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Dynamic programming equation for a class of nonlinear boundary control problems of parabolic type

by

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Abstract. The paper is concerned with the Hamilton–Jacobi–Bellman equation of an infinite horizon optimal control problem for a class of evolution equations in an Hilbert space. Such systems include semilinear parabolic equations under various kinds of boundary conditions.

Keywords: boundary control, viscosity solutions, Hamilton–Jacobi equation, parabolic equations.

1. Introduction

In this paper we are concerned with the existence and uniqueness of viscosity solutions to the infinite dimensional Hamilton–Jacobi equation

$$\lambda v(x) + H(A^{\beta+\mu}x, Dv(x)) + \langle Ax + A^{-\beta}F(A^{\beta+\mu}x), Dv(x)\rangle = 0 \tag{1}$$

where X is a real Hilbert space, $\lambda > 0$ and $H : X \times X \to \mathbb{R}$ is continuous. Moreover, $A : D(A) \subset X \to X$ is a closed linear operator with a compact and dense inclusion $D(A) \subset X$. Also, we assume A to be positive and self–adjoint. We denote by A^{β} the fractional power of A and we assume $\beta, \mu > 0$ and $\beta + \mu < 1$. Finally $F : X \to X$ is Lipschitz continuous.

In order to explain our interest in the above equation, let us consider the problem of minimizing the functional

$$J(x_0;\gamma) = \int_0^{\infty} e^{-\lambda t} L(A^{\mu}x(t), \gamma(t))dt, \qquad \lambda > 0, \tag{2}$$

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over all trajectory–control pairs (x,γ), $\gamma : [0,\infty) \to \tilde{U}$, subject to the state equation

$$\begin{cases} x'(t) + Ax(t) + F(A^{\mu}x(t)) = A^{\beta}B\gamma(t) \\ x(0) = x_0. \end{cases} \tag{3}$$

Here U is a real Hilbert space, $\tilde{U} \subset U$ is closed and bounded and $B : U \to D(A^{\rho})$ is a bounded linear operator for some $\rho > 0$. For any control γ and initial state x_0 we denote by $x(\cdot; x_0, \gamma)$ the unique mild solution in $L^1(0,T; D(A^{\eta}))$ to problem (3), see Proposition 2.1.

If we denote by u the value function of problem (3)–(2), that is

$$u(x_0) = \inf\left\{ J(x_0,\gamma) \,\middle|\, \gamma : [0,+\infty) \to \tilde{U} \text{ measurable} \right\}, \tag{4}$$

then u is a candidate solution to the dynamic programming equation

$$\lambda u(x) + H(A^{\mu}x, A^{\beta}Du(x)) + \langle Ax + F(A^{\mu}x), Du(x)\rangle = 0. \tag{5}$$

where

$$H(x,p) = \sup_{\gamma\in\tilde{U}} \left[- < B\gamma, p > -L(x,\gamma) \right]. \tag{6}$$

Now, the change of variable $u(x) = v(A^{-\beta}x)$ transforms equation (5) into equation (1). Therefore, u is uniquely determined once v has been characterized. For this purpose we are interested in proving that equation (1) admits a unique solution.

Hamilton–Jacobi equations in infinite dimensions were first studied by Barbu and Da Prato (1982), in convex classes, and then by Crandall and Lions (1985, 1986) using the viscosity solution approach. Additional contributions to the viscosity solution method were obtained by Soner (1988), Ishii (1992) and Tataru (1992a, 1992b). On the other hand, the results proved in these papers apply to equation (1) only in the case of $\beta = \mu = 0$.

An existence and uniqueness result for equation (5) with $\mu = 0$ and $\beta \in (\frac{1}{4}, \frac{1}{2})$ was obtained in Cannarsa, Gozzi and Soner (1993), adapting the viscosity solution approach. In Cannarsa and Tessitore (to appear) we studied equation (1) for $\beta \in (0,1)$ and $\mu = 0$.

In the present paper we extend the method of Cannarsa and Tessitore (to appear) to the case $\beta, \mu > 0$ under the assumption $\beta + \mu < 1$.

It is well known that equation (3) is a possible abstract formulation for modelling parabolic systems controlled at the boundary. In particular, taking $\mu = \frac{1}{2}$ and $\beta \in (\frac{1}{4}, \frac{1}{2})$ the above equation can be used to study the Neumann boundary control problem

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi) + f(\xi, x(t,\xi), \nabla_\xi x(t,\xi)) & \text{in } (0,\infty)\times\Omega \\ x(0,\xi) = x_0(\xi) & \text{on } \Omega \\ \dfrac{\partial x}{\partial \nu}(t,\xi) = \gamma(t,\xi) & \text{on } (0,\infty)\times\partial\Omega \end{cases} \tag{7}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary.

Finally, we give a brief outline the paper. In §2 we state the main assumptions on the data and recall the basic material on boundary control problems. In §3 we define viscosity solutions and we derive a comparison result which implies a uniqueness theorem. We note that this result is obtained for a more general class of equations than (1). In §4 we show an existence and uniqueness result for (1), proving that the value function v is a viscosity solution of such an equation.

2. Preliminaries

Let X and U be two real Hilbert spaces and let $\tilde{U} \subset U$ be closed and bounded. We set $R = \sup_{\gamma \in \tilde{U}} |\gamma|$. Let $x_0 \in X$ and let $\gamma : [0,\infty) \to \tilde{U}$ be a measurable function. We are concerned with the controlled system

$$\begin{cases} x'(t) + Ax(t) + F(A^\mu x(t)) = A^\beta B\gamma(t) \\ x(0) = x_0 \end{cases} \tag{8}$$

where

(i) $A : D(A) \subset X \to X$ is a closed linear operator such that $A = A^*$ and $< Ax, x > \geq \omega |x|^2$ for some $\omega > 0$ and all $x \in D(A)$;

(ii) the inclusion $D(A) \subset X$ is dense and compact;

(iii) $F : X \to X$, $|F(x) - F(y)| \leq K_F|x - y|$, $|F(x)| \leq C_F \quad \forall x, y \in X$; (9)

(iv) $\beta > 0$, $\mu > 0$ and $\beta_\mu := \beta + \mu \in (0,1)$;

(v) there exists $\rho > 0$, such that $B \in \mathcal{L}(U, D(A^\rho))$.

for some constants K_F, $C_F > 0$.

We note that (i) and (ii) imply that $-A$ is the infinitesimal generator of an analytic semigroup satisfying $||e^{-tA}|| \leq e^{-\omega t}$ for some $\omega > 0$ and all $t \geq 0$. Hence, fractional powers A^α, $\alpha \in \mathbb{R}$, are well defined, see e.g. Pazy (1983). Moreover for every $\alpha \in [0,1]$ there exists a constant $M_\alpha > 0$ such that

$$|A^\alpha e^{-tA} x| \leq \frac{M_\alpha}{t^\alpha}|x|, \quad \forall t > 0, \forall x \in X. \tag{10}$$

Let $\gamma \in (0,1]$ and $\alpha \in (0,\gamma)$. Then, a well known interpolation inequality, see e.g. Pazy (1983), states that for every $\sigma > 0$ there exists $C_\sigma > 0$ such that

$$|A^\alpha x| \leq \sigma |A^\gamma x| + C_\sigma |x|, \quad \forall x \in D(A^\gamma) \tag{11}$$

and there exists $C_{\alpha\gamma} > 0$ such that

$$|A^\alpha x| \le C_{\alpha\gamma} |A^\gamma x|^{\frac{\alpha}{\gamma}} |x|^{1-\frac{\alpha}{\gamma}}, \quad \forall x \in D(A^\gamma). \tag{12}$$

In assumption (v) above, we have denoted by $\mathcal{L}(U, D(A^\rho))$ the Banach space of all bounded linear operators $B : U \to D(A^\rho)$, where $D(A^\rho)$ is equipped with the graph norm. We have the following existence and uniqueness result on the solution to problem (8).

PROPOSITION 2.1 *Assume that (9) holds. Let $\gamma : [0, \infty) \to U$ be a bounded measurable control and fix $T > 0$. Then for any $x_0 \in X$ and for for any η such that $\mu \le \eta < 1 - \beta$, there exists a unique mild solution of* (8)*, denoted by $x(\cdot; x_0, \gamma)$, such that*

$$x \in L^1(0, T; D(A^\eta)). \tag{13}$$

Proof. We recall that x is a mild solution of (8) if

$$x(t) = e^{-tA} x_0 - \int_0^t e^{-(t-s)A} F(A^\mu x(s)) ds + A^\beta \int_0^t e^{-(t-s)A} B\gamma(s) ds\ , \tag{14}$$

for a. e. $t \ge 0$. Now we define the map Φ on $L^1(0, T; D(A^\eta))$ by

$$\Phi x(t) = e^{-tA} x_0 - \int_0^t e^{-(t-s)A} F(A^\mu x(s)) ds + A^\beta \int_0^t e^{-(t-s)A} B\gamma(s) ds$$

for any $0 \le t \le T$. Let us prove that

$$\Phi : L^1(0, T; D(A^\eta)) \to L^1(0, T; D(A^\eta))\ .$$

Indeed, recalling (10), we have

$$\int_0^T |A^\eta \Phi x(t)| dt \le \int_0^T |A^\eta e^{-tA} x_0| dt$$

$$+ \int_0^T \left| A^\eta \int_0^t e^{-(t-s)A} F(A^\mu x(s)) ds \right| dt + \int_0^T \left| A^{\eta+\beta} \int_0^t e^{-(t-s)A} B\gamma(s) ds \right| dt$$

$$\le M_\eta \int_0^T \frac{|x_0|}{t^\eta} dt + \int_0^T \int_0^t \frac{C_F M_\eta}{(t-s)^\eta} ds dt + M_{\beta,\eta} \int_0^T \int_0^t \frac{|B\gamma(s)|}{(t-s)^{\eta+\beta}} ds dt$$

$$\le M_\eta |x_0| T^{1-\eta} + C_F M_\eta T^{1-\eta} + M_{\beta,\eta} R ||B|| T^{1-(\beta+\eta)},$$

recalling that $|\gamma(s)| \le R$. Hence $\Phi x \in L^1(0, T; D(A^\eta))$.

Next we prove that Φ is a contraction if T is sufficiently small. For any $x, z \in L^1(0,T;D(A^\eta))$ we have

$$\int_0^T |A^\eta(\Phi x(s) - \Phi z(s))|\, ds \le K_F M_\eta \int_0^T \int_0^t \frac{|A^\mu(x(s)-z(s))|}{(t-s)^\eta} ds dt$$

$$= K_F M_\eta \int_0^T |A^\mu(x(s)-z(s))|\, ds \int_s^T \frac{1}{(t-s)^\eta} dt$$

$$\le K_F M_\eta T^{1-\eta} \|x-z\|_{L^1(0,T;D(A^\mu))} \le K_F M_\eta T^{1-\eta} \|x-z\|_{L^1(0,T;D(A^\eta))}.$$

By the Contraction Map Theorem it follows that equation (20) has a unique solution $x \in L^1(0,T;D(A^\eta))$. The conclusion for general T follows repeating the previous argument. □

Now let us assume $\theta \in [0,1)$ and consider the problem of minimizing the functional

$$J(x_0;\gamma) = \int_0^\infty e^{-\lambda t} L(A^\theta x(t;x_0,\gamma),\gamma(t))dt \tag{15}$$

over all measurable functions $\gamma : [0,\infty) \to \tilde{U}$ (usually called controls). The discount factor λ in (15) is positive and L satisfies the following assumptions

$$\begin{aligned} &\text{(i)}\ \ L \in C(X\times\tilde{U}),\ \ |L(x,\gamma)| \le C_L\ ,\ \forall (x,\gamma)\in X\times\tilde{U};\\ &\text{(ii)}\ \ |L(x,\gamma)-L(y,\gamma)| \le K_L|x-y|,\ \forall\gamma\in\tilde{U},\ x,y\in X;\\ &\text{(iii)}\ \ \beta_\theta := \beta+\theta \in (0,1),\ \theta \ge \mu, \end{aligned} \tag{16}$$

for some $C_L > 0$ and $K_L > 0$. We define the value function of problem (15)–(8) as

$$u(x_0) = \inf\left\{ \int_0^\infty e^{-\lambda t} L(A^\theta x(t;x_0,\gamma),\gamma(t))dt \,\middle|\, \gamma : [0,+\infty) \to \tilde{U} \text{ measurable} \right\} \tag{17}$$

We transform (8) by the change of variable

$$y(t) = A^{-\beta} x(t). \tag{18}$$

More precisely, let $y_0 \in X$ and denote by $y(\cdot;y_0,\gamma)$ the solution of

$$\begin{cases} y'(t) + Ay(t) + A^{-\beta}F(A^{\beta_\mu}y(t)) = B\gamma(t) \\ y(0) = y_0 \in X \end{cases} \tag{19}$$

Again the above equation has to be understood in mild form

$$y(t) = e^{-tA}y_0 - A^{-\beta}\int_0^t e^{-(t-s)A}F(A^{\beta_\mu}y(s))ds + \int_0^t e^{-(t-s)A}B\gamma(s)ds\ . \tag{20}$$

The existence and uniqueness of the solution to (20) is guarantee by the following result, see Cannarsa and Tessitore (to appear).

PROPOSITION 2.2 *Assume that (9) holds. Let $\gamma : [0,\infty) \to U$ be a bounded measurable control and fix $T > 0$. Then for any $y_0 \in X$ there exists a unique solution*

$$y \in C([0,T];X) \cap L^1(0,T;D(A^{\beta_\mu})). \tag{21}$$

Moreover, if $y_0 \in D(A^{\frac{1}{2}})$, then

$$y \in C([0,T];D(A^{\frac{1}{2}})) \cap L^2(0,T;D(A)) \cap W^{1,2}(0,T;X). \tag{22}$$

Finally, if $y_0 \in D(A)$, then

$$y \in C([0,T];D(A)). \tag{23}$$

By inserting the change of variable (18) in the cost functional (15), we obtain a new optimal control problem whose value function v is given by

$$v(y_0) = \inf_{\gamma(t)\in\tilde{U}} \int_0^\infty e^{-\lambda t} L(A^{\beta_\theta} y(t;y_0,\gamma),\gamma(t))dt. \tag{24}$$

It is easy to realize that value functions v and u are related by the formula

$$u(x) = v(A^{-\beta}x)\ ,\ \forall x \in X. \tag{25}$$

In particular, u is uniquely determined once v has been characterized. Therefore, we will study problem (19)–(24) instead of (8)–(17).

We will show that if $\theta = \mu$, then v is the unique solution of the following Hamilton–Jacobi–Bellman equation

$$\lambda v(x) + H(A^{\beta_\mu}x, Dv(x)) + \langle Ax + A^{-\beta}F(A^{\beta_\mu}x), Dv(x)\rangle = 0 \tag{26}$$

where

$$H(x,p) = \sup_{\gamma\in\tilde{U}} \left[- < B\gamma, p > - L(x,\gamma) \right]. \tag{27}$$

Clearly, one needs a suitable notion of weak solution of problem (26), since v is not everywhere differentiable and the coefficients of the equation are discontinuous. In the sequel, we use viscosity solutions to overcome these difficulties.

3. Definition of viscosity solution and comparison result

In this Section we study the Hamilton–Jacobi equation

$$\lambda u(x) + H(A^{\beta_\theta}x, Du(x)) + \langle Ax + A^{-\beta}F(A^{\beta_\mu}x), Du(x)\rangle = 0. \tag{28}$$

We assume that (9) holds and that $H : X \times X \to \mathbb{R}$ is a function, not necessarily given by (27), satisfying

$$|H(x,p) - H(y,q)| \le K_H\left(|x-y| + |p-q|\right) \text{ for some } K_H > 0. \tag{29}$$

Let $w, \varphi : D(A^{\frac{1}{2}}) \to \mathbb{R}$ be given. For any $\delta > 0$ we define $M_\delta^+(w,\varphi)$ to be the set of all points $x \in D(A^{\frac{1}{2}})$ such that

$$w(x) - \varphi(x) - \frac{\delta}{2}|A^{\frac{1}{2}}x|^2 \geq w(y) - \varphi(y) - \frac{\delta}{2}|A^{\frac{1}{2}}y|^2 \tag{30}$$

for all $y \in D(A^{\frac{1}{2}})$. Similarly, we denote by $M_\delta^-(w,\varphi)$ the set of all points $x \in D(A^{\frac{1}{2}})$ such that

$$w(x) - \varphi(x) + \frac{\delta}{2}|A^{\frac{1}{2}}x|^2 \leq w(y) - \varphi(y) + \frac{\delta}{2}|A^{\frac{1}{2}}y|^2 \tag{31}$$

for all $y \in D(A^{\frac{1}{2}})$.

DEFINITION 3.1 *We say that a bounded continuous function $w : X \to \mathbb{R}$ is a viscosity subsolution of* (28) *if w is sequentially weakly upper semicontinuous, and, for every $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $\delta > 0$,*

(i) $M_\delta^+(w,\varphi) \subset D(A)$;

(ii) $$\lambda w(x) + H(A^{\beta_\theta}x, D\varphi(x) + \delta Ax) + \langle Ax + A^{-\beta}F(A^{\beta_\mu}x), D\varphi(x)\rangle + \delta|Ax|^2 + \delta\langle Ax, A^{-\beta}F(A^{\beta_\mu}x)\rangle \leq 0 \,, \forall x \in M_\delta^+(w,\varphi). \tag{32}$$

We say that w is a viscosity supersolution of (28) *if w is sequentially weakly lower semicontinuous, and, for every $\varphi \in C^1(D(A^{\frac{1}{2}}))$ and $\delta > 0$,*

(i) $M_\delta^-(w,\varphi) \subset D(A)$;

(ii) $$\lambda w(x) + H(A^{\beta_\theta}x, D\varphi(x) - \delta Ax) + \langle Ax + A^{-\beta}F(A^{\beta_\mu}x), D\varphi(x)\rangle - \delta|Ax|^2 - \delta\langle Ax, A^{-\beta}F(A^{\beta_\mu}x)\rangle \geq 0 \,, \forall x \in M_\delta^-(w,\varphi). \tag{33}$$

We say that w is a viscosity solution of (28) *if it is both a viscosity subsolution and a supersolution of* (28).

Now we give a comparison result between viscosity subsolutions and supersolutions of (28).

THEOREM 3.1 *Assume that (9) and* (29) *hold true, and define $\alpha_{\beta_\mu} \in (0,1)$ as*

$$\alpha_{\beta_\mu} = \frac{4\beta_\mu - 3}{2\beta_\mu - 1}. \tag{34}$$

Let u and v be a viscosity subsolution and supersolution of the Hamilton–Jacobi equation (28) *respectively. If u and v are Hölder continuous of exponent $\alpha > \alpha_{\beta_\mu}$, then*

$$u(x) \leq v(x), \quad \forall x \in X. \tag{35}$$

Proof. For simplicity we take $\lambda = 1$ and we consider $\beta_\mu \in \left(\frac{3}{4}, 1\right)$. If $\beta_\mu \in \left(0, \frac{3}{4}\right]$, the proof can be easily derived, adapting the same technique we use in the sequel.

For ε and δ positive, we define a function $\phi : D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \to \mathbb{R}$ as

$$\phi(x,y) = u(x) - v(y) - \frac{1}{2\varepsilon} < A^{\frac{1}{2}}(x-y), x-y > -\frac{\delta}{2}[< Ax, x > + < Ay, y >]. \quad (36)$$

Notice that ϕ is weakly upper–semicontinuous. Let $(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$ be such that

$$\phi(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) = \max_{D(A^{\frac{1}{2}})\times D(A^{\frac{1}{2}})} \phi(x,y) .$$

First of all we prove that

$$|A^{\frac{1}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \le C_1 \varepsilon^{\frac{1}{2-\alpha}} , \quad (37)$$

where $C_1 > 0$ and α is the Hölder exponent of u and v. Since

$$\phi(x_{\varepsilon,\delta}, x_{\varepsilon,\delta}) + \phi(y_{\varepsilon,\delta}, y_{\varepsilon,\delta}) \le 2\phi(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) ,$$

from the Hölder continuity of u and v we derive

$$\frac{1}{\varepsilon}|A^{\frac{1}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 \le C|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}|^\alpha, \quad (38)$$

for some positive constant C. Therefore (37) holds.

Now let us consider

$$\varphi(x) = v(y_{\varepsilon,\delta}) + \frac{1}{2\varepsilon} < A^{\frac{1}{2}}(x - y_{\varepsilon,\delta}), x - y_{\varepsilon,\delta} > + \frac{\delta}{2} < Ay_{\varepsilon,\delta}, y_{\varepsilon,\delta} >$$

$$\psi(y) = u(x_{\varepsilon,\delta}) - \frac{1}{2\varepsilon} < A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y), x_{\varepsilon,\delta} - y > - \frac{\delta}{2} < Ax_{\varepsilon,\delta}, x_{\varepsilon,\delta} >$$

Notice that $\varphi, \psi \in C^1(D(A^{\frac{1}{2}}))$. Also, $x_{\varepsilon,\delta} \in M_\delta^+(u,\varphi)$ and $y_{\varepsilon,\delta} \in M_\delta^-(v,\psi)$ by construction. Since u is a viscosity subsolution, using φ as a test function, we have

$$u(x_{\varepsilon,\delta}) + H\left(A^{\beta_\theta} x_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon} + \delta A x_{\varepsilon,\delta}\right) + \delta|Ax_{\varepsilon,\delta}|^2$$

$$+\delta\left\langle Ax_{\varepsilon,\delta}, A^{-\beta}F(A^{\beta_\mu}x_{\varepsilon,\delta})\right\rangle + \left\langle Ax_{\varepsilon,\delta} + A^{-\beta}F(A^{\beta_\mu}x_{\varepsilon,\delta}), \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon}\right\rangle \le 0 \quad (39)$$

Since v is a viscosity supersolution, using ψ as a test function, we have

$$v(y_{\varepsilon,\delta}) + H\left(A^{\beta_\theta} y_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon} - \delta A y_{\varepsilon,\delta}\right) - \delta|Ay_{\varepsilon,\delta}|^2$$

$$-\delta\left\langle Ay_{\varepsilon,\delta}, A^{-\beta}F(A^{\beta_\mu}y_{\varepsilon,\delta})\right\rangle + \left\langle Ay_{\varepsilon,\delta} + A^{-\beta}F(A^{\beta_\mu}y_{\varepsilon,\delta}), \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon}\right\rangle \ge 0 \quad (40)$$

Subtracting (40) from (39), we obtain

$$\begin{aligned}
&u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \delta\left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2\right] + \frac{1}{\varepsilon}|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 \\
&\le H\left(A^{\beta_\theta}y_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon} - \delta A y_{\varepsilon,\delta}\right) \\
&-H\left(A^{\beta_\theta}x_{\varepsilon,\delta}, \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon} + \delta A x_{\varepsilon,\delta}\right) \\
&-\delta\left[\langle Ax_{\varepsilon,\delta}, A^{-\beta}F(A^{\beta_\mu}x_{\varepsilon,\delta})\rangle + \langle Ay_{\varepsilon,\delta}, A^{-\beta}[F(A^{\beta_\mu}y_{\varepsilon,\delta})\rangle\right] \\
&+\left\langle A^{-\beta}\left[F(A^{\beta_\mu}y_{\varepsilon,\delta}) - F(A^{\beta_\mu}x_{\varepsilon,\delta})\right], \frac{A^{\frac{1}{2}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})}{\varepsilon}\right\rangle. \qquad (41)
\end{aligned}$$

Recalling assumption (29) on H and assumption (9) on F, the above inequality yields

$$\begin{aligned}
&u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \delta\left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2\right] + \frac{1}{\varepsilon}|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 \\
&\le K_H\delta\left[|Ax_{\varepsilon,\delta}| + |Ay_{\varepsilon,\delta}|\right] + K_H|A^{\beta_\theta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \\
&+\delta C_F||A^{-\beta}||[|Ax_{\varepsilon,\delta}| + |Ay_{\varepsilon,\delta}|] + K_F|A^{\beta_\mu}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|\frac{|A^{\frac{1}{2}-\beta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon}. \qquad (42)
\end{aligned}$$

Now we estimate the right hand side of (42). We derive

$$K_H\delta\left[|Ax_{\varepsilon,\delta}| + |Ay_{\varepsilon,\delta}|\right] \le \frac{\delta}{8}\left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2\right] + C_2\delta, \qquad (43)$$

where C_2 is a positive constant.

For the second term of (42) we have two cases. If $\beta_\theta \in (\frac{1}{4}, \frac{3}{4})$ then exploiting inequality (12) we obtain

$$K_H|A^{\beta_\theta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \le K_H|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \le \frac{|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2}{4\varepsilon} + K_H^2\varepsilon \qquad (44)$$

If $\beta_\theta \in (\frac{3}{4}, 1)$ then recalling the interpolation inequality (12), we get

$$K_H|A^{\beta_\theta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \le C_3|A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{4\beta_\theta - 3}|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{4-4\beta_\theta}, \qquad (45)$$

for some $C_3 > 0$. Moreover recall the following well known inequality

$$ab \le \frac{\sigma^p}{p}a^p + \frac{1}{\sigma^q q}b^q \qquad (46)$$

for every $a, b \in \mathbb{R}^+$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\sigma > 0$. Choosing $\sigma = \left(\frac{\delta}{16}\right)^{\frac{4\beta_\theta - 3}{2}}$ and $p = \frac{2}{4\beta_\theta - 3}$ in (46) and applying it to (45) we derive

$$C_3 |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{4\beta_\theta - 3} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{4-4\beta_\theta}$$

$$\leq \frac{\delta}{16} |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_4}{\delta^{\frac{4\beta_\theta - 3}{5-4\beta_\theta}}} \left| A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta}) \right|^{\frac{8-8\beta_\theta}{5-4\beta_\theta}}, \tag{47}$$

where C_4 is some positive constant. On the other hand, again applying (46) with $p = \frac{5 - 4\beta_\theta}{4 - 4\beta_\theta}$ and $\sigma = \left(\frac{1}{4\varepsilon}\right)^{\frac{4-4\beta_\theta}{5-4\beta_\theta}}$, we find

$$\frac{C_4}{\delta^{\frac{4\beta_\theta - 3}{5-4\beta_\theta}}} \left| A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta}) \right|^{\frac{8-8\beta_\theta}{5-4\beta_\theta}} \leq \frac{1}{4\varepsilon} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_5 \varepsilon^{4-4\beta_\theta}}{\delta^{4\beta_\theta - 3}}, \tag{48}$$

for $C_5 > 0$. From estimates (47) and (48), inequality (45) can be rewritten as

$$K_H |A^{\beta_\theta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \leq \frac{\delta}{8} \left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2 \right] + \frac{1}{4\varepsilon} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2$$
$$+ \frac{C_6 \varepsilon^{4-4\beta_\theta}}{\delta^{4\beta_\theta - 3}}. \tag{49}$$

On the other hand we get

$$\delta C_F ||A^{-\beta}|| [|Ax_{\varepsilon,\delta}| + |Ay_{\varepsilon,\delta}|] \leq \frac{\delta}{8} \left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2 \right] + C_7 \delta, \tag{50}$$

for $C_7 > 0$. Finally, from estimate (37), it follows that

$$K_F |A^{\beta_\mu}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \frac{|A^{\frac{1}{2} - \beta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon} \leq \frac{C_8 |A^{\beta_\mu}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}}, \tag{51}$$

where $C_8 > 0$. Applying the interpolation inequality (12) and inequality (46) to (51) as we did in (45) we find

$$\frac{C_8 |A^{\beta_\mu}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}} \leq \frac{\delta}{16} |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_9 |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^{\frac{8-8\beta_\mu}{5-4\beta_\mu}}}{\delta^{\frac{4\beta_\mu - 3}{5-4\beta_\mu}} \varepsilon^{\frac{2-2\alpha}{(2-\alpha)(5-4\beta_\mu)}}},$$

for some positive constant C_9. Again, using (46) in the last term of the above inequality we rewrite (51) as

$$K_F |A^{\beta_\mu}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})| \frac{|A^{\frac{1}{2} - \beta}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|}{\varepsilon}$$

$$\leq \frac{\delta}{16} |A(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{1}{4\varepsilon} |A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2 + \frac{C_{10} \varepsilon^{4-4\beta_\mu}}{\delta^{4\beta_\mu - 3} \varepsilon^{\frac{2-2\alpha}{2-\alpha}}}, \tag{52}$$

with C_{10} positive constant. Substituting estimates (43), (44) or (49), (50) and (52) in inequality (42) we get

$$u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \frac{\delta}{2}\left[|Ax_{\varepsilon,\delta}|^2 + |Ay_{\varepsilon,\delta}|^2\right] + \frac{1}{2\varepsilon}|A^{\frac{3}{4}}(x_{\varepsilon,\delta} - y_{\varepsilon,\delta})|^2$$
$$\leq C_{11}\delta + \frac{C_5\varepsilon^{4-4\beta_\theta}}{\delta^{4\beta_\theta-3}} + \frac{C_{10}\varepsilon^\gamma}{\delta^{4\beta_\mu-3}}\,, \tag{53}$$

where $C_{11} > 0$ and $\gamma = 4 - 4\beta_\mu - \dfrac{2-2\alpha}{2-\alpha}$ is positive as $\alpha > \alpha_{\beta_\mu}$. Therefore, if $x \in D(A^{\frac{1}{2}})$ we have

$$u(x) - v(x) = \phi(x,x) + \delta < Ax, x > \leq \phi(x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) + \delta < Ax, x >$$
$$\leq u(x_{\varepsilon,\delta}) - v(y_{\varepsilon,\delta}) + \delta < Ax, x > \leq C_{11}\delta + \frac{C_5\varepsilon^{4-4\beta_\theta}}{\delta^{4\beta_\theta-3}} + \frac{C_{10}\varepsilon^\gamma}{\delta^{4\beta_\mu-3}} + \delta < Ax, x > \ .$$

Letting $\varepsilon \to 0$ and then $\delta \to 0$ we conclude that

$$u(x) \leq v(x)\ ,\ \forall x \in D(A^{\frac{1}{2}})\ .$$

Since $D(A^{\frac{1}{2}})$ is dense in X, we have $u(x) \leq v(x)$ for every $x \in X$. □

4. Properties of the value function and existence result

In this Section we assume

$$\mu = \theta. \tag{54}$$

Using exactly the same arguments as in Cannarsa and Tessitore (to appear) one can show that the value function v of problem (19)–(24) is the unique viscosity solution of

$$\lambda v(x) + H(A^{\beta_\mu}x, Dv(x)) + \langle Ax + A^{-\beta}F(A^{\beta_\mu}x), Dv(x)\rangle = 0 \tag{55}$$

where $H(x,p) = \sup_{\gamma\in\tilde{U}}\left[- < B\gamma, p > -L(x,\gamma)\right]$.

The precise statements are given below and proofs are omitted.

PROPOSITION 4.1 *Assume (9), (16) and* (54). *Then the value function v defined in* (24) *is Hölder continuous in X with any exponent $\alpha \in (0,1]$ satisfying $\alpha < \frac{\lambda}{K_F}$. Moreover for any $\rho \in [0, 1-\beta_\mu)$ there exists a constant $C_{\alpha\rho} > 0$ such that*

$$|v(x) - v(y)| \leq C_{\alpha\rho}|A^{-\rho}(x-y)|^\alpha \qquad \textit{for all } x, y \in X. \tag{56}$$

THEOREM 4.1 *Assume that (9), (16) and* (54) *hold true. Then the value function v is a viscosity solution of* (26) *in the sense of Definition* 3.1.

Combining Theorem 4.1 with Theorem 3.1 we obtain the following existence and uniqueness result for the Hamilton–Jacobi equation (55).

COROLLARY 4.1 *Assume that (9), (16) and* (54) *hold true. Let* $\lambda_F = \min\left\{1, \dfrac{\lambda}{K_F}\right\}$ *and fix*

$$\beta_\mu \in \left(\frac{3}{4}, \frac{3-\lambda_F}{4-2\lambda_F}\right). \tag{57}$$

Then the value function v *defined in* (24) *is the unique viscosity solution of the Hamilton–Jacobi equation* (55) *satisfying a Hölder condition with exponent* $\alpha \in (\alpha_{\beta_\mu}, 1)$, *where* α_{β_μ} *is defined in* (34).

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