilbert space. We ask under what conditions on B, V the range of the solution mapping for (6.3) at time T:

$$\Phi(T) : V \ni v \to \varphi(\cdot,T) \in H,$$

namely, the set $\mathbf{R}(\Phi(T)) = \{\varphi(\cdot,T) \mid \varphi \text{ satisfies (6.3) for some } v \in V\}$ is dense in H? If it is so, (6.3) is said to be *approximately controllable* in H at time T. It is well known that the approximate controllability property is a direct consequence of the corresponding uniqueness result, which yields the following.

Given $\varepsilon \in (0,T)$, let

$$V = \{v \in L^2(0,T) \mid v(t) = 0 \qquad \text{a.e. in } (T-\varepsilon, T)\}. \tag{6.4}$$

THEOREM 6.1 *Given $T > \varepsilon > 0$, let $S(t)$, $t \in (0,T)$ be an arbitrary set-valued map which satisfies Theorem 4.1. Then system (6.2)-(6.4), (2.2) is approximately controllable in $L^2(\Omega)$ at time T.*

Proof. First of all observe that $B(T-\cdot)v(\cdot) \in L^2(Q)$. Hence, system (6.2)-(6.4), (2.2) admits a unique solution from $C([0,T];L^2(\Omega)) \bigcap H_0^{1,0}(Q)$ (see, e.g., Ladyzhenskaya, Solonnikov and Ural'ceva (1968), p. 160). Take any solution u of the system (1.1), ((2.1)) (2.2). In view of the latter assumption it lies in

$H^1(Q)$ (see Ladyzhenskaya, Solonnikov and Ural'ceva (1968), pp. 178, 180-181 for details). From (6.3) it follows that:

$$\int_\Omega u_*(x,T)\varphi(x,T)dx - \int_0^T\int_\Omega \frac{\partial u_*}{\partial t}\varphi dxdt =$$
$$\int_0^T\int_\Omega(-\sum_{i,j=1}^n a_{ij}(x,T-t)\frac{\partial u_*}{\partial x_i}\frac{\partial \varphi}{\partial x_j} - \sum_{i=1}^n \frac{\partial u_*}{\partial x_i}b_i(x,T-t)\varphi$$
$$-a(x,T-t)u_*\varphi)dxdt + \int_0^T\int_\Omega u_* B(T-t)v(t)dxdt, \tag{6.5}$$

where $u_*(x,t) = u(x,T-t)$. From here and (1.1) we deduce:

$$\int_\varepsilon^T\int_\Omega u(x,t)B(t)v(T-t)dxdt = \int_\Omega u(x,0)\varphi(x,T)dx, \tag{6.6}$$

which is valid for any $u(\cdot,0) \in L^2(\Omega)$ and for any solution φ of system (6.2)-(6.4), (2.2) (or, that is the same, for any $v \in V$). By the classical duality argument, applying the uniqueness result of Theorem 5.1 to (6.6) yields the assertion of Theorem 6.1. □

THEOREM 6.2 *Let $n \le 3$. Given $T > \varepsilon > 0$, let $\hat{x}(t)$, $t \in (0,T)$ be an arbitrary curve satisfying Theorem 3.1. Then system (6.3), (6.1), (6.4), (2.2), (2.3) is approximately controllable in $H^{-1}(\Omega)$ dual of $H_0^1(\Omega)$ at time T.*

Proof. The generalized solution of the system (6.3), (6.1), (6.4), (2.2), (2.3) can be defined by transposition (see Lions, 1971, p. 202) as a unique element of $C([0,T];H^{-1}(\Omega))\bigcap L^2(Q)$, which, in particular, satisfies the following identity:

$$-\int_0^T\int_\Omega \varphi\frac{\partial u_*}{\partial t}dxdt + [\varphi(\cdot,T),u_*(\cdot,T)] = \int_0^T\int_\Omega(\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,T-t)\frac{\partial u_*}{\partial x_i})\varphi$$
$$-\sum_{i=1}^n \frac{\partial u_*}{\partial x_i}b_i(x,T-t)\varphi - a(x,T-t)u_*\varphi)dxdt + \int_0^T\int_\Omega u_* B(T-t)v(t)dxdt,$$

$$\forall u_* \in \{u_* \mid u_* \in H^{2,1}(Q), u_*(x,t) = u(x,T-t),\ u \text{ is a solution of (1.1), (2.3)}\},$$

where $[\cdot,\cdot]$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. (Here $L^2(\Omega)$ is identified with its dual space, so one can write $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.) Hence, by (1.1),

$$[\varphi(\cdot, T), u(\cdot, 0)] = \int_{\varepsilon}^{T} u(\hat{x}(t), t) v(T - t) dt,$$

which is valid for all $v \in V$, $u(\cdot, 0) \in H_0^1(\Omega)$, where u is the solution of (1.1), (2.2), (2.3). The conclusion of Theorem 6.2 now follows by applying the uniqueness result of Theorem 5.1. □

REMARK 6.1 *Based on suitable treatment of regularity, the estimates (3.1) and (4.2), by the direct duality method, also point out at the exact null-controllability (to the zero-state) of the systems (6.3), (6.1), (6.4), (2.2), (2.3) and (6.2)-(6.4), (2.2) (where, to employ the estimate (4.2), the control operator in (6.2) has to be complemented by a multiplier* $\text{mes}^{-1}\{S(T-t)\}$*) with controls from the space dual of* $L^\infty(0, T)$ *in accordingly* $H^{-1}(\Omega)$ *and* $L^2(\Omega)$*, see, e.g., the relevant constructions in Khapalov, 1995, where the same control space was used in the context of the controlled wave equation.*

References

AUBIN, J.-P. AND FRANKOWSKA, H. (1990) *Set-Valued Analysis.* Birkhäuser, Basel.

BARDOS, C. AND L. TARTAR, L. (1973) Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines. *Arch. Rational Mech. Anal.*, **50**, 10 -25.

DOLECKI, SZ. (1973) Observation for the one-dimensional heat equation. *Studia Math.*, **48**, 291-305.

KHAPALOV, A.Y. (1994) L^∞-exact observability of the heat equation with scanning pointwise sensor. *SIAM J. Control Optim.*, **32**, 1037-1051.

KHAPALOV, A.Y. (1995) Controllability of the wave equation with moving point control. *Appl. Math. Optim.*, **31**, 155-175.

KHAPALOV, A.Y. (1995) Exact observability of the time-varying hyperbolic equation with finitely many moving internal observations. *SIAM J. Control Optim.*, **33**, 1256-1269.

LADYZHENSKAYA, O.H., SOLONNIKOV, V.A. AND URAL'CEVA, N.N. (1968) *Linear and Quasilinear Equations of Parabolic Type.* Transl. of Mathematical Monographs, **23**, AMS, Providence, Rhode Island.

LASIECKA, I.M. AND TRIGGIANI, R. (1991) *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory.* Lecture Notes in Control and Info. Sci., **164**, Springer-Verlag, Berlin.

LIONS, J.-L. (1971) *Optimal Control of Systems Governed by Partial Differential Equations.* Springer-Verlag, Berlin, Heidelberg, New York.

LIONS, J.-L. (1988) *Contrôlabilité Exacte et Stabilisation de Systémes Distribués, I, Contrôlabilité Exacte.* Masson, Paris.

LIONS, J.-L. (1992) Pointwise control for distributed systems, in H.T. Banks (Ed.) *Control and Estimation in Distributed Parameter Systems, Frontiers in Applied Mathematics*, v. 11, SIAM Publications, Philadelphia, 1-39.

SAUT, J.-C. AND SCHEURER, B. (1987) Unique continuation for some evolution equations. *J. Diff. Equat.,* **66** , 118-139.

SCHWARTZ, L. (1943) *Étude des sommes d'exponentielles rélees,* Actualités Sci. Indust., no. 959, Hermann, Paris.