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Some dynamical inverse problems for hyperbolic systems[1]

by

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Abstract. The problems of identification of unknown controls for systems described by hyperbolic equations and variational inequalities are considered. For some classes of such systems identification algorithms of identification for distributed and boundary controls are designed. The algorithms are stable with respect to informational noises and computational errors. Estimates of convergence rate are derived. The problem of approximation of point source intensities is discussed especially.

Keywords: hyperbolic systems, identification, control

1. Introduction

The problem of dynamical identification of controls in a hyperbolic system through inaccurate measurements of some state characteristics is considered. It is supposed that a system is described by a linear hyperbolic equation

$$\ddot{x}(t) + A\dot{x}(t) + A_1 x(t) = Bu(t) + f(t), \quad t \in T = [t_0, \vartheta], \tag{1}$$

or a variational inequality

$$\begin{aligned}&(\ddot{x}(t) + Ax(t), \dot{x}(t) - v) + \varphi(\dot{x}(t)) - \varphi(v) \\ &\quad \le (Bu(t) + f(t), \dot{x}(t) - v) \quad \text{for a. a. } t \in T \quad \forall v \in V.\end{aligned} \tag{2}$$

The initial state $x(t_0) = x_0$, $\dot{x}(t_0) = x_{10}$ is given. Evolution of system state $x(t)$ is determined by an unknown control $u(\cdot)$ belonging to a given functional set $U_T \subset L_2(T; U)$. Here $(U, |\cdot|_U)$ is a uniformly convex real Banach space. At discrete time instants $\tau_i \in T$, $\tau_i < \tau_{i+1}$ (sufficiently frequent) some characteristics of the history $\{x_{t_0,\tau_i}(\cdot), \dot{x}_{t_0,\tau_i}(\cdot)\}$, namely $z(\tau_i) = C\{x_{t_0,\tau_i}(\cdot), \dot{x}_{t_0,\tau_i}(\cdot)\}$ are measured inaccurately. A certain $u(\cdot) \in U_T$ generating $z(\cdot)$ $(u(\cdot) = u(\cdot; z(\cdot)))$

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is to be calculated in "real time". The precise identification (reconstruction) of $u(\cdot)$ is, in general, impossible due to measurement noises. Therefore we suppose a certain approximation to $u(\cdot)$ to be designed. The former is required to be arbitrary close to the latter provided measurement errors and steps between τ_i are sufficiently small.

The problem belongs to the class of inverse problems of dynamics for control systems (reconstruction of a system input through output observations). For systems with distributed parameters the inverse problems, in a posteriori setting, have been studied in Lavrentyev et al. (1980), Banks and Kunisch (1982), Hoffmann and Spekels (1986), Kurzhanskii and Khapalov (1989), Barbu (1991). An approach to problems of the above type based on the ideas of the theory of positional control has been suggested in Kryazhimskii and Osipov (1983). Basing on this approach algorithms of dynamical identification of distributed and boundary controls and coefficients of an elliptic operator have been suggested for linear hyperbolic equations and for equations with monotonous operator in Osipov and Korotkii (1991), Maksimov (1990). In the papers of Osipov (1991) and Maksimov (1990, 1993b) analogous problem (reconstruction of a distributed control) has been discussed for hyperbolic variational inequalities. All attempts to apply this approach for solving of problems of reconstruction of boundary controls and coefficients of an elliptic operator in hyperbolic variational inequalities were unsuccessful up to now.

In the present paper the approach mentioned above has received further development. In Sect. 2 the problem of reconstruction of point sources intensity for the linear system (1) is treated. The solution algorithm which is stable with respect to informational and computational hindrances is constructed. Analogous problem for parabolic systems has been discussed in Maksimov (1993a) and Kryazhimskii et al. (1995). In Sect. 3 algorithms of control identification based on dynamical modification of the discrepancy method are suggested. It is supposed that the system is described by the variational inequality (2). The cases when distributed (boundary) disturbances or unknown coefficients of an elliptic operator fulfill the role of controls are considered.

Let $(H, |\cdot|_H)$ be a real Hilbert space with inner product $(\cdot,\cdot)$, and let $(V, \|\cdot\|)$ be a separable and reflexive Banach space. We assume that V is densely and continuously embedded in H. Identify H with its dual and denote a duality between V and V^* by $\langle\cdot,\cdot\rangle$. Let Δ_h be a family of partitions of the interval $T = [t_0, \vartheta]$ with diameters $\delta(h) : \Delta_h = \{\tau_{i,h}\}_{i=0}^{m_h}$, $\tau_{i,h} = \tau_{i-1,h} + \delta$, $\delta = \delta(h)$, $\tau_{0,h} = t_0$, $\tau_{m_h,h} = \vartheta (h > 0)$.

2. Reconstruction of point sources intensity for linear systems

Consider the system whose evolution is described by equation (1). Let $A_1 : V \to V^*$ be a linear continuous selfadjoint operator, $A : V \to V^*$ be a linear continuous operator satisfying, with a certain $c > 0$, the condition

$$\langle Ay, y\rangle \geq c\|y\|^2 \qquad \forall y \in V,$$

$u(t)$ be a n-dimensional value of time-varying input (a control) inducing the system motion, $f(\cdot) \in L_2(T;H)$ be a given disturbance, $B : U = \mathbf{R}^n \to V$, $Bu = \sum_{j=1}^{n} \omega_j u_j$, $\omega_j \in V$, $u_j \in \mathbf{R}$. We assume $t_0 = 0$, $x(t_0) = x_0 \in V$, $Ax_0 = 0$, $\dot{x}(t_0) = x_{10} \in H$.

A function $x(\cdot) = x(\cdot; x_0, x_{10}, u(\cdot))$ is called a solution of (1) on T if
a) $x(\cdot) \in C(T;V)$, $\dot{x}(\cdot) \in \{y(\cdot) \in L_2(T;V) : \dot{y}(\cdot) \in L_2(T;V^*)\}$;
b) for a. a. $t \in T$ the equality (1) is true, i. e. the equality

$$\langle \ddot{x}(t) + A\dot{x}(t) + A_1 x(t), v\rangle = (Bu(t) + f(t), v) \quad \forall\, v \in V$$

holds.

By the Theorem 1.2 (see Gajewski et al., 1974, p. 285) for any $u(\cdot) \in L_2(T;U)$ there exists a unique solution $x(\cdot)$ of (1).

Discuss the following problem. Let $C : H \times H \to \mathbf{R}^n$ be a linear continuous operator:

$$C\{x, y\} = Px + Qy, \quad Px = \{(p_j, x)\}_{j=1}^{n}, \quad Qx = \{(q_j, x)\}_{j=1}^{n},$$

$p_j \in H$, $q_j \in V$, $A^* q_j = 0$, $j \in [1:n]$. Let $x(\cdot) = x(\cdot; u(\cdot)) = x(\cdot; x_0, x_{10}, u(\cdot))$ be the solution of the system (1) depending on an unknown control $u(\cdot) \in U_T = L_2(T;U)$. At time instants $\tau_i \in \Delta = \{\tau_i\}_{i=0}^{m}$ the phase coordinates $\{x(\tau_i), \dot{x}(\tau_i)\}$ of the system (1) are measured approximately. The measurement results are values ξ_i such that

$$|\xi_i - z(\tau_i)|_n \leq h, \qquad z(t) = C\{x(t), \dot{x}(t)\}$$

(h is a bound for informational noise, $|\cdot|_n$ is a norm in $\mathbf{R}^n$). The problem is to construct an algorithm restoring an unknown control $u(\cdot) = u(\cdot; z(\cdot))$ on the basis of inaccurate measurements of $z(\tau_i)$.

Before describing the algorithm we indicate the set of inputs compatible with output $z(\cdot)$, i. e. the set

$$U(z(\cdot)) = \{v(\cdot) \in U_T : z(t) = Cx(t; x_0, x_{10}, v(\cdot))\ \forall\, t \in T\}.$$

For any $k \in [1:n]$ and $\sigma \geq 0$, define the function $w_k(\cdot; \sigma)$ to be the solution of the Cauchy problem

$$\ddot{w}(t) - A^*\dot{w}(t) + A_1 w(t) = 0, \quad w(\sigma) = q_k \in V, \quad \dot{w}(\sigma) = -p_k \in H \tag{3}$$

on $]-\infty, \sigma]$ and zero on $]\sigma, \infty[$. Existence and uniqueness of the solution of the adjoint system (3) follow from Lions (1968), Chapter III. Let

$$(\phi_k(t;\sigma))_j = (w_k(t;\sigma), \omega_j), \qquad j \in [1:n],$$
$$g_k(a, \sigma) = a_k + (\dot{w}_k(0;\sigma), x_0) - (w_k(0;\sigma), x_{10}), \quad a = \{a_k\}_{k=1}^{n} \in \mathbf{R}^n,\ k \in [1:n].$$

THEOREM 1 *An input $u(\cdot)$ is compatible with an observation result $z(\cdot)$ on T $(u(\cdot) \in U(z(\cdot)))$ if and only if*

$$\int_0^\sigma (\phi_k(t,\sigma), u(t))_{\mathbf{R}^n} dt = g_k(z(\sigma),\sigma) \quad \textit{for all } \sigma \in T \textit{ and } k \in [1:n]. \tag{4}$$

Proof of necessity. Let $u(\cdot)$ be compatible with $z(\cdot)$ and $x(\cdot) = x(\cdot;\, x_0, x_{10}, u(\cdot))$. Then for all $t \in T$

$$z^{(k)}(t) = (p_k, x(t)) + (q_k, \dot{x}(t)) \qquad \forall\, k \in [1:n].$$

Take an arbitrary $\sigma \in T$ and $k \in [1:n]$. Let $w(\cdot) = w_k(\cdot;\, \sigma)$. It is easily seen that

$$I_1 \equiv \int_0^\sigma \{\langle \ddot{x}(t), w(t)\rangle - \langle \ddot{w}(t), x(t)\rangle\}\, dt =$$
$$= (\dot{x}(\sigma), w(\sigma)) - (x_{10}, w(0)) - (\dot{w}(\sigma), x(\sigma)) + (\dot{w}(0), x_0), \qquad \sigma \in T,$$
$$I_2 \equiv \int_0^\sigma \{\langle A\dot{x}(t), w(t)\rangle + \langle A^*\dot{w}(t), x(t)\rangle\}\, dt = \langle Ax(\sigma), w(\sigma)\rangle - \langle Ax_0, w(0)\rangle.$$

Note that conditions $Ax_0 = 0$ and $A^*q_j = 0$ imply $I_2 = 0$. In virtue of selfadjoitness of the operator A_1 we have

$$\int_0^\sigma \{\langle A_1 x(t), w(t)\rangle - \langle A_1 w(t), x(t)\rangle\}\, dt = 0.$$

Multiply scalarly (1) by $w(t)$ and (3) by $x(t)$ respectively, subtract (3) from (1) and integrate from 0 to σ. We get

$$I_1 = \int_0^\sigma (w(t), Bu(t))\, dt.$$

The above equality may be rewritten as

$$\int_0^\sigma (\varphi_k(t,\sigma), u(t))_{\mathbf{R}^n}\, dt = (q_k, \dot{x}(\sigma)) + (p_k, x(\sigma)) +$$
$$+ (x_0, \dot{w}(0)) - (x_{10}, w(0)) = z^{(k)}(\sigma) + (x_0, \dot{w}(0)) - (x_{10}, w(0)).$$

This equality is equivalent to (4). Proof of sufficiency is analogous.
Let the following condition hold.

CONDITION 1 Rank $D(q) = n$.

We describe the algorithm approximating an unknown control $u(\cdot) = u(\cdot; z(\cdot))$. Introduce the control system

$$\begin{cases} \dot{w}^{(1)}(t) = D(q)v^h(t), \\ \dot{w}^{(2)}(t) = w^{(3)}(t), \quad w^{(3)}(t) = \int\limits_0^t K(t,\tau)v^h(\tau)d\tau, \\ \ddot{w}_k^{(1)}(t) + A^*\dot{w}_k^{(1)}(t) + A_1 w_k^{(1)}(t) = 0, \quad t \in T, \end{cases} \tag{5}$$

$$w^{(1)}(0) = w^{(2)}(0) = 0, \quad w_k^{(1)}(0) = q_k, \quad \dot{w}_k^{(1)}(0) = p_k, \quad k \in [1:n],$$

where $D(q)$ and $K(s,t)$ are $n\times n$-dimensional matrices of the forms:

$$D(q) = \{(\omega_j, q_k)\}_{j,k=1}^n, \quad K(s,t) = \{b_{k,j}(s,t)\}_{j,k=1}^n,$$

$$b_{k,j}(s,t) = \begin{cases} (\omega_j, \frac{d}{ds}w_k^{(1)}(s-t,0)), & \text{if } s > t \\ 0, & \text{in the opposite case.} \end{cases}$$

Before the initial time of the process, the value of $h \in (0,1)$ and a partition $\Delta = \Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$, $m = m_h$ are fixed. The algorithm is decomposed into $m_h - 1$ steps. At the ith step carried out during the time interval $\delta_{h,i} = [\tau_{h,i}, \tau_{h,i+1})$, $i \geq 1$, the following operations are carried out. At time $\tau_i = \tau_{h,i}$ we calculate

$$\nu_i = (g^*(\tau_{i+1}) - g^*(\tau_i))/\delta - w^{(3)}(\tau_i).$$

Here we have

$$g^*(\tau_i) = \{g_k^*(\tau_i)\}_{k=1}^n \in \mathbf{R}^n, \quad \xi = \{(\xi_i)_k\}_{k=1}^n \in \mathbf{R}^n,$$
$$g_k^*(\tau_i) = (\xi_i)_k + (w_k^{(1)}(0,\tau_i), x_0) - (\dot{w}_k^{(1)}(0,\tau_i), x_{10}), \quad k \in [1:n].$$

Then we determine the control $v^h(t) = v^h(t; \xi_{0,t}(\cdot))$,

$$v^h(t) = v_i^h = \begin{cases} |\nu_i|_n D^{-1}(q)s_i \,/\, |s_i|_n, & |s_i|_n \neq 0 \\ 0, & |s_i|_n = 0, \quad t \in \delta_{h,i}, \end{cases} \tag{6}$$

$$s_i = g^*(\tau_i) - g^*(0) - w^{(1)}(\tau_i) - w^{(2)}(\tau_i)$$

and transform the state $w(\tau_i) = \{w^{(1)}(\tau_i), w^{(2)}(\tau_i), w^{(3)}(\tau_i)\} \in \mathbf{R}^{3n}$ of the model (5) into $w(\tau_{i+1})$. The procedure stops at time ϑ.

THEOREM 2 *If $h/\delta(h) \to 0$, $\delta(h) \to 0$ as $h \to 0$, then $v^h(\cdot;\xi(\cdot)) \to u(\cdot; z(\cdot))$ weakly in $L_2(T;\mathbf{R}^n)$.*

Proof. By virtue of Theorem 1 the following equality is true

$$U(z(\cdot)) = \{v(\cdot) \in L_2(T;\mathbf{R}^n) : \ \varepsilon(t; v(\cdot), z(\cdot)) = 0 \qquad \forall t \in T\}, \tag{7}$$

where

$$\varepsilon(t; v(\cdot), z(\cdot)) = \int\limits_0^t \{\frac{d}{ds} g(z(s), s) - D(q)v(s) - \int\limits_0^t K(s, \tau)v(\tau)\, d\tau\}\, ds,$$

$$g(z(s), s) = \{g_k(z(s), s)\}_{k=1}^n.$$

Further we shall write $\dot{g}(s)$ instead of $\dfrac{d}{ds} g(z(s)s)$ and $g(s)$ instead of $g(z(s), s)$ for the sake of simplicity. Let Condition 1 be fulfilled. Then it follows from (7) that the set $U(z(\cdot))$ contains one element, i. e. $U(z(\cdot)) = \{u(\cdot; z(\cdot))\}$. Let us estimate the evolution of

$$\varepsilon(t) = \varepsilon(t; v^h(\cdot), z(\cdot)) = |g(t) - g(0) - w^{(1)}(t) - w^{(2)}(t)|^2, \qquad t \in [\tau_1, \vartheta].$$

We have

$$\varepsilon(\tau_{i+1}) = \varepsilon(\tau_i) + 2r_i' \mu_i + |\mu_i|^2. \tag{8}$$

Here

$$r_i = g(\tau_i) - g(0) - w^{(1)}(\tau_i) - w^{(2)}(\tau_i),$$
$$\mu_i = g(\tau_{i+1}) - g(\tau_i) - \delta^2\{2K(\tau_{i+1}, \tau_i) + K(\tau_i, \tau_i)\}v_i^h -$$
$$- \delta^2 \sum_{j=1}^{i} K(\tau_{i+1}, \tau_{j-1}) v_{j-1}^h - \delta D(q) v_i^h.$$

It can easily be shown that the following inequalities hold:

$$|r_i| \le C_1 + C_2 \delta \sum_{j+1}^{i} |v_{j-1}^h|, \quad i \ge 1, \tag{9}$$

$$|\mu_i|^2 \le C_3 \delta\{ \int\limits_{\tau_i}^{\tau_{i+1}} |\dot{g}(s)|^2\, ds + \sum_{j+1}^{i+1} \delta^2 |v_{j-1}^h|^2 + \delta |v_i^h|^2\}. \tag{10}$$

Here, constants C_j, $j \in [1:3]$ do not depend on i, δ. Hence, taking into account the definition of v_i^h (see (6)) we deduce from (8)

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + 4h|\mu_i| + C_4(1 + |r_i|)\delta^2 |v_i^h| + |\mu_i|^2 + C_5 h(|r_i| + h). \tag{11}$$

Note that

$$d_i \equiv \delta_i |v_{i-1}^h|^2 \le a_i + C_6 \delta \sum_{j=1}^{i-1} d_j, \qquad \sum_{j=1}^{m_h - 1} a_j < +\infty.$$

Consequently,

$$\sum_{j=1}^{m_h - 1} \delta |v_{j-1}^h|^2 \equiv |v^h(\cdot)|^2_{\mathbf{L}_2(T;U)} \le C_7 < +\infty. \tag{12}$$

Thus, by (9) – (12) we have

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + C_8(h/\delta + \delta), \qquad i \in [0 : m_h - 1].$$

Therefore the following estimation is true

$$\varepsilon(t; v^h(\cdot), z(\cdot)) \le \lambda(h, \delta) \to 0 \quad \text{as} \quad h \to 0, \ \delta \to 0, \ h/\delta \to 0. \tag{13}$$

Validity of the theorem follows from (12), (13). Theorem 2 is proved. □

3. Control identification for variational inequalities

Now for the hyperbolic variational inequality (2) we construct a dynamical procedure for identification of a control $u(\cdot)$. We assume that $\phi : H \to \overline{\mathbf{R}} = \mathbf{R}^+ \bigcup \{+\infty\}$ is a convex, lower semicontinuous, proper function, $\mathbf{R}^+ = \{r \in \mathbf{R} : r \ge 0\}$, $P \subset U$ is a convex, bounded and closed set, $U_T = \{u(\cdot) \in L_2(T; U) : u(t) \in P \text{ for a. a. } t \in T\}$, $x_0 \in V$, $x_{10} \in H$.

Consider two cases. In the first case we suppose $B \in L(U; H)$, $f(\cdot) \in W(T; H) = \{x(\cdot) \in L_2(T; H) : \dot{x}(\cdot) \in L_2(T; H)\}$ and an unknown real control $u(\cdot) = u(\cdot; x(\cdot)) \in U_T$ is such that

$$x(\cdot) = x(\cdot\,; x_0, x_{10}, u(\cdot)) \in W_1(T; V^*) = \Big\{y(\cdot) \in C(T; V) :$$
$$\dot{y}(\cdot) \in L_\infty(T; V) \bigcap C(T; H), \ \ddot{y}(\cdot) \in L_2(T; H)\Big\}, \tag{14}$$

where $L(U; H)$ is a space of linear continuous operators acting from U to H.

The inclusion (14) takes place (see, for example, Tiba (1985)) if $t \to Bu(t) = Bu(t; x(\cdot) \in L_2(T; V)$ and the following condition is fulfilled.

CONDITION 2 $V = H_0^1(\Omega)$, $H = L_2(\Omega)$, $\phi(y) = \int\limits_\Omega j(y(\eta))d\eta$, *if* $y \in V$, $\eta \to j(y(\eta)) \in L_1(\Omega)$, $\phi(y) = +\infty$, *in the opposite case,* $j : \mathbf{R} \to \overline{\mathbf{R}}$ *is a convex, lower semicontinuous, proper function,* $f(\cdot) \in W(T; V)$, $B = I$ *(identity operator),* $U = H$, $x_0 \in V$, $\Delta_L x_{10} \in H$, $x_{10}(\eta) \in \mathrm{dom}(\partial j)$ *for a. a.* $\eta \in \Omega$.

Here Δ_L is Laplace operator, ∂j is subdifferential of j.

In the second case $B \in L(U; V^*)$, $\phi : V \to \overline{\mathbf{R}}$ and the system is described by the variational inequality

$$(\ddot{x}(t) + A_1\dot{x}(t) + Ax(t), \dot{x}(t) - v) + \phi(\dot{x}(t)) - \phi(v) \le$$
$$\le \langle Bu(t), \dot{x}(t) - v\rangle + (f(t), \dot{x}(t) - v) \quad \text{for a. a. } t \in T \quad \forall v \in V. \tag{15}$$

Here $A_1 : V \to V^*$ is a linear continuous and coercivity operator. The solution of the system (15) is the function $x(\cdot) \in \{y(\cdot) \in L_2(T; V) : \dot{y}(\cdot), \ddot{y}(\cdot) \in L_2(T; V)\}$.

The sufficient conditions for existence and uniqueness of solutions of (15) with indicated smoothness have been obtained, for example, in Duvant and Lions (1972), ch. III, § 6.

Let us focus on the first case. Consider the following problem: Let $x(\cdot) = x(\cdot\,; u(\cdot)) = x(\cdot\,; x_0, x_{10},\, u(\cdot)) \in W_1(T; V^*)$ be the solution of the system (2) depending on an unknown control $u(\cdot) \in U_T$. At time instants $\tau_i \in \Delta = \{\tau_i\}_{i=0}^m$ the phase coordinates $\{z(\tau_i) = x(\tau_i), \dot{x}(\tau_i)\}$ of the system (2) are measured approximately. The measurements results are values $\xi_i = \{\xi_i^{(1)}, \xi_i^{(2)}\} \in V \times H$ with the properties

$$|\xi_i^{(2)} - \dot{x}(\tau_i)|_H \le h, \quad \|\xi_i^{(1)} - x(\tau_i)\| \le h, \quad \xi_{i+1}^{(1)} - \xi_i^{(1)} \in D(\phi). \tag{16}$$

The problem is to construct an algorithm restoring an unknown control $u(\cdot) = u(\cdot; x(\cdot))$ on the basis of inaccurate measurements of $x(\tau_i)$.

REMARK 1 With Condition 2, the inclusion $t \to Bu(t) = Bu(t; x(\cdot)) \in L_2(T; V)$ means the following. A real (unknown) control $u(\cdot) = u(\cdot; x(\cdot))$ possesses two properties: $u(t) \in P \subset H$ for a. a. $t \in T$ and $t \to u(t) \in L_2(T; V)$. It generates the output $x(\cdot) = x(\cdot; u(\cdot))$ (the solution of the system (2)). This output is measured inaccurately. A control $u(\cdot)$ is to be reconstructed.

Let $U(x(\cdot))$ be the set of all controls $v(\cdot) \in U_T$, generating $x(\cdot)$:

$$U(x(\cdot)) = \{v(\cdot) \in U_T : x(t) = x(t; x_0, x_{10}, v(\cdot)) \qquad \forall t \in T\}.$$

Let $S(v) = \{z \in D(\phi) : \|z - v\| \le 1\}$, and let $\phi(\cdot,\cdot)$ be a function such that $\phi(h,\ \delta) \to 0$ as $h \to 0+$, $\delta \to 0+$, $h/\delta \to 0+$,

$$\phi(h,\delta) \ge f_x(c_1\delta^{1/2} + 2ch/\delta) + \sup\{\int\limits_{\tau_i}^{\tau_{i+1}} |\ddot{x}(\tau)|_H^2 d\tau :\ i \in [0 : m-1]\},$$

$$m = m_h.$$

We denote $\sup\{|\phi(\dot{x}(\tau)) - \phi(\psi)| : \psi \in D(\phi),\ |\dot{x}(\tau) - \psi|_H \le \varepsilon,\ \tau \in T\}$ by $f_x(\varepsilon)$ and the element of the set $U(x(\cdot))$, whose $L_2(T; U)$-norm is minimal, by $u_*(\cdot; x(\cdot))$. The numbers c and c_1 are such that $|z|_H \le c\|z\| \quad \forall z \in V$, $|\ddot{x}(\cdot; u(\cdot))|_{L_2(T;H)} \le c_1$. Introduce the convex bounded sets

$$U_k^{h,\delta}(v, w, \mu, \nu) = \Big\{u \in P : \sup_{z \in S((\mu-\nu)/\delta)} \{-F_{i,\delta}(w, v, \mu, \nu; z - (\mu-\nu)/\delta) +$$

$$+ (Bu, z - (\mu-\nu)/\delta) + \phi((\mu-\nu)/\delta) - \phi(z)\} \le \nu(k, h, \delta)\Big\},$$

where

$$F_{i,\delta}(w, v, \mu, \nu; z) = ((w - v)\delta^{-1} - f(\tau_i), z) + \langle A(\mu+\nu)/2, z\rangle,$$
$$\nu(k, h, \delta) = k(h\delta^{-2} + \delta^{1/2} + \phi(h,\delta)).$$

Assume that the following condition is fulfilled.

CONDITION 3 *The function $\phi(\cdot)$ is continuous on $D(\phi)$. There exists a number $N \geq 1$ such that for any $v \in D(\phi)$, $\lambda \in [0,1]$ the inequality $\phi(\lambda v) \leq \lambda^N \phi(v)$ holds.*

To calculate (approximately) $u(\cdot) = u_*(\cdot;\, x(\cdot))$ we apply dynamical modification of the discrepancy method. Let us describe the algorithm, i. e. the sequence of actions forming an approximation to $u_*(\cdot) = u_*(\cdot; x(\cdot))$. First, a family Δ_h of partitions of the interval T with diameters $\delta(h)$, $\delta(h) \to 0$, $h/\delta^2(h) \to 0$ as $h \to 0$, is chosen. Before the initial time of the process, values h, k and the partition $\Delta = \Delta_h$ are fixed. The work of the algorithm starting at time t_0 is decomposed into $m_h - 1$ steps. At the ith step carried out during the time interval $\delta_i = \delta_{h,i} = [\tau_{h,i}, \tau_{h,i+1})$, the control $v^h(t) = v^h(t; \xi_{t_0,t}(\cdot)) = v_i^h$, $t \in \delta_{h,i}$, $i \geq 1$,

$$v_i^h = \begin{cases} \arg\min\{|u|_U : u \in U_k^{h,\delta}(p_i(\xi))\}, & \text{if } U_k^{h,\delta}(p_i(\xi)) \neq \emptyset \\ 0, \quad \text{in the opposite case,} & \end{cases} \tag{17}$$

$p_i(\xi) = \{\xi_{i-1}^{(2)}, \xi_i^{(2)}, \xi_{i-1}^{(1)}, \xi_i^{(1)}\}$, is calculated. The procedure stops at time ϑ. Let $v(t) = v_0^h = \arg\min\{|u|_U : u \in P\}$, $t \in \delta_{h,0}$.

THEOREM 3 *There exists a number $k_* > 0$ such that for every $k \in [k_*, +\infty)$*

$$|v^h(\cdot; \xi(\cdot)) - u_*(\cdot; x(\cdot))|_{L_2(T;U)} \to 0 \quad \text{as } h \to 0. \tag{18}$$

Proof of the theorem is performed by analogy with proof of corresponding statements in the papers by Osipov and Kryazhimskii (1983) and Osipov et al. (1991). It is based on the lemmas we will formulate here. First, introduce the sets

$$U_{\Delta,k}^h(\xi(\cdot)) = \{u(\cdot) \in U_T \mid u(t) = u_i \text{ for a. a. } t \in [\tau_{i-1}, \tau_i),$$
$$i \in [1:m],\ u_i \in U_{k,i}^{h,\delta}(p_i(\xi))\},$$
$$U_{k,i}^{h,\delta}(v, w, \mu, \nu) = \{u \in P : \sup_{z \in S((\mu-\nu)/\delta)} \{-((w-v)/\delta - f(\tau_i),\ z - (\mu-\nu)/\delta)$$
$$- \langle A(\mu+\nu)/2,\ z - (\mu-\nu)/\delta\rangle + (Bu, z - (\mu-\nu)/\delta)$$
$$+ \varphi((\mu-\nu)/\delta) - \varphi(z)\} \leq \nu(k,h,\delta)\},\quad i \in [1:m],$$
$$U_\Delta(x(\cdot)) = \{u(\cdot) \in U_T \mid \int_{\tau_i}^{\tau_{i+1}} (Bu(t), v - \dot{x}(t))\, dt \leq \psi(\tau_i, \tau_{i+1}, v, x(\cdot))$$
$$\forall v \in V,\quad i \in [0:m-1]\}.$$

Here $\xi(\cdot) \in \Xi^h(x(\cdot))$, $\tau_i = \tau_{h,i}$, $m = m_h$, $\Xi^h(x(\cdot))$ is the set of piecewise constant functions $\xi(t) = \{\xi^{(1)}(t), \xi^{(2)}(t)\} \in V \times H$, $t \in T$, $\xi^{(1)}(t) = \xi_i^{(1)}$, $\xi^{(2)}(t) = \xi_i^{(2)}$, $t \in [\tau_i, \tau_{i+1})$, satisfying (16),

$$\psi(\tau_i, \tau_{i+1}, v, x(\cdot)) = \int_{\tau_i}^{\tau_{i+1}} \{(\ddot{x}(t) - f(t), v - \dot{x}(t)) + \langle Ax(t), v - \dot{x}(t)\rangle + \varphi(v)$$

$-\varphi(\dot{x}(t))\}\,dt.$

LEMMA 1 *Let* $v(\cdot) \in U_\Delta(x(\cdot))$, $v_*(t) = \delta^{-1} \int\limits_{\tau_i}^{\tau_{i+1}} v(t)\,dt$ *for a. a.* $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : m-1]$. *Then there exists a value* $k_* > 0$ *such that the inclusion*

$$v_*(\cdot) \in U^h_{\Delta,k}(\xi(\cdot)) \quad \forall\, k \geq k_* \tag{19}$$

holds uniformly with respect to all $h \in (0,1)$, $\delta \in (0, \vartheta - t_0)$ *and* $\xi(\cdot) \in \Xi^h(x(\cdot))$.

Proof. Let $z \in D(\varphi)$, $\xi(\cdot) \in \Xi^h(x(\cdot))$, $\xi(t) = \{\xi_i^{(1)}, \xi_i^{(2)}\}$ as $t \in [\tau_i, \tau_{i+1})$, $\chi_i = (\xi^{(1)}_{i+1} - \xi^{(1)}_i)/\delta$,

$$\begin{aligned}\psi^*(\tau_i, \tau_{i+1}, z, \xi(\cdot)) &= (\xi^{(2)}_{i+1} - \xi^{(2)}_i - \delta f(\tau_i), z - \chi_i) + \\ &+ \delta\{\langle A(\xi^{(1)}_{i+1} + \xi^{(1)}_i)/2, z - \chi_i\rangle + \varphi(z) - \varphi(\chi_i)\},\end{aligned} \tag{20}$$

$$|\dot{x}(\cdot)|_{L_\infty(T;V)} \leq c_2 < +\infty. \tag{21}$$

Due to equality $A = A^*$ and (21) we have

$$\begin{aligned}&\Big|\int\limits_{\tau_i}^{\tau_{i+1}} \langle Ax(t), z - \dot{x}(t)\rangle\,dt - \delta\langle A(x_{i+1} + x_i)/2, z - (x_{i+1} - x_i)/\delta\rangle\Big| \\ &\quad\leq\; k_0\delta^2(1 + \|z - (x_{i+1} - x_i)/\delta\|), \qquad x_i = x(\tau_i).\end{aligned} \tag{22}$$

Using inclusions $\dot{x}(\cdot)$, $f(\cdot) \in W^{1,2}(T;H)$ and continuity of embedding V into H, one can easily deduce inequality

$$\begin{aligned}&\Big|\int\limits_{\tau_i}^{\tau_{i+1}} (\ddot{x}(t) - f(t), z - \dot{x}(t))\,dt - (\dot{x}_{i+1} - \dot{x}_i - \delta f(\tau_i), z - (x_{i+1} - x_i)/\delta)\Big| \\ &\quad\leq k_1\delta\{\delta + \int\limits_{\tau_i}^{\tau_{i+1}} |\ddot{x}(t)|^2_H\,dt + \delta^{1/2}\|z - (x_{i+1} - x_i)/\delta\|\}, \qquad \dot{x}_i = \dot{x}(\tau_i).\end{aligned} \tag{23}$$

It follows from (16) that the inequalities

$$|\varphi(\dot{x}(t)) - \varphi(\chi_i)| \leq f_x(c_1\delta^{1/2} + 2ch/\delta), \tag{24}$$

$$\begin{aligned}&|(\xi^{(2)}_{i+1} - \xi^{(2)}_i - \delta f(\tau_i), z - \chi_i) - (\dot{x}_{i+1} - \dot{x}_i - \delta f(\tau_i), z - (x_{i+1} - x_i)/\delta)| \\ &\quad\leq k_2|\xi^{(2)}_{i+1} - \xi^{(2)}_i - \delta f(\tau_i)|_H h/\delta + 2h|z - (x_{i+1} - x_i)/\delta|_H \\ &\quad\leq k_3 h\delta^{-1}\{h + \delta + \int\limits_{\tau_i}^{\tau_{i+1}} |\ddot{x}(t)|^2_H\,dt\} + 2h|z - (x_{i+1} - x_i)/\delta|_H \\ &\quad\leq k_4 h\{\delta^{-1}(h + \delta + \int\limits_{\tau_i}^{\tau_{i+1}} |\ddot{x}(t)|^2_H\,dt) + \|z - (x_{i+1} - x_i)/\delta\|\},\end{aligned} \tag{25}$$

$$\delta|\langle A(\xi_{i+1}^{(1)}+\xi_i^{(1)})/2, z-\chi_i\rangle - \langle A(x_{i+1}+x_i)/2, z-(x_{i+1}-x_i)/\delta\rangle|$$
$$\leq\ k_5 h\{1+\delta\|z-(x_{i+1}-x_i)/\delta\|\} \tag{26}$$

hold for $t \in \delta_i$. Taking into account (22) – (26) we have

$$|\psi(\tau_i,\tau_{i+1},z,x(\cdot)) - \psi^*(\tau_i,\tau_{i+1},z,\xi(\cdot))| \leq k_6\{h+\delta^2+h^2/\delta+$$
$$(\delta+h/\delta)\int\limits_{\tau_i}^{\tau_{i+1}} |\ddot{x}(t)|_H^2\,dt\} + \delta f_x(c_1\delta^{1/2}+2ch/\delta)$$
$$+k_7\delta(h\delta^{-1}+\delta^{1/2})\,\|z-(x_{i+1}-x_i)/\delta\|. \tag{27}$$

Besides,

$$\int\limits_{\tau_i}^{\tau_{i+1}} |\ddot{x}(t)|_H^2\,dt \leq c_1^2, \qquad |\int\limits_{\tau_i}^{\tau_{i+1}} \{(Bv(t),\dot{x}(t)) - (Bv(t),\chi_i)\}\,dt|$$
$$\leq k_8\delta\{\delta^{1/2}+h/\delta\}\ \ \forall v(\cdot)\in U_T,\ \ i\in[0:m-1]. \tag{28}$$

From (27), (28) we conclude that there exists a value $k_* > 0$ such that the inequality

$$\delta^{-1}(B\int\limits_{\tau_i}^{\tau_{i+1}} v(t)\,dt, z-\chi_i) \leq \delta^{-1}\psi^*(\tau_i,\tau_{i+1},z,\xi(\cdot)) + \nu(k,h,\delta)$$
$$\forall\, k\geq k_*,\ \ z\in S(\chi_i)$$

is true. (The constant k_* is written out explicitly.) This implies (19). Lemma 1 is proved. □

LEMMA 2 *Let $h_j \to 0$, $\delta_j \to 0$, $h_j\delta_j^{-2} \to 0$ as $j\to\infty$, $\Delta_j = \Delta_{h_j}$, $u_j(\cdot) \in U^{h_j}_{\Delta_j,k}(\xi_j(\cdot))$, $\xi_j(\cdot) \in \Xi^{h_j}(x(\cdot))$, $u_j(\cdot) \to u_0(\cdot)$ weakly in $L_2(T,U)$. Then $u_0(\cdot) \in U(x(\cdot))$.*

Proof. Let $u_0(\cdot) \notin U(x(\cdot))$. There exist $v_* \in V$, $t_1, t_2 \in T$, $t_1 < t_2$ and $\alpha_* > 0$ such that

$$\int\limits_{t_1}^{t_2} (Bu_0(t), v_*-\dot{x}(t))\,dt > \psi(t_1,t_2,v_*,x(\cdot)) + \alpha_*. \tag{29}$$

Let j_1 be such that for $j \geq j_1$

$$\delta_j \leq (t_2-t_1)/3, \tag{30}$$

$$\sup\{\int\limits_{t_*}^{t^*} |(Bu(t),v_*-\dot{x}(t))|\,dt,\quad \int\limits_{t_*}^{t^*}\{|(\ddot{x}(t)-f(t),v_*-\dot{x}(t))| + \langle Ax(t),v_*-\dot{x}(t)\rangle$$
$$+\varphi(v_*)-\varphi(\dot{x}(t))\}\,dt:\quad t_*,t^*\in T,\quad 0\leq t^*-t_*\leq\delta_j,$$
$$u(\cdot)\in U_T\} \leq \alpha_*/16. \tag{31}$$

Let $\tau_{i_*(j)} = \max\{\tau_{i(j)} \in \Delta_j : \tau_{i(j)} \le t_2\}$ and $\tau_{i^*(j)} = \min\{\tau_{i(j)} \in \Delta_j : \tau_{i(j)} \ge t_1\}$, $\tau_i = \tau_{i(j)} = \tau_{h_j,i(j)}$. Due to (29) – (31) we have $i_*(j) > i^*(j)$ and for $j > j_1$

$$\int\limits_{\tau_{i^*(j)}}^{\tau_{i_*(j)}} (Bu_0(t), v_* - \dot{x}(t))\, dt \ge \psi(\tau_{i^*(j)}, \tau_{i_*(j)}, v_*, x(\cdot)) + 3\alpha_*/4. \tag{32}$$

By the definition of $u_j(\cdot)$, $u_j(t) = u_j^{(i)} = \delta_j^{-1} u_{ij}$ for a. a. $t \in \delta_{h_j,i(j)} = [\tau_i, \tau_{i+1})$, we get

$$(Bu_{ij}, v - \chi_i^{(j)}) \le \psi^*(\tau_i, \tau_{i+1}, v, \xi_j(\cdot)) + \delta_j \nu_j \quad \forall v \in S(\chi_i^{(j)})$$
$$(\nu_j = \nu(k, h_j, \delta_j)). \tag{33}$$

Let $\lambda_i = 1$, $v_i = v_*$, if $v_* \in S(\chi_i^{(j)})$, and $\lambda_i = \|v_* - \chi_i^{(j)}\|$, $v_i = \chi_i^{(j)} + (v_* - \chi_i^{(j)})/\|v_* - \chi_i^{(j)}\|$ — otherwise. Note that with $v_* \notin S(\chi_i^{(j)})$ the relation $v_i \in D(\varphi)$, $\|v_i - \chi_i^{(j)}\| = 1$ is true. Using Condition 1, convexity and nonnegativity of φ we have

$$\lambda_i\{\varphi(v_i) - \varphi(\chi_i^{(j)})\} - \varphi(v_*) + \varphi(\chi_i^{(j)}) \le 0.$$

Consequently

$$\lambda_i \psi^*(\tau_i, \tau_{i+1}, v_i, \xi_j(\cdot)) \le \psi^*(\tau_i, \tau_{i+1}, v_*, \xi_j(\cdot)). \tag{34}$$

Taking into account (20), inequalities $|\xi_{ij}^{(2)} - \dot{x}(\tau_i)|_H \le h_j$ and inclusions $u_j(\cdot) \in U^{h_j}_{\Delta_j,k}(\xi_j(\cdot))$, we conclude that

$$|\chi_i^{(j)} - \dot{x}(t)|_H \le 2ch_j/\delta_j + c_1\delta_j^{1/2} \quad \text{for } t \in \delta_{h_j,i(j)},$$
$$l_j = \int\limits_{\tau_{i^*(j)}}^{\tau_{i_*(j)}} (Bu_j(t), v_* - \dot{x}(t))\, dt \le \int\limits_{\tau_{i^*(j)}}^{\tau_{i_*(j)}} (Bu_j(t), v_* - \chi_i^{(j)})\, dt + \mu_j,$$
$$\mu_j = k_0(h_j/\delta_j + \delta_j^{1/2}).$$

Therefore it follows from (33), (34) that

$$l_j \le \sum_{i=i^*(j)}^{i_*(j)-1} \lambda_i (Bu_{ij}, v_i - \chi_i^{(j)}) + \mu_j \le \sum_{i=i^*(j)}^{i_*(j)-1} \lambda_i\{\psi^*(\tau_i, \tau_{i+1}, v_i, \xi_j(\cdot)) + \delta_j\nu_j\} + \mu_j. \tag{35}$$

Due to (34), (27) we deduce

$$l_j \le \psi(\tau_{i^*(j)}, \tau_{i_*(j)}, v_*, x(\cdot)) + f_1(h_j, \delta_j), \tag{36}$$
$$f_1(h_j, \delta_j) = c_1 h_j \delta_j^{-2} + c_2 \delta_j^{1/2} + c_3 \varphi_x(h_j, \delta_j).$$

Let $j_2 \geq j_1$ be such that for $j \geq j_2 \quad f_1(h_j, \delta_j) \leq \alpha_*/4$ and

$$\int\limits_{\tau_{i^*(j)}}^{\tau_{i_*(j)}} (B(u_0(t) - u_j(t)), v_* - \dot{x}(t))\, dt$$

$$\leq \int\limits_{t_1}^{t_2} (B(u_0(t) - u_j(t)), v_* - \dot{x}(t))\, dt + \alpha_*/8 \leq \alpha_*/4. \tag{37}$$

Therefore for $j \geq j_2$ it follows from (36), (37) that

$$\int\limits_{\tau_{i^*(j)}}^{\tau_{i_*(j)}} (Bu_0(t), v_* - \dot{x}(t))\, dt \leq \psi(\tau_{i^*(j)}, \tau_{i_*(j)}, v_*, x(\cdot)) + \alpha_*/2. \tag{38}$$

However, (38) contradicts (32). Lemma 2 is proved. □

Introduce the following

Condition 4 *Function ∂j is single-valued and Lipschitz.*

We assume that conditions 2 and 4 are fulfilled. Let elements ξ_i satisfy (16), a family Δ_h with diameters $\delta(h)$ be such that $\delta(h) \to 0$, $h\delta^{-1}(h) \to 0$ as $h \to 0$. Theorem 3 is also valid, if we assume in (17)

$$U_k^{h,\delta}(p_i(\xi)) = \{u \in P : |\delta^{-1}(\xi_i^{(2)} - \xi_{i-1}^{(2)}) + A\xi_i^{(1)} + A_2\xi_i^{(2)} - f(\tau_i) - Bu|_{V^*} \leq k(\delta^{1/2} + h\delta^{-1})\},$$

where $A_2 : H \to H$ is an operator of the form $(A_2 x)(\eta) = \partial j(x(\eta))$ for a. a. $\eta \in \Omega$. In this case proof of Theorem 3 differs by some technical details.

Let under Conditions 2, 4 the set of admissible controls U_T be of the form:

$$U_T = \{v(\cdot) \in L_2(T; V) : v(t) \in P,\ |\dot{v}(t)|_H \leq a \text{ for a. a. } t \in T\},\ a < +\infty.$$

At time instants $\tau_i \in \Delta$ the history of the motion $x_{\tau_{i-1},\tau_i}(\cdot)$ is measured approximately, i. e. a piecewise constant function $\xi_{\tau_{i-1},\tau_i}(\cdot)$ being an approximation to $x_{\tau_{i-1},\tau_i}(\cdot)$ is calculated:

$$|\xi_i - \dot{x}(\tau_i)|_H \leq h, \quad \Big| \int\limits_{\tau_{i-1}}^{\tau_i} A(x(t) - \psi(t))dt \Big|_H \leq h. \tag{39}$$

Here $\psi(t) = x_0 + \int\limits_0^t \xi(\tau)d\tau$, $\xi_i = \xi(\tau_i)$, $\xi(\cdot) \in \Xi(x(\cdot), h)$, $\Xi(x(\cdot), h)$ is the set of all piecewise constant functions $\xi(\cdot) : T \to H_0^1(\Omega) \bigcap H^2(\Omega)$ such that the inequalities (39) hold (the set of all possible measurement results for $x(\cdot)$).

Let X_T be the bundle of all solutions of the variational inequality (2), i. e. $X_T = \{x(\cdot; x_0, x_{10}, v(\cdot)) : v(\cdot) \in U_T\}$, and let $u(\cdot; x(\cdot))$ be a control generating the motion $x(\cdot) \in X_T$. Denote

$$\sup\{|v^h(\cdot;\xi(\cdot)) - u(\cdot;x(\cdot))|^2_{L_2(T;H)} : x(\cdot) \in X_T,\ \xi(\cdot) \in \Xi(x(\cdot), h)\}$$

by $\nu(h)$. We assume in (17)

$$U_k^{h,\delta}(p_i(\xi)) = \Big\{u \in P : \Big|\delta^{-1}(\xi_i - \xi_{i-1}) + \delta^{-1}\int\limits_{\tau_{i-1}}^{\tau_i} A\psi(t)dt + A_2\xi_i - f(\tau_i) - u\Big|_H$$

$$\leq k(\delta^{1/2} + h\delta^{-1})\Big\}, \quad p_i(\xi) = \xi_{t_0,\tau_i}(\cdot).$$

Then the following theorem is true.

THEOREM 4 *Let* $\operatorname{int} P \neq \emptyset$. *Then there exist values* $k_* > 0$ *and* $h_* \in (0,1)$ *such that for every* $k \geq k_*$ *and* $h \in (0, h_*)$ *it holds*

$$C_1(h\delta^{-1} + \delta^{1/2})^2 \leq \nu(h) \leq C_2(h\delta^{-1} + \delta^{1/2}).$$

Here the constants C_1 and C_2 are found explicitly.

Proof of Theorem 4 is performed by analogy with proof of Theorem 2.2 from Maksimov (1994). It is based on the ideas from Osipov and Kryazhimskii (1995).

Consider the second case. We assume $\phi : V \to \overline{\mathbf{R}}$. At time instants τ_i the coordinates $\dot{x}(\tau_i)$ of the system (15) are measured approximately, i. e. the elements $\xi_i \in V$ close to $\dot{x}(\tau_i)$ in the following sense

$$\|\xi_i - \dot{x}(\tau_i)\| \leq h \tag{40}$$

are found. Let $\phi_1(\cdot,\cdot)$ be a function with the properties: $\phi_1(h,\delta) \to 0$ as $h \to 0+$, $\delta \to 0+$,

$$\phi_1(h,\delta) \geq \nu_x(\delta) + f_x^{(1)}(h) + \sup\{\int\limits_{\tau_i}^{\tau_{i+1}} \|\ddot{x}(\tau)\|^2 d\tau : i \in [0 : m-1]\},$$

$\nu_x(\cdot)$ be the modulo of continuity of the function $t \to \phi(\dot{x}(t))$, $t \in T$, $f_x^{(1)}(\varepsilon) = \sup\{|\phi(\dot{x}(\tau)) - \phi(\psi)| : \psi \in D(\phi),\ \|\dot{x}(\tau) - \psi\| \leq \varepsilon,\ \tau \in T\}$. Introduce the convex bounded and closed sets:

$$V_k^{h,\delta}(\xi_{t_0,\tau_i}(\cdot)) = \Big\{u \in P : \sup_{z \in S(\xi_{i-1})} \{F_{i,\delta}^{(1)}(u, \xi_i, \xi_{i-1}, \xi_{i-1} - z)$$

$$+\langle A(x_0 + \delta\sum_{j=1}^{i-1}\xi_j), \xi_{i-1} - z\rangle + \phi(\xi_{i-1}) - \phi(z)\} \leq k(h\delta^{-1} + \delta^{1/2} + \phi_1(h,\delta))\Big\},$$

where

$F_{i,\delta}^{(1)}(u,w,v,z) = ((w-v)\delta^{-1} - f(\tau_i), z) + \langle A_1 v - Bu, z\rangle.$

In this case the algorithm of approximation to $u(\cdot) = u_*(\cdot; x(\cdot)) \in U(x(\cdot)) \subset U_T$ is analogous to the one described above. Let $\delta(h) \to 0$, $h\delta^{-1}(h) \to 0$ as $h \to 0+$. If in (17) we replace sets $U_k^{h,\delta}(p_i(\xi))$ by sets $V_k^{h,\delta}(\xi_{t_0,\tau_i}(\cdot))$, then the following theorem is true.

THEOREM 5 *Let Condition 3 be fulfilled. Then the convergence (18) takes place.*

Proof of Theorem 5 is similar to proof of Theorem 3. Instead of the inequality (27) we use in the process the following inequality

$$\begin{aligned}
&|\psi(\tau_i,\tau_{i+1},z,x(\cdot)) - \psi_1^*(\tau_i,\tau_{i+1},z,\xi(\cdot))| \\
&\quad \le \delta\{\mu(\delta) + |\varphi(\dot{x}_i) - \varphi(\xi_i)|\} + k_1\delta\{\delta + \int_{\tau_i}^{\tau_{i+1}} \|\dot{x}(\tau)\|^2\, d\tau\} \\
&\quad + k_2\delta\|v - x_i\| \int_{\tau_i}^{\tau_{i+1}} \{|\dot{f}(\tau)|_H + \|\dot{x}(\tau)\|\} + k_3 h(1 + \|v - \xi_i\|),
\end{aligned}$$

where

$$\begin{aligned}
&\mu(\delta) = \sup\{|\varphi(\dot{x}(t_2)) - \varphi(\dot{x}(t_1))| \,:\, t_1, t_2 \in T,\ |t_2 - t_1| \le \delta\}, \\
&\psi_1^*(\tau_i,\tau_{i+1},z,\xi(\cdot)) = (\xi_{i+1} - \xi_i - \delta f(\tau_i), z - \xi_i) + \delta\langle A\xi_i, z - \xi_i\rangle \\
&\qquad + \delta\{\varphi(z) - \varphi(\xi_i)\},
\end{aligned}$$

$\xi(\cdot) \in \Xi_1^h(x(\cdot))$, $\Xi_1^h(x(\cdot))$ is the set of piecewise constant functions $\xi(t) \in V$, $t \in T$, $\xi(t) = \xi_i$, $t \in [\tau_i, \tau_{i+1})$, satisfying (40).

Suppose that in (15) $B = B(\dot{x})$ (or $B = B(x)$), $V = H_1^0(\Omega)$, $H = L_2(\Omega)$, $\Omega \subset \mathbf{R}^n$ is a bounded region with a smooth border, $U = L_2(\Omega; \mathbf{R}^{n\times n})$, a family $B(y)u : U \to V^*$ $(\forall y \in V)$ of operators is of the form

$$\langle B(y)u, z\rangle = \sum_{k,l=1}^{n} \int_{\Omega} u_{k,l}(\eta) y_{\eta_k}(\eta) z_{\eta_l}(\eta) d\eta \quad \forall y, z \in V, \quad u = \{(u_{k,l}(\eta))_{k,l=1}^n\},$$

$P \subset U$ is a convex bounded (in $L_\infty(\Omega; \mathbf{R}^{n\times n})$) and closed set. Theorem 3 is also true, if in definition $F_{i,\delta}^{(1)}$ we replace B by $B(v)$.

REMARK 2 The given case corresponds to the following problem. There is some dynamical system described by the variational inequality (15). Several leading coefficients of elliptic operator (they respond to item $A_1\dot{x}$) are known. The remaining coefficients $(u(t) = \{(u_{k,l}(t,\eta))_{k,l=1}^n\})$ are to be defined by use of approximate measurements of elements $\dot{x}(\tau_i)$. Namely, it is required to calculate some coefficients $u(t) = \{(u_{k,l}(t,\eta))_{k,l=1}^n\} \in P$ for a. a. $t \in T$ such that the relation (15) is true provided $Bu(t) = B(\dot{x}(t))u(t)$ for a. a. $t \in T$.

REMARK 3 Let $B = B(x)$ in the inequality (15). Theorem 3 is also true if in definition $F^{(1)}_{i,\delta}(u, \xi_i, \xi_{i-1}, \xi_{i-1} - z)$ we replace $\langle Bu, z\rangle$ by $\langle B(x_0 + \delta \sum_{j=1}^{i-1} \xi_j), \xi_{i-1} - z\rangle$

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