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On the design and control of systems governed by differential equations on submanifolds

by

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Abstract. Using ideas and constructions recently developed for intrinsic modelling of shells, we discuss their extension to submanifolds of codimension larger than one. Intrinsic modelling and differential calculus makes it possible to do Optimal Design, Shape Sensitivity Analysis and Control Theory of systems on submanifolds in much the same way as in the N-dimensional Euclidean space. To illustrate this point we compute the shape derivative of the compliance for the Laplace-Beltrami and the membrane equations by the Velocity Method. This method finds a surprising application in the Multiplier Method which gives the hidden boundary smoothness, and controllability results for second order hyperbolic systems. It extends to the wave equation for the Laplace-Beltrami, the membrane, and the shell operators where the form of the basic identity would be more difficult to guess.

Keywords: intrinsic differential operators, tangential calculus, Laplace-Beltrami, membrane, shell, distance function, hidden boundary smoothness, multiplier method, controllability

1. Introduction

This paper builds up on recent results which link the geometrical and smoothness properties of the boundary of a subset of $\mathbb{R}^N$ with the corresponding properties of its oriented distance function in a neighbourhood of the boundary. This has many interesting applications. It can be used to define intrinsic tangential derivatives which coincide with classical covariant derivatives (cf. Delfour and Zolésio, 1995 and the references therein). Classical models of membranes and shells can now be rewritten using intrinsic differential operators on the associated submanifold of $\mathbb{R}^N$ of codimension one. This technique is not limited to the codimension one and readily extends to codimensions higher than one.

The intrinsic formulations present definite advantages over the "parametrized ones" when it comes to the Shape Sensitivity Analysis. They go hand in hand with the Velocity Method (cf. Sokołowski and Zolésio, 1992) and theorems on the differentiation of a Min or a Min Max with respect to a parameter (cf. Delfour and Zolésio, 1988, 1991). We illustrate this point by computing the shape derivative of the compliance associated with the Laplace-Beltrami and the membrane equations. The same technique applies to various models of shells but the complexity of the expression increases with the complexity of the model.

Another unexpected but very interesting application of the Velocity Method and the oriented distance function arises in the Multiplier Method for finite time controllability of second order hyperbolic equations associated with an elliptic operator. More specifically the shape derivative of the elliptic part of the energy term naturally yields the basic identity (cf. for instance the recent book of Komornik, 1994, p. 20, Lemma 2.3) which is used to obtain the hidden boundary smoothness and the basic isomorphism. The equivalence between the smoothness properties of the boundary and the oriented distance function in a neighbourhood of the boundary provides the extension of the outward normal which is used as the "multiplier". But of course the beauty of this is that it extends to second order hyperbolic equations associated with elliptic operators defined on submanifolds. We illustrate this point on the "wave equation" associated with the Laplace-Beltrami and the membrane operator where the exact form of the identity would be difficult to guess. It also extends to various models of linear shells and further details will be provided in a subsequent paper.

2. Intrinsic differential equations on submanifolds

2.1. Tangential calculus on submanifolds

We have seen in Delfour and Zolésio (1995) how to construct tangential differential operators on a C^2-submanifold $\partial\Omega$ of $\mathbb{R}^N$ of codimension one and how they relate to classical covariant derivatives by composition with the projection operator onto $\partial\Omega$. In fact the same constructions extend to submanifolds of codimension larger than one. Consider sets Ω such that

$$b_\Omega(x) \geq 0 \quad \text{and} \quad \mathrm{m}\,(\partial\Omega) = 0.$$

By construction

$$d_\Omega(x) = b_\Omega^+(x) = b_\Omega(x) = |b_\Omega(x)| = d_{\partial\Omega}(x) \quad \Rightarrow \quad \partial\Omega = \overline{\Omega}$$

and since b_Ω is differentiable almost everywhere and $\mathrm{m}\,(\partial\Omega) = 0$

$$|\nabla b_\Omega(x)| = |\nabla d_{\partial\Omega}(x)| = 1.$$

Moreover the projection onto $\partial\Omega$ is still given by

$$p_{\partial\Omega}(x) = x - \frac{1}{2}\nabla b_\Omega^2(x) = x - \frac{1}{2}\nabla d_{\partial\Omega}^2(x).$$

Therefore instead of assuming that $b_\Omega \in C^2(U_h(\partial\Omega))$ for some $h > 0$ and

$U_h(\partial\Omega) = \{x \in \mathbb{R}^N \ : \ |b_\Omega(x)| = d_{\partial\Omega}(x) < h\}$

we assume that

$d^2_{\partial\Omega} = b^2_\Omega \in C^2(U_h(\partial\Omega))$.

Then the tangential gradient of a function w on Γ is defined as

$\nabla_\Gamma w \stackrel{\text{def}}{=} \nabla(w \circ p)|_\Gamma$

Finally since $\mathrm{m}\,(\partial\Omega) = 0$, $|\nabla b_\Omega(x)| = |\nabla d_{\partial\Omega}(x)| = 1$ a.e. and Federer's decomposition of the measure yields

$$\int_{U_h(\partial\Omega)} f\,dx = \int_0^h dz \int_{\partial\Omega_z} d\Gamma_z\, f, \quad \partial\Omega_z = \{x \in \mathbb{R}^N \ : \ b_\Omega(x) = z\}$$

where $\partial\Omega_z$ is still a submanifold of codimension $N-1$. Then the intrinsic tangential calculus is the same and coincides with covariant derivatives as was shown in Delfour and Zolésio (1995).

2.2. Laplace-Beltrami equation

Assume that the domain Γ is Lipschitzian in $\partial\Omega$ and let $y \in H^1_0(\Gamma)$ be the solution of the variational problem: for all $\varphi \in H^1_0(\Gamma)$

$$\int_\Gamma \nabla_\Gamma y \cdot \nabla_\Gamma \varphi\, d\Gamma = \int_\Gamma f\varphi\, d\Gamma$$
$$\Rightarrow \quad -\Delta_\Gamma y = -\mathrm{div}_\Gamma(\nabla_\Gamma y) = f \text{ in } \Gamma \quad y = 0 \text{ on } \partial\Gamma. \tag{1}$$

Let ν be the unit exterior normal to the relative boundary $\partial\Gamma$ of Γ in $\partial\Omega$ (that is $\nu \cdot \nabla b = 0$).

2.3. Membrane and shell equations

2.3.1. Membrane equation

There are several models for membranes which typically involve the tangential tensor $\varepsilon_\Gamma(u)$ of the displacement vector u of the membrane Γ. In some cases constraints are introduced such as the *inextensibility – incompressibility* of the membrane which translates into the condition $\mathrm{div}_\Gamma u = 0$ (cf. Clariond, 1993, Clariond and Zolésio, 1994). For our purpose we choose the simplest model which is characterized by the tangential equation: to find $u \in H^1_0(\Gamma)^N$ such that for all $v \in H^1_0(\Gamma)^N$

$$\int_\Gamma \varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v)\, d\Gamma = \int_\Gamma f \cdot v\, d\Gamma$$
$$\Rightarrow -\overrightarrow{\mathrm{div}}_\Gamma\, (\varepsilon_\Gamma(u)) + H\,\varepsilon_\Gamma(u)\,n = f \text{ in } \Gamma, \quad u = 0 \text{ on } \partial\Gamma, \tag{2}$$

where $n = \nabla b_\Omega$ is the outward unit normal to Ω on Γ.

2.3.2. Intrinsic shell equation

The intrinsic model described in Delfour and Zolésio (1995) is of the form

$$\tilde{\varepsilon}(u,\ell) = \varepsilon^0(u,\ell) + \varepsilon^1(u,\ell)\, z^1 + \varepsilon^2(u,\ell)\, z^2, \tag{3}$$

where

$$\begin{aligned} 2\,\varepsilon^0(u,\ell) &= 2\,\varepsilon_\Gamma(u) + \ell\ {}^*n + n\ {}^*\ell \\ 2\,\varepsilon^1(u,\ell) &= 2\,\varepsilon_\Gamma(\ell) - D_\Gamma(u)\, D^2b - D^2b\ {}^*D_\Gamma(u) \\ 2\,\varepsilon^2(u,\ell) &= [D_\Gamma(\ell) - D_\Gamma(u)\, D^2b](-D^2b) + (-D^2b)\,[{}^*D_\Gamma(\ell) - D^2b\,{}^*D_\Gamma(u)] \end{aligned} \tag{4}$$

with the simple rheological law

$$\sigma = 2\,\mu\,\tilde{\varepsilon} + \lambda\, \mathrm{tr}\, \tilde{\varepsilon} I, \quad \mu > 0,\ \lambda \geq 0.$$

It contains the basic elementary terms of the vectorial case which are present in both linear and nonlinear models.

2.3.3. Naghdi's linear model

The intrinsic reformulation of the variational Naghdi's model is

$$\begin{aligned} &\frac{h\,E}{1+\nu} \int_\Gamma \varepsilon^0(u, \ell - \nabla_\Gamma(u\cdot n)) \cdot\cdot\, \varepsilon^0(v, \bar{\ell} - \nabla_\Gamma(v\cdot n)) + \frac{h^2}{12}\, \varepsilon^1_t(u,\ell) \cdot\cdot\, \varepsilon^1_t(v,\bar{\ell}) \\ &\qquad + \frac{1}{1-\nu} \left\{ \mathrm{div}_\Gamma\, u\, \mathrm{div}_\Gamma\, v + \frac{h^2}{12}\, \mathrm{div}_\Gamma\, \ell\, \mathrm{div}_\Gamma\, \bar{\ell} \right\} d\Gamma \\ &= \int_\Gamma p\cdot v\, d\Gamma + \int_{\partial\Gamma} N\cdot v - M\cdot \bar{\ell}\, d\gamma, \end{aligned} \tag{5}$$

where

$$\varepsilon^0(u,\beta) \stackrel{\text{def}}{=} \varepsilon_\Gamma(u) + \frac{1}{2}\,[\beta\ {}^*n + n\ {}^*\beta], \tag{6}$$

$$\varepsilon^1_t(u,\ell) \stackrel{\text{def}}{=} \varepsilon_\Gamma(\ell) + \frac{1}{2}\,[D^2b\, D_\Gamma u + {}^*D_\Gamma u\, D^2b] - \frac{1}{2}\,[D^2b\,\ell\ {}^*n + n\ {}^*(D^2b\,\ell)].$$

2.3.4. Asymptotic membrane equation model

The intrinsic version of the model of Ciarlet and Sanchez-Palencia (1993) is

$$\begin{aligned} &\int_\Gamma 4\,\mu\,\varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\, n) \cdot\cdot\, \varepsilon^0(v, -2\,\varepsilon_\Gamma(v)\, n) \\ &\qquad + \frac{4\,\mu\,\lambda}{\lambda + 2\,\mu}\, \mathrm{tr}\, \varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\, n)\, \mathrm{tr}\, \varepsilon^0(v, -2\,\varepsilon_\Gamma(v)\, n)\, d\Gamma = \int_\Gamma f\cdot v\, d\Gamma. \end{aligned} \tag{7}$$

With the tangential operator $\varepsilon^P_\Gamma(u) = \varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\, n)$, it further simplifies

$$\int_\Gamma 4\mu\,\varepsilon^P_\Gamma(u) \cdot\cdot\, \varepsilon^P_\Gamma(v) + \frac{4\,\mu\ \lambda}{\lambda + 2\,\mu}\, \mathrm{tr}\, \varepsilon^P_\Gamma(u)\, \mathrm{tr}\, \varepsilon^P_\Gamma(v)\, d\Gamma = \int_\Gamma f\cdot v\, d\Gamma. \tag{8}$$

3. Application to shape sensitivity analysis

We compute the derivatives of the compliance for the Laplace-Beltrami and the membrane equations via the Velocity method and theorems on the derivative of a Min (cf. Delfour and Zolésio, 1998, 1991).

3.1. Basic shape tangential calculus

It is convenient to first summarize a number of basic identities which will be used in subsequent computations. Given a sufficiently smooth vector field $\{V(z)\}$ on $\mathbb{R}^N$, define the following transformation, $T_z(V)$, of $\mathbb{R}^N$

$$\frac{dx}{dz}(z) = V(z, x(z)), \quad x(0) = x, \quad T_z(V)(x) \stackrel{\text{def}}{=} x(z). \tag{9}$$

For a domain $\Omega \subset \mathbb{R}^N$ with a boundary $\partial\Omega$ which is a C^2 submanifold of codimension one, let $p(x)$ and $p_z(x)$ be the respective projections of x onto $\Gamma \stackrel{\text{def}}{=} \partial\Omega$ and $\Gamma_z \stackrel{\text{def}}{=} \partial\Omega_z = T_z(\partial\Omega)$. In particular $T_z \circ p = p_z \circ T_z \circ p$ and

$$\begin{aligned}\frac{d}{dz}(T_z \circ p) &= V(z) \circ (T_z \circ p) = V(z) \circ p_z \circ (T_z \circ p) \\ &= (V(z) \circ p_z) \circ (T_z \circ p)\end{aligned} \tag{10}$$

$$\frac{d}{dz} D(T_z \circ p) = D(V(z) \circ p_z) \circ (T_z \circ p)\, D(T_z \circ p). \tag{11}$$

By restriction to Γ

$$\begin{aligned}&\frac{d}{dz} D_\Gamma T_z = D_{\Gamma_z} V(z) \circ (T_z \circ p)\, D_\Gamma T_z \\ &\Rightarrow \quad \frac{d}{dz} D_\Gamma T_z|_{z=0} = D_\Gamma V(0)\, Dp = D_\Gamma V(0)\, P_{\partial\Omega} \quad (= D_\Gamma V(0) \ \text{ on } \ T_\Gamma p(x))\end{aligned} \tag{12}$$

where $P_{\partial\Omega}$ is the orthogonal projector onto the tangent plane $T_\Gamma p(x)$ to Γ in $p(x)$. If $M(DT_z)$ is the matrix of cofactors of DT_z the canonical density on Γ_z is given by

$$\begin{aligned}&w_z = \det M(DT_z) = \det DT_z \, \|{}^*DT_z^{-1} \nabla b_\Omega\| \\ &\Rightarrow \quad \frac{dw_z}{dz} = \operatorname{div}_\Gamma V(0)\, w_z, \quad w_0 = 1.\end{aligned} \tag{13}$$

It can also be seen as the determinant of the transformation $D_\Gamma T_z(x)$ between the tangent spaces $T_x\partial\Omega$ and $T_{T_z(x)}\partial\Omega_z$.

$$\begin{aligned}&\frac{d}{dz}(T_z \circ p) = V \circ (T_z \circ p) = V \circ p_z \circ T_z \circ p \\ &\quad \Rightarrow \frac{d}{dz} D(T_z \circ p) = D(V \circ p_z) \circ (T_z \circ p)\, D(T_z \circ p) \\ &\quad \Rightarrow \frac{d}{dz} D(T_z \circ p)|_{z=0} = D(V \circ p) \circ p\, Dp \quad \Rightarrow (D_\Gamma V \text{ on } \Gamma).\end{aligned}$$

Moreover $(T_z^{-1}\circ p_z)\circ(T_z\circ p)=p$

$$\frac{d}{dz}(T_z^{-1}\circ p_z)\circ(T_z\circ p)+D(T_z^{-1}\circ p_z)\circ(T_z\circ p)\,\frac{d}{dz}(T_z\circ p)=0 \tag{14}$$

$$\frac{d}{dz}(T_z^{-1}\circ p_z)|_{z=0}\circ p=-[Dp\,V]\circ p \quad \Rightarrow (-P_{\partial\Omega}\,V \text{ on } \Gamma) \tag{15}$$

$$D(T_z^{-1}\circ p_z)\circ(T_z\circ p)\,D(T_z\circ p)=Dp \tag{16}$$

$$\begin{aligned}&\frac{d}{dz}[D(T_z^{-1}\circ p_z)\circ(T_z\circ p)]|_{z=0}\\ &\quad=-D_pD(V\circ p)\circ p\Rightarrow(-P_{\partial\Omega}D_\Gamma V \text{ on } T_\Gamma p(x)).\end{aligned} \tag{17}$$

Finally with $b=b_\Omega$ and $n=\nabla b_\Omega=\nabla b$

$$p'\stackrel{\text{def}}{=}\frac{d}{dz}p_z|_{z=0}=\nabla b\cdot V\circ p\;\nabla b+b\,\nabla(\nabla b\cdot V\circ p)\Rightarrow p'|_\Gamma=(n\cdot V)\,n. \tag{18}$$

3.2. Laplace-Beltrami

It is the simplest example, but it contains all the elements and techniques for more complex equations. Let ω be a domain in Γ such that its relative boundary γ in Γ be sufficiently smooth. Let $\omega_z=T_z(\omega)$ and $\gamma_z=T_z(\gamma)$. Consider the compliance

$$J(\omega_z)=\min_{\varphi\in H_0^1(\omega_z)}E(\omega_z,\varphi)=E(\omega_z,y_z)$$

where y_z in $H_0^1(\omega_z)$ is the unique minimizing element of the energy functional

$$E(\omega_z,\varphi)=\int_{\omega_z}\frac{1}{2}|\nabla_{\Gamma_z}\varphi|^2-f\,\varphi\,d\Gamma_z,\Rightarrow\;-\Delta_{\Gamma_z}y_z=f \text{ in } \omega_z\;\; y_z=0 \text{ on } \gamma_z. \tag{19}$$

We can use a theorem on the derivative of a Min with respect to z by parametrizing the elements of the function space $H_0^1(\omega_z)$ by those of $H_0^1(\omega)$ (cf. Delfour and Zolésio, 1988, 1991)

$$\varphi\mapsto\varphi\circ T_z^{-1}\;:\;H_0^1(\omega)\to H_0^1(\omega_z).$$

Since $p\circ T_z^{-1}\circ p_z=T_z^{-1}\circ p_z$ we have

$$\nabla_{\Gamma_z}(\varphi\circ T_z^{-1})=\nabla(\varphi\circ T_z^{-1}\circ p_z)|_{\omega_z}=\nabla(\varphi\circ p\circ T_z^{-1}\circ p_z)|_{\omega_z}$$

and

$$\nabla(\varphi\circ p\circ T_z^{-1}\circ p_z)={}^*D(T_z^{-1}\circ p_z)\,\nabla(\varphi\circ p)\circ T_z^{-1}\circ p_z.$$

Therefore introduce the following new functional for $\varphi\in H_0^1(\omega)$

$$\begin{aligned}\tilde{E}(z,\varphi)&\stackrel{\text{def}}{=}E(\omega_z,\varphi\circ T_z^{-1})\\ &=\int_{\omega_z}\frac{1}{2}|\nabla(\varphi\circ p\circ T_z^{-1}\circ p_z)|^2-f\,(\varphi\circ p)\circ T_z^{-1}\,d\Gamma_z\end{aligned} \tag{20}$$

and after a change of variable

$$\tilde{E}(z,\varphi) = \int_\omega \left[\frac{1}{2}|{}^*D(T_z^{-1}\circ p_z)\nabla(\varphi\circ p)|^2 - (f\circ T_z)\,(\varphi\circ p)\right]\, w_z\, d\Gamma. \tag{21}$$

Then compute the "volume" and "boundary" expressions of the shape derivative

$$\partial_z\tilde{E}(0) \stackrel{\text{def}}{=} \frac{d}{dz}\tilde{E}(z,y_z)|_{z=0} = \frac{d}{dz}\min_{\varphi\in H_0^1(\omega)}\tilde{E}(z,\varphi)|_{z=0} = \frac{\partial}{\partial z}\tilde{E}(0,y) \tag{22}$$

since the minimizing element y is unique.

In order to compute the partial derivative of the two expressions (20) and (21) of $\tilde{E}(z,\varphi)$, we need the derivatives of the "transported" of $\varphi\circ p$

$$\varphi' \stackrel{\text{def}}{=} \frac{d}{dz}(\varphi\circ p\circ T_z^{-1})|_{z=0} = -\nabla(\varphi\circ p)\cdot V \Rightarrow \quad \varphi'|_\omega = -\nabla_\Gamma\varphi\cdot V \tag{23}$$

$$\Phi' \stackrel{\text{def}}{=} \frac{d}{dz}(\varphi\circ p\circ T_z^{-1}\circ p_z)|_{z=0} = \varphi'\circ p + \nabla(\varphi\circ p)\circ p\cdot p'. \tag{24}$$

It can be shown that

$$\Phi' = -\nabla(\varphi\circ p)\circ p\cdot P_{\partial\Omega}V\circ p + b_\Omega\,\nabla(\nabla b_\Omega\cdot V\circ p)\cdot\nabla(\varphi\circ p)\circ p$$

$$\Phi'|_\omega = \varphi'|_\omega = -\nabla_\Gamma\varphi\cdot V = -\nabla_\Gamma\varphi\cdot V_\Gamma$$

(V_Γ is the tangential component of V)

$$\begin{aligned}\nabla\Phi' &= -\nabla(\nabla(\varphi\circ p)\circ p\cdot V\circ p) + \nabla(\nabla(\varphi\circ p)\circ p\cdot\nabla b_\Omega\; V\circ p\cdot\nabla b_\Omega)\\ &\quad +\nabla b_\Omega\,\nabla(\nabla b_\Omega\cdot V\circ p)\cdot\nabla(\varphi\circ p)\circ p + b_\Omega\,\nabla(\nabla(\nabla b_\Omega\cdot V\circ p)\cdot\nabla(\varphi\circ p)\circ p)\\ \nabla\Phi'|_\omega &= -\nabla_\Gamma(\nabla_\Gamma\varphi\cdot V) + \nabla b_\Omega\;\nabla_\Gamma(\nabla b_\Omega\cdot V)\cdot\nabla_\Gamma\varphi\\ &\Rightarrow \quad \nabla_\Gamma\varphi\cdot\nabla\Phi'|_\omega = \nabla_\Gamma\varphi\cdot\nabla_\Gamma\varphi'.\end{aligned}$$

From (21) we get the "volume" expression of the derivative

$$\partial_z\tilde{E}(0,\varphi) = \int_\omega \frac{1}{2}\left[\operatorname{div}_\Gamma V\, I - 2\,\varepsilon_\Gamma V\right]\,\nabla_\Gamma\varphi\cdot\nabla_\Gamma\varphi - f\,\varphi\operatorname{div}_\Gamma V - \nabla f\cdot V\,\varphi\, d\Gamma, \tag{25}$$

where $2\varepsilon_\Gamma V = D_\Gamma V + {}^*D_\Gamma V$. The derivative of (20) is a little more complicated:

$$\begin{aligned}\partial_z\tilde{E}(0,\varphi) &= \int_\gamma \left\{\frac{1}{2}|\nabla_\Gamma\varphi|^2 - f\,\varphi\right\}\, V\cdot\nu\, d\gamma\\ &\quad -\int_\omega \nabla_\Gamma\varphi\cdot\nabla_\Gamma\,(\nabla_\Gamma\varphi\cdot V) - f\,\nabla_\Gamma\varphi\cdot V\, d\Gamma\\ &\quad +\int_\omega \left\{H\left[\frac{1}{2}|\nabla_\Gamma\varphi|^2 - f\,\varphi\right] + \nabla(\frac{1}{2}|\nabla_\Gamma\varphi|^2\circ p - f\,\varphi\circ p)\cdot\nabla b\right\}\, V\cdot n\, d\Gamma\\ &= \int_\gamma \left\{\frac{1}{2}|\nabla_\Gamma\varphi|^2 - f\,\varphi\right\}\, V\cdot\nu - \nabla_\Gamma\varphi\cdot\nu\,(\nabla_\Gamma\varphi\cdot V)\, d\gamma\end{aligned}$$

$$+\int_\omega (\Delta_\Gamma \varphi + f)\,(\nabla_\Gamma \varphi \cdot V)\, d\Gamma$$
$$+\int_\omega \left\{ H\left[\frac{1}{2}|\nabla_\Gamma \varphi|^2 - f\,\varphi\right] - \nabla f \cdot n\,\varphi \right\} V \cdot n\, d\Gamma. \tag{26}$$

For $\varphi = y$, $y = 0$ and $\nabla_\Gamma y = (\partial y / \partial \nu)\, \nu$ on γ and we get the "boundary" expression

$$dJ(\omega; V) = \int_\gamma -\frac{1}{2}\left|\frac{\partial y}{\partial \nu}\right|^2 V \cdot \nu\, d\gamma$$
$$+\int_\omega \left\{ H\left[\frac{1}{2}|\nabla_\Gamma y|^2 - f\,y\right] - \nabla f \cdot n\,u \right\} V \cdot n\, d\Gamma. \tag{27}$$

and from (25) we get the "volume" expression

$$dJ(\omega; V) = \int_\omega \frac{1}{2}\,[\mathrm{div}_\Gamma V\, I - 2\,\varepsilon_\Gamma V]\, \nabla_\Gamma y \cdot \nabla_\Gamma y - \nabla f \cdot V\, y - f\, y\, \mathrm{div}_\Gamma V\, d\Gamma. \tag{28}$$

The formulae are similar to the ones for the Laplacian and they coincide when the mean curvature H is zero. For more details including the shape and material derivatives of the state, the reader is referred to Desaint (1995), and Desaint and Zolésio (1995).

3.3. Membranes and shells

We concentrate on the membrane which is the prototype for the vectorial case. The shell is the same except that the complexity of the model is more important. Assume that ω and γ are as in the previous section and consider the membrane problem (2)

$$\exists u \in H^1_0(\omega)^N, \text{ such that } \forall v \in H^1_0(\omega)^N,\ \int_\omega \varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v)\, d\Gamma = \int_\omega f \cdot v\, d\Gamma \tag{29}$$

$$\Rightarrow -\overrightarrow{\mathrm{div}}_\Gamma\,(\varepsilon_\Gamma(u)) + H\,\varepsilon_\Gamma(u)\, n = f \text{ in } \omega, \quad u = 0 \text{ on } \gamma. \tag{30}$$

As for the Laplace-Beltrami operator consider the associated compliance. The solution u of (29) is the minimizing element of the energy functional

$$E(\omega, v) \stackrel{\mathrm{def}}{=} \int_\omega \frac{1}{2}\|\varepsilon_\Gamma v\|^2 - f \cdot v\, d\Gamma, \quad E(\omega, u) = \min_{v \in H^1_0(\omega)^N} E(\omega, v). \tag{31}$$

With a parametrization of the form $v \circ T_z^{-1}$ of the elements of $H^1_0(\omega_z)^N$ by elements $v \in H^1_0(\omega)^N$, the energy functional on ω_z becomes

$$\tilde{E}(z, v) \stackrel{\mathrm{def}}{=} E(\omega_z, v \circ T_z^{-1}) = \int_{\omega_z} \frac{1}{2}\|\varepsilon_\Gamma(v \circ T_z^{-1})\|^2 - f \cdot v \circ T_z^{-1}\, d\Gamma, \tag{32}$$

$$\tilde{E}(z, u_z) = \min_{v \in H^1_0(\omega)^N} \tilde{E}(z, v). \tag{33}$$

On ω_z we have

$$D_{\Gamma_z}(v \circ T_z^{-1}) = D(v \circ p \circ T_z^{-1} \circ p_z) = D(v \circ p) \circ (T_z^{-1} \circ p_z)\, D(T_z^{-1} \circ p_z) \tag{34}$$

and we obtain the following two expressions for $\tilde{E}(z, v)$

$$= \int_{\omega_z} \frac{1}{2} \|\varepsilon(v \circ p \circ T_z^{-1} \circ p_z)\|^2 - f \cdot (v \circ p \circ T_z^{-1})\, d\Gamma \tag{35}$$

$$\begin{aligned} &= \int_{\omega} \frac{1}{8} \|D(v \circ p)\, D(T_z^{-1} \circ p_z) \circ (T_z \circ p) \\ &\quad +^* D(T_z^{-1} \circ p_z) \circ (T_z \circ p)\ {}^*D(v \circ p)\|^2\, w_z \\ &\quad -(f \circ T_z \circ p) \cdot (v \circ p)\, w_z\, d\Gamma. \end{aligned} \tag{36}$$

As for the Laplace-Beltrami operator, the shape derivative of the compliance is given by

$$\partial_z \tilde{E}(0) \stackrel{\text{def}}{=} \frac{d}{dz} \tilde{E}(z, u_z)|_{z=0} = \frac{d}{dz} \min_{w \in H_0^1(\omega)^N} \tilde{E}(z, w)|_{z=0} = \frac{\partial}{\partial z} \tilde{E}(0, u). \tag{37}$$

The derivatives of the extension of $w \circ p$ are similar to the ones of the scalar case.

$$\begin{aligned} &w' \stackrel{\text{def}}{=} \frac{d}{dz}(w \circ p \circ T_z^{-1})|_{z=0} = -D(w \circ p)\, V \quad \Rightarrow \quad w'|_\omega = -D_\Gamma w\, V \\ &W' \stackrel{\text{def}}{=} \frac{d}{dz}(w \circ p \circ T_z^{-1} \circ p_z)|_{z=0} = D(w \circ p)\, p' + w' \circ p \\ &\quad = -D(w \circ p)\, [V - V \cdot \nabla b_\Omega\, \nabla b_\Omega] + b_\Omega\, D(w \circ p)\, \nabla(\nabla b_\Omega \cdot V \circ p) \\ &W'|_\omega = -D_\Gamma w\, V_\Gamma = -D_\Gamma w\, V \\ &\quad (V_\Gamma \text{ is the tangential component of } V) \\ &DW' = -D(D(w \circ p)\, V \circ p) + D(V \cdot \nabla b_\Omega\ D(w \circ p)\, \nabla b_\Omega) \\ &\quad + b_\Omega\, D(D(w \circ p)\, \nabla(\nabla b_\Omega \cdot V \circ p)) + (D(w \circ p)\, \nabla(\nabla b_\Omega \cdot V \circ p))\ {}^*\nabla b_\Omega) \\ &DW'|_\omega = -D_\Gamma(D_\Gamma w\, V) + D_\Gamma w\, \nabla_\Gamma(V \cdot n)\ {}^*n. \end{aligned}$$

The derivative of (36) gives the easy "volume" expression

$$\begin{aligned} \partial_z \tilde{E}(0) = &\int_\omega \frac{1}{2} [\operatorname{div}_\Gamma V\, \varepsilon_\Gamma v - D_\Gamma v\, D_\Gamma V -^* D_\Gamma V\, {}^*D_\Gamma v] \cdot\cdot\, \varepsilon_\Gamma v \\ &-f \cdot v \operatorname{div}_\Gamma V - Df\, V \cdot v\, d\Gamma. \end{aligned} \tag{38}$$

As for the "boundary" expression the derivative of (35) first yields

$$\begin{aligned} \partial_z \tilde{E}(0) = &\int_\gamma \left\{ \frac{1}{2} \|\varepsilon_\Gamma v\|^2 - f \cdot v \right\} V \cdot \nu\, d\gamma \\ + &\int_\omega \left\{ H \left[\frac{1}{2} \|\varepsilon_\Gamma v\|^2 - f \cdot v \right] + \nabla \left[\frac{1}{2} \|\varepsilon_\Gamma v\|^2 \circ p - f \cdot v \circ p \right] \cdot n \right\} V \cdot n\, d\Gamma \end{aligned}$$

$$+\int_\omega \varepsilon_\Gamma v\cdot\cdot\frac{1}{2}(DV'+^* DV')-f\cdot v'\,d\Gamma$$
$$=\int_\gamma\left\{\frac{1}{2}\|\varepsilon_\Gamma v\|^2-f\cdot v\right\}V\cdot\nu\,d\gamma$$
$$+\int_\omega\left\{H\left[\frac{1}{2}\|\varepsilon_\Gamma v\|^2-f\cdot v\right]-Df\,n\cdot v\right\}V\cdot n\,d\Gamma$$
$$+\int_\omega \varepsilon_\Gamma v\cdot\cdot\varepsilon_\Gamma(-D_\Gamma v\,V)+\varepsilon_\Gamma(v)\,n\cdot D_\Gamma v\,\nabla_\Gamma(V\cdot n)-f\cdot(-D_\Gamma v\,V)\,d\Gamma$$
$$=\int_\gamma\left[\frac{1}{2}\|\varepsilon_\Gamma v\|^2-f\cdot v\right]V\cdot\nu-\varepsilon_\Gamma v\,\nu\cdot D_\Gamma v\,V+\varepsilon_\Gamma(v)\,n\cdot\,D_\Gamma(v)\,\nu\,(V\cdot n)\,d\gamma$$
$$+\int_\omega\left\{H\left[\frac{1}{2}\|\varepsilon_\Gamma v\|^2-f\cdot v\right]-Df\,n\cdot v-\mathrm{div}_\Gamma\left[{}^*D_\Gamma(v)\,\varepsilon_\Gamma(v)\,n\right]\right\}V\cdot n\,d\Gamma$$
$$+\int_\omega\left[\overrightarrow{\mathrm{div}}_\Gamma\,(\varepsilon_\Gamma v)-H\,\varepsilon_\Gamma(v)\,n+f\right]\cdot D_\Gamma v\,V\,d\Gamma. \tag{39}$$

Then we set $v=u$ and since $u=0$ on γ

$$D_\Gamma u=D_\Gamma u\,\nu\,{}^*\nu\quad\Rightarrow\quad {}^*D_\Gamma u\,\nu=(D_\Gamma u\,\nu\cdot\nu)\,\nu \tag{40}$$
$$\|\varepsilon_\Gamma(u)\|^2\,(V\cdot\nu)=\varepsilon_\Gamma(u)\,\nu\cdot D_\Gamma(u)\,V$$
$$=\frac{1}{2}\left[|D_\Gamma(u)\,\nu|^2+|D_\Gamma(u)\,\nu\cdot\nu|^2\right]\,V\cdot\nu \tag{41}$$
$$\|\varepsilon_\Gamma(u)\,\nu\|^2=\frac{1}{4}\left[|D_\Gamma(u)\,\nu|^2+3\,|D_\Gamma(u)\,\nu\cdot\nu|^2\right] \tag{42}$$
$$\Rightarrow\quad |D_\Gamma(u)\,\nu|^2+|D_\Gamma(u)\,\nu\cdot\nu|^2$$
$$=\|\varepsilon_\Gamma(u)\,\nu\|^2+\frac{1}{4}\,|D_\Gamma(u)\,\nu\cdot\nu|^2+\frac{3}{4}|D_\Gamma(u)\,\nu|^2. \tag{43}$$

In the vectorial case there are several ways to rearrange the above terms. This will be used in Section 6 to get the hidden boundary smoothness of the respective intrinsic vectors and quantity $D_\Gamma(u)\,\nu$, ${}^*D_\Gamma(u)\,\nu$, and $D_\Gamma(u)\,\nu\cdot\nu$. Coming back to the "boundary" expression

$$dJ(\omega;V)=\int_\gamma\frac{1}{4}\left[|D_\Gamma(u)\,\nu|^2+|D_\Gamma(u)\,\nu\cdot\nu|^2\right]\,V\cdot\nu$$
$$+\frac{1}{2}\nabla_\Gamma(u\cdot n)\cdot\,D_\Gamma(u)\,\nu\,(V\cdot n)\,d\gamma$$
$$+\int_\omega\left\{H\left[\frac{1}{2}\|\varepsilon_\Gamma u\|^2-f\cdot u\right]-Df\,n\cdot u-\mathrm{div}_\Gamma\left[{}^*D_\Gamma(u)\,\varepsilon_\Gamma(u)\,n\right]\right\}\,V\cdot n\,d\Gamma. \tag{44}$$

As for the "volume" expression from (38)

$$dJ(\omega;V)=\int_\omega\frac{1}{2}\left[\mathrm{div}_\Gamma V\,\varepsilon_\Gamma u-D_\Gamma u\,D_\Gamma V-^*D_\Gamma V\,{}^*D_\Gamma u\right]\cdot\cdot\,\varepsilon_\Gamma u$$
$$-f\cdot u\,\mathrm{div}_\Gamma V-Df\,V\cdot u\,d\Gamma \tag{45}$$

4. Hidden boundary smoothness via shape derivatives

In this section we show how the Velocity Method can be used to obtain the fundamental identity (cf. for instance Komornik, 1994, p. 20, Lemma 2.3 and its references) which is the key to obtain the hidden boundary smoothness, observability and controllability of the wave equation by the Multiplier Method. This technique naturally extends to the tangential wave equation and to equations of linear vibrations of membranes and shells. More details can be found in Delfour and Zolésio (1995-5).

4.1. Boundary smoothness of the solution of the Laplace equation revisited

Consider an open bounded Lipschitzian domain Ω of $\mathbb{R}^N$ and $V \in W^{1,\infty}(N(\Omega))^N$ for some bounded neighbourhood $N(\Omega)$ of Ω. Let $T_z = T_z(V)$ be the transformation associated with the velocity field V as in (9). Given $\varphi \in H^1(\Omega)$ consider the $L^2(\Omega_z)$-norm of the gradient of its "transported" $\varphi \circ T_z^{-1}(V)$ onto $\Omega_z = T_z(\Omega)$

$$E_z(V,\varphi) = \int_{\Omega_z} |\nabla(\varphi \circ T_z^{-1}(V))|^2 \, dx. \tag{46}$$

After a change of variable this functional can be expressed as an integral over Ω

$$E_z(V,\varphi) = \int_{\Omega} |{}^*D(T_z^{-1}(V)) \, \nabla\varphi|^2 \, \det(T_z(V)) \, dx. \tag{47}$$

Now the technique consists in computing the Shape derivative

$$E(V,\varphi) \stackrel{\text{def}}{=} \frac{d}{dz} E_z(V,\varphi)|_{z=0} \tag{48}$$

from the two expressions (46) and (47) of $E_z(V,\varphi)$. By equating the resulting "boundary" and "volume" expressions we get the basic identity used in the Multiplier Method where V is the vector of multipliers. Since this identity will then be used to "extract" the hidden smoothness of the solution on the boundary, we shall refer to $E(V,\varphi)$ as the "Extractor". It can be modified to suit the problem at hand and will usually coincide with the quadratic term of the natural static energy of the problem. The following theorem yields an elliptic boundary smoothness and also an hyperbolic boundary smoothness which will be discussed in the next section.

THEOREM 4.1 *Given $V \in W^{1,\infty}(N(\Omega))^N$, and $\varphi \in H^1(\Omega)$, we have*

$$E(V,\varphi) = \int_{\Omega} [\operatorname{div}(V) \, I - 2\,\varepsilon(V)] \, \nabla\varphi \cdot \nabla\varphi \, dx, \tag{49}$$

where $2\varepsilon(V) = DV + {}^*DV$. *Moreover if* $\nabla\varphi \cdot V \in H^1(\Omega)$ *or* $\Delta\varphi \in L^2(\Omega)$, *we have*

$$E(V,\varphi) = \int_{\partial\Omega} |\nabla\varphi|^2\, V\cdot n\, d\gamma - 2\int_\Omega \nabla\varphi\cdot\nabla(\nabla\varphi\cdot V)\,dx$$

$$= \int_{\partial\Omega} |\nabla\varphi|^2\, V\cdot n - 2\frac{\partial\varphi}{\partial n}\nabla\varphi\cdot V\,d\gamma + 2\int_\Omega \Delta\varphi\,\nabla\varphi\cdot V\,dx. \tag{50}$$

If $\partial\Omega$ *is* $C^{1,1}$, *there exists a neighbourhood* $U(\partial\Omega)$ *of* $\partial\Omega$ *where*

$b_\Omega \in W^{1,\infty}(U(\partial\Omega))$,

and if, in addition, $\nabla\varphi\cdot\nabla b_\Omega \in H^1(\Omega\cap U(\partial\Omega))$, *then there exists a constant* $c \geq 0$ *such that*

$$\int_{\partial\Omega} |\nabla\varphi|^2\,d\gamma \leq c\left[\|\nabla\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\cdot\nabla b_\Omega\|^2_{H^1(\Omega\cap U(\partial\Omega))}\right]. \tag{51}$$

In particular this is true when $\Delta\varphi \in L^2(\Omega)$.

Proof. By standard techniques from Sokołowski and Zolésio (1992) for shape derivatives and from Delfour and Zolésio (1994) for the fact that $b_\Omega \in W^{2,\infty}(U(\partial\Omega))$. □

COROLLARY 4.1 *(i) For* $\varphi \in H^1_0(\Omega)$, $\nabla\varphi\cdot V \in H^1(\Omega)$ *or* $\Delta\varphi \in L^2(\Omega)$, *and* $\partial\Omega$ *of class* $C^{1,1}$

$$\int_{\partial\Omega} \left|\frac{\partial\varphi}{\partial n}\right|^2 V\cdot n\,d\gamma$$

$$= \int_\Omega [\operatorname{div}(V)\,I - 2\varepsilon(V)]\;\nabla\varphi\cdot\nabla\varphi + 2\,\nabla\varphi\cdot\nabla(\nabla\varphi\cdot V)\,dx \tag{52}$$

$$= \int_\Omega -[\operatorname{div}(V)\,I - 2\varepsilon(V)]\;\nabla\varphi\cdot\nabla\varphi + 2\,\Delta\varphi\,(\nabla\varphi\cdot V)\,dx.$$

If $\Delta\varphi\cdot\delta b_\Omega \in H^1(\Omega\cap U(\partial\Omega))$, *there exists a constant* $c > 0$ *such that*

$$\left\|\frac{\partial\varphi}{\partial n}\right\|^2_{L^2(\partial\Omega)} \leq c\left[\|\nabla\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\cdot\nabla b_\Omega\|^2_{H^1(\Omega\cap U(\partial\Omega))}\right]. \tag{53}$$

(ii) For $\varphi \in H^1(\Omega)$ *such that* $\partial\varphi/\partial n = 0$ *and* $\nabla\varphi\cdot\delta b_\Omega \in H^1(\Omega\cap U(\partial\Omega))$, *the boundary term in (51) becomes and*

$$\int_{\partial\Omega} |\nabla_\Gamma\varphi|^2\,d\gamma \leq c\left[\|\nabla\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\cdot\nabla b_\Omega\|^2_{H^1(\Omega\cap U(\partial\Omega))}\right]. \tag{54}$$

In particular, for $\varphi \in H^1(\Omega)$ *such that* $\Delta\varphi \in L^2(\Omega)$ *and* $\partial\varphi/\partial n = 0$, *and* $\partial\Omega$ *of class* $C^{1,1}$

$$\int_{\partial\Omega} |\nabla_\Gamma\varphi|^2\,d\gamma \leq \left[\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2\right]. \tag{55}$$

Recall that $\partial\varphi/\partial n$ is defined as the element of $H^{-\frac{1}{2}}(\Gamma)$ verifying:

$$\forall\mu \in H^1(\Omega),\quad <\frac{\partial\varphi}{\partial n},\mu> = \int_\Omega \nabla\varphi\cdot\nabla\mu + \Delta\varphi\,\mu\,dx.$$

4.2. Multiplier method for the wave equation

Now consider a time dependent function $\varphi(t)$ and use identities (49)-(50) to recover the central identity in the Multiplier Method.

THEOREM 4.2 *Let $V \in W^{1,\infty}(N(\Omega))^N$ in some neighbourhood $N(\Omega)$ of Ω and $\tau > 0$.*

(i) Let $\varphi \in C(0,\tau;H^1(\Omega)) \cap C^1(0,\tau;L^2(\Omega))$ such that

$$\varphi_{tt} - \Delta\varphi \in L^2(Q), \quad Q = \Omega\times\,]0,\tau[, \quad \Sigma = \partial\Omega\times\,]0,\tau[. \tag{56}$$

Then

$$\int_\Sigma 2\,\frac{\partial\varphi}{\partial n}\,\nabla\varphi\cdot V + ((\varphi_t)^2 - |\nabla\varphi|^2)\,V\cdot n\,d\Sigma$$
$$= \int_Q \varphi_t^2\,\mathrm{div}\,V - [\mathrm{div}\,(V)\,I - 2\,\varepsilon(V)]\;\nabla\varphi\cdot\nabla\varphi$$
$$+2\,(\Delta\varphi - \varphi_{tt})\,\nabla\varphi\cdot V\,dx\,dt$$
$$+2\int_\Omega \varphi_t(\tau)\,\nabla\varphi(\tau)\cdot V - \varphi_t(0)\,\nabla\varphi(0)\cdot V\,dx. \tag{57}$$

(ii) If, in addition, $\partial\Omega$ is $C^{1,1}$ and $\varphi \in C(0,\tau;H_0^1(\Omega))$, there exist $c > 0$ and a neighbourhood $U(\partial\Omega)$ of $\partial\Omega$ such that

$$\left\|\frac{\partial\varphi}{\partial n}\right\|^2_{L^2(\Sigma)} \le c\,\left[\|\varphi_t\|^2_{C(0,\tau;L^2(\Omega))} + \|\nabla\varphi\|^2_{C(0,\tau;L^2(\Omega))} + \|\Delta\varphi - \varphi_{tt}\|^2_{L^2(Q)}\right]. \tag{58}$$

This is the first step of the Multiplier Method. The other steps remain unchanged.

5. Wave equation for the Laplace-Beltrami operator

We use the same notation, definitions and assumptions as in sections 3.1 and 3.2. Consider the wave equation for the Laplace-Beltrami operator on the global cylindrical evolution domain $Q_0 = \Gamma\times\,]0,\tau[$. Further assume that ω is a smooth open subset in Γ with sufficiently smooth relative boundary γ in Γ. Define $Q = \omega\times\,]0,\tau[\, \subset Q_0$ and its lateral boundary $\Sigma = \gamma\times\,]0,\tau[$. The field V is again an element of $W^{1,\infty}(N(\Omega))^N$ for some bounded neighbourhood $N(\Omega)$ of Ω.

5.1. Boundary smoothness for the Laplace-Beltrami operator

From this point on we proceed as in the previous section. Define the "tangential extractor"

$$\mathcal{E}(V,\varphi) = \frac{\partial}{\partial z}\mathcal{E}_z(V,\varphi)|_{z=0},$$
$$\mathcal{E}_z(V,\varphi) = \int_{\omega_z(V)} |\nabla_{\Gamma_z}(\varphi\circ T_z(V)^{-1})|^2 d\Gamma, \quad \varphi \in H_0^1(\omega). \tag{59}$$

In view of section 3.2 this is twice the quadratic part of the energy functional $\tilde{E}(z,\varphi)$ in (21) and we can use our previous computations by setting $f=0$ in (25) and (26) and multiplying by 2.

THEOREM 5.1 *Given $V \in W^{1,\infty}(N(\Omega))^N$ and $\varphi \in H^1(\omega)$, we have*

$$\mathcal{E}(V,\varphi)=\int_\omega [\operatorname{div}_\Gamma(V)\, I-2\,\varepsilon_\Gamma(V)]\;\nabla_\Gamma\varphi\cdot\nabla_\Gamma\varphi\, d\Gamma, \tag{60}$$

where $2\varepsilon_\Gamma(V)=D_\Gamma V+{}^*D_\Gamma V$. Moreover if $\varphi\in H^2(\omega)$,

$$\begin{aligned}\mathcal{E}(V,\varphi)&=\int_\gamma |\nabla_\Gamma\varphi|^2\, V\cdot\nu\, d\gamma+\int_\omega H\,|\nabla_\Gamma\varphi|^2\, V\cdot n-2\,\nabla_\Gamma\varphi\cdot\nabla_\Gamma(\nabla_\Gamma\varphi\cdot V)\, d\Gamma\\ &=\int_\gamma |\nabla_\Gamma\varphi|^2\, V\cdot\nu-2\,\nabla_\Gamma\varphi\cdot\nu\,(\nabla_\Gamma\varphi\cdot V)\, d\gamma\\ &+\int_\omega H\,|\nabla_\Gamma\varphi|^2\, V\cdot n+2\,\Delta_\Gamma\varphi\,\nabla_\Gamma\varphi\cdot V\, d\Gamma.\end{aligned}$$

If, in addition, $\varphi\in H^2(\omega)\cap H^1_0(\omega)$, then

$$\begin{aligned}&\int_\gamma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 V\cdot\nu\, d\gamma\\ &=\int_\omega [\operatorname{div}_\Gamma(V)\, I-2\,\varepsilon_\Gamma(V)]\;\nabla_\Gamma\varphi\cdot\nabla_\Gamma\varphi+2\nabla_\Gamma\varphi\cdot\nabla_\Gamma(\nabla_\Gamma\varphi\cdot V)\\ &\qquad -H\,|\nabla_\Gamma\varphi|^2\, V\cdot n\, d\Gamma\\ &=\int_\omega -[\operatorname{div}_\Gamma(V)\, I-2\varepsilon_\Gamma(V)]\;\nabla_\Gamma\varphi\cdot\nabla_\Gamma\varphi+H\,|\nabla_\Gamma\varphi|^2\, V\cdot n\\ &\qquad +2\,\Delta_\Gamma\varphi\,(\nabla_\Gamma\varphi\cdot V)\, d\Gamma\end{aligned} \tag{61}$$

5.2. Tangential wave equation

THEOREM 5.2 *Let $V\in W^{1,\infty}(N(\Omega))^N$ in some neighbourhood $N(\Omega)$ of Ω and $\tau>0$ be a real number. Let $\varphi\in C(0,\tau;H^1(\omega))\cap C^1(0,\tau;L^2(\omega))$ such that*

$$\varphi_{tt}-\Delta_\Gamma\varphi\in L^2(Q),\qquad Q=\omega\times\,]0,\tau[,\qquad \Sigma=\gamma\times\,]0,\tau[. \tag{62}$$

Then

$$\begin{aligned}&\int_\Sigma 2\,\frac{\partial\varphi}{\partial\nu}\,\nabla_\Gamma\varphi\cdot V+((\varphi_t)^2-|\nabla_\Gamma\varphi|^2)\, V\cdot\nu\, d\Sigma\\ &=\int_Q \varphi_t^2\operatorname{div}_\Gamma V-[\operatorname{div}_\Gamma(V)\, I-2\,\varepsilon_\Gamma(V)]\;\nabla_\Gamma\varphi\cdot\nabla_\Gamma\varphi\\ &\qquad +2\,(\Delta_\Gamma\varphi-\varphi_{tt})\nabla_\Gamma\varphi\cdot V\, d\Gamma\, dt\end{aligned}$$

$$+\int_Q H\left(|\nabla_\Gamma \varphi|^2 - \varphi_t^2\right) V \cdot n \, d\Gamma \, dt$$

$$+2\int_\omega \varphi_t(\tau)\, \nabla_\Gamma \varphi(\tau) \cdot V - \varphi_t(0)\, \nabla_\Gamma \varphi(0) \cdot V \, d\Gamma$$

and for $\varphi \in C(0,\tau; H_0^1(\omega))$ *the left-hand side of the above identity reduces to*

$$\int_\Sigma |\partial\varphi/\partial\nu|^2 \, V \cdot \nu \, d\Sigma, \quad \partial\varphi/\partial\nu \stackrel{\text{def}}{=} \nabla_\Gamma \varphi \cdot \nu.$$

In order to conclude as for the wave equation it must be shown that for a given smoothness of the relative boundary γ (for instance $C^{1,1}$), there exists a vector field V with support in a neighbourhood of γ such that $V \cdot \nu = 1$ on γ. In section 4.1. we chose $V = \nabla b_\Omega \, \psi$ for some $\psi \in \mathcal{D}(U(\Gamma))$ for a neighbourhood $U(\Gamma)$ of Γ where $b_\Omega \in C^{1,1}(U(\Gamma)) = W^{2,\infty}(U(\Gamma))$. An adaptation of this technique is available in Defour and Zolésio (1995-5).

6. Wave equations for membranes and shells

Consider now the tangential vector case. For simplicity we limit our analysis to membranes. The technique readily carries over to various linear and nonlinear models of membranes and shells with an unavoidable increase in the number of terms.

6.1. Boundary smoothness for the membrane

Γ, ω, γ and V are as in the previous section. Define the "vectorial tangential extractor"

$$\mathcal{E}(V,u) = \frac{\partial}{\partial z}\mathcal{E}_z(V,u)|_{z=0} \quad \mathcal{E}_z(V,u) = \int_{\omega_z(V)} \|\varepsilon_{\Gamma_z}(u \circ T_z^{-1}(V))\|^2 \, d\Gamma,$$
$$u \in H_0^1(\omega)^N. \tag{63}$$

In view of section 3.3. this is equal to twice the quadratic term in expression (32) of the energy functional $\tilde{E}(z,u)$ with $f = 0$. So we can use expressions (35) and (36) and the respective "volume" (38) and "boundary" (39) expressions of their derivatives along with the identities (41) and (43) to compute the term on γ for $u \in H_0^1(\omega)^N \cap H^2(\omega)^N$.

THEOREM 6.1 *Given* $V \in W^{1,\infty}(N(\Omega))^N$ *and* $u \in H^1(\omega)^N$, *we have*

$$\mathcal{E}(V,u) = \int_\omega [\operatorname{div}_\Gamma V \, \varepsilon_\Gamma u - D_\Gamma u \, D_\Gamma V -^* D_\Gamma V \, ^*D_\Gamma u] \cdot\cdot \, \varepsilon_\Gamma u \, d\Gamma. \tag{64}$$

Moreover if $u \in H^2(\omega)^N$, $\mathcal{E}(V,u)$ *is equal to*

$$\int_\gamma \|\varepsilon_\Gamma u\|^2 \, V \cdot \nu \, d\gamma$$

$$+\int_\omega H\,\|\varepsilon_\Gamma u\|^2\,V\cdot n - 2\varepsilon_\Gamma u\cdot\cdot\,\varepsilon_\Gamma(D_\Gamma u\,V) + 2\varepsilon_\Gamma u\,n\cdot D_\Gamma u\,\nabla_\Gamma(V\cdot n)\,d\Gamma$$

$$= \int_\gamma \|\varepsilon_\Gamma u\|^2\,V\cdot\nu - 2\,\varepsilon_\Gamma u\,\nu\cdot D_\Gamma u\,V + 2\,\varepsilon_\Gamma u\,n\cdot D_\Gamma u\,\nu\,(V\cdot n)\,d\gamma$$

$$+\int_\omega 2\left[\overrightarrow{\mathrm{div}}_\Gamma\,(\varepsilon_\Gamma u) - H\,\varepsilon_\Gamma u\,n\right]\cdot D_\Gamma u\,V\,d\Gamma$$

$$+\int_\omega H\,\|\varepsilon_\Gamma u\|^2\,V\cdot n - 2\,\mathrm{div}_\Gamma\,[{}^*D_\Gamma u\,\varepsilon_\Gamma u\,n]\,V\cdot n\,d\Gamma.$$

If $u\in H^2(\omega)^N\cap H^1_0(\omega)^N$, *then*

$$\int_\gamma \frac{1}{2}\left\{\|\varepsilon_\Gamma(u)\,\nu\|^2 + \frac{1}{4}\,|D_\Gamma(u)\,\nu\cdot\nu|^2 + \frac{3}{4}|D_\Gamma(u)\,\nu|^2\right\}\,V\cdot\nu\,d\gamma$$

$$= \int_\omega [\mathrm{div}_\Gamma\,V\,\varepsilon_\Gamma u - D_\Gamma u\,D_\Gamma V -^*D_\Gamma V\,{}^*D_\Gamma u]\cdot\cdot\,\varepsilon_\Gamma u\,d\Gamma$$

$$-\int_\omega H\,\|\varepsilon_\Gamma u\|^2\,V\cdot n - 2\,\varepsilon_\Gamma u\cdot\cdot\,\varepsilon_\Gamma(D_\Gamma u\,V) + 2\,\varepsilon_\Gamma u\,n\cdot D_\Gamma u\,\nabla_\Gamma(V\cdot n)\,d\Gamma$$

$$= -\int_\omega [\mathrm{div}_\Gamma\,V\,\varepsilon_\Gamma u - D_\Gamma u\,D_\Gamma V -^*D_\Gamma V\,{}^*D_\Gamma u]\cdot\cdot\,\varepsilon_\Gamma u\,d\Gamma$$

$$+\int_\omega H\,\|\varepsilon_\Gamma u\|^2\,V\cdot n + 2\,\varepsilon_\Gamma u\,n\cdot D_\Gamma u\,\nabla_\Gamma(V\cdot n)$$

$$+2\left[\overrightarrow{\mathrm{div}}_\Gamma\,(\varepsilon_\Gamma u) - H\,\varepsilon_\Gamma u\,n\right]\cdot D_\Gamma u\,V\,d\Gamma.$$

6.2. The wave equation for the membrane

THEOREM 6.2 *Let* $V\in W^{1,\infty}(N(\Omega))^N$ *in some neighbourhood* $N(\Omega)$ *of* Ω *and* $\tau>0$ *be a real number. Let* $u\in C(0,\tau;H^1(\omega))^N\cap C^1(0,\tau;L^2(\omega))^N$ *such that*

$$u_{tt} - \overrightarrow{\mathrm{div}}_\Gamma\,(\varepsilon_\Gamma u) + H\,\varepsilon_\Gamma u\,n\in L^2(Q),\quad Q=\omega\times\,]0,\tau[,\quad \Sigma=\gamma\times\,]0,\tau[. \tag{65}$$

Then

$$\int_\Sigma 2\,\varepsilon_\Gamma u\,\nu\cdot D_\Gamma u\,V + (|u_t|^2 - \|\varepsilon_\Gamma u\|^2)\,V\cdot\nu\,d\Sigma$$

$$= \int_Q |u_t|^2\,\mathrm{div}_\Gamma\,V - [\mathrm{div}_\Gamma\,V\,\varepsilon_\Gamma u - D_\Gamma u\,D_\Gamma V -^*D_\Gamma V\,{}^*D_\Gamma u]\cdot\cdot\,\varepsilon_\Gamma u\,d\Gamma\,dt$$

$$+\int_Q H\,(\|\varepsilon_\Gamma u\|^2 - |u_t|^2).V\cdot n + 2\,\varepsilon_\Gamma u\,n\cdot D_\Gamma u\,\nabla_\Gamma(V\cdot n)$$

$$+2\left[\overrightarrow{\mathrm{div}}_\Gamma\,(\varepsilon_\Gamma u) - H\,\varepsilon_\Gamma u\,n - u_{tt}\right]\cdot D_\Gamma u\,V\,d\Gamma\,dt$$

$$+2\int_\omega u_t(\tau)\,D_\Gamma u(\tau)\,V - u_t(0)\,D_\Gamma u(0)\,V\,d\Gamma$$

and for $u \in C(0,\tau;H_0^1(\omega))^N$ *the left-hand side reduces to*

$$\int_\Sigma \frac{1}{2}\left\{\|\varepsilon_\Gamma(u)\,\nu\|^2 + \frac{1}{4}\,|D_\Gamma(u)\,\nu\cdot\nu|^2 + \frac{3}{4}|D_\Gamma(u)\,\nu|^2\right\}\,V\cdot\nu\,d\Sigma.$$

Again, to conclude on the hidden boundary smoothness, we need the technique developed in Delfour and Zolésio (1995-5).

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