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# Proper periodic and quasi-periodic control for semi-linear hyperbolic systems[1]

by

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**Abstract.** Continuously operated semi-linear hyperbolic systems endowed with boundary lumped and spatially distributed controls are considered. The question of improving a (locally) optimal time-invariant process by time-periodic or time-quasi-periodic operation is investigated. Sufficient conditions guaranteeing the existence of improving time-dependent control are derived with the help of second-order necessary optimality conditions in Banach spaces applied to an abstract interpretation of the problem exploiting trigonometric semi-groups of bounded linear operators.

**Keywords**: semi-linear hyperbolic system, proper periodic control, sufficient condition, semi-group approach, abstract optimization.

## 1. Introduction

A suitably chosen periodic control may improve the averaged performance of certain industrial production systems and environmental engineering systems as compared to their performance for (locally) optimal steady-state control. Useful criteria guaranteeing the existence of such a beneficial control (called proper periodic control) have been obtained for lumped parameter systems on the basis of second-order necessary optimality conditions (Bittanti et al., 1973; Bernstein, 1985; Colonius, 1988). These criteria, taking the form of the so-called $\pi$-tests exploiting weak harmonic control variations yield important information about a nearly optimal operation period and an advantageous configuration of control variables. They can also be used for the determination of a range of process parameters, for which a proper periodic control exists.

A concept of time periodic control is applicable for many distributed-parameter processes such as the conversion of chemical substances in tubular apparatus,

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the growth of age-structured populations or spatially interacting species, etc. (Douglas, 1972; Carlson, Haurie, 1987; Metz, Diekmann, 1986; Styczeń, Nitka-Styczeń, 1994). The diversity of potential formulations of time-periodic control problems is here wide and the question of existence of a proper time-periodic control is investigated to a lesser degree (Grabmuller et al., 1985; Colonius, 1987).

We consider the discussed problem for a class of semi-linear hyperbolic systems endowed with boundary lumped and spatially distributed controls. The dynamics of such systems is governed by first-order partial differential equations of the form

$$x_t(t,z) + vx_z(t,z) = f(x(t,z), w(t,z), z), \quad (t,z) \in [0,\tau] \times [0,1]. \tag{1}$$

$$x(t,0) = Bu(t) + b, \quad t \in [0,\tau], \tag{2}$$

$$x(0,z) = x(\tau,z), \quad z \in [0,1], \tag{3}$$

where $x(t,z)$ is the state at time $t$ and the point $z$, $w(t,z)$ is the spatial control at time $t$ and the point $z$, $u(t)$ is the boundary control at time $t$. The equation (1) is referred to piston flow processes, the equation (2) is the boundary condition, and the equation (3) mirrors the time-periodicity condition. These equations may depict, for example, the dynamics of chemical reactions performed in tubular reactors, where the control $u$ is the inlet concentration and the control $w$ is the temperature distribution, or the dynamics of age-structured populations, where the control $u$ is the birth rate and the control $w$ is the harvesting effort.

We connect with the process (1)–(3) the vector of characteristics

$$y = \int_0^1 \frac{1}{\tau} \int_0^\tau g(x(t,z), w(t,z), z)dtdz + \frac{1}{\tau} \int_0^\tau h(u(t))dt \tag{4}$$

describing such quantities as the average consumption of raw material and energy, the average production of a desired component, the average content of a waste component etc. The aim of the process optimization may be now formulated as the minimization of the scalar goal function $\phi_0(y)$ subject to the scalar constraints $\phi_p(y) = 0,\ p = 1, ..., q$.

Such a general form of the objective function and the constraints allows us to take into account the so-called ratio-integral performance indices important for many chemical processes. Another reason motivating the above objective function is its polyoptimal interpretation as the deviation from the desired point (or the utopia point), i.e.

$$\phi_0(y) = \mid y - y^0 \mid,$$

where the vector $y^0$ is obtained by the solution of several scalar optimization problems with particular characteristics (4) as objective functions.

Since we hope to obtain conditions involving the existence of a proper time-periodic control by the use of second-order optimality conditions, we need to formulate our problem in a form containing sufficiently smooth functions in appropriately chosen spaces. One of possible approaches to this question is to rewrite the equations (1)–(3) as the following abstract differential equation in the space of time-periodic functions $\mathcal{B}$:

$$x'(z) + Ax(z) = f(x(z), w(z), z), \quad z \in [0,1], \tag{5}$$

$$x(0) = Bu + b \tag{6}$$

where $A$ is the unbounded operator of differentiation of time-periodic functions. Since the operator $-A$ is known to be the generator of the trigonometric semi-group of bounded linear operators (Butzer, Berens, 1967), the semi-group treatment of the equations (5),(6) may be applied to our aim, because it ensures a sufficiently smooth formulation of the problem under discussion.

This approach can be generalized to the case of time-quasi-periodic controls by the consideration of the equations (5),(6) in the space of such functions.

We use the following notation:

$I_a = [0, a]$, $I = I_1$, $I_\infty = [0, \infty)$ – the intervals on the real axis,

$R^n$ the $n$-dimensional Euclidean space,

$C(I; \mathcal{X})$ the space of continuous functions with values in a Banach space $\mathcal{X}$,

$L^\infty(I; \mathcal{X})$ the space of essentially bounded functions with values in a Banach space $\mathcal{X}$,

$C_\tau^{\rho,n}$ the space of $\tau$-periodic $n$-dimensional $\rho$-times continuously differentiable functions,

$L_\tau^{2,n}$ the space of $\tau$-periodic $n$-dimensional square-integrable functions,

$C_\omega^{\rho,n}$ the space of quasi-periodic $n$-dimensional $\rho$-times continuously differentiable functions with the frequency basis $\omega \in R^\gamma$,

$\alpha_\tau, \alpha$ the time-averaging functionals defined as

$$\alpha_\tau : C_\tau^{0,n} \to R^r, \quad \alpha_\tau \circ \chi = \frac{1}{\tau}\int_0^\tau \chi(t)dt \quad \text{for} \quad \chi \in C_\tau^{0,n},$$

$$\alpha : C_\omega^{0,n} \to R^r, \quad \alpha \circ \chi = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \chi(t)dt \quad \text{for} \quad \chi \in C_\omega^{0,n}.$$

Functions differentiable in the Frechet sense are called $F$-differentiable, and integrals of function-valued mappings are understood in the Bochner sense.

## 2. Proper periodic control

The arguments discussed in Introduction motivate the formulation of the optimal time-periodic control (OTPC) problem for semi-linear hyperbolic systems as the following non-periodic abstract control problem: minimize the objective function

$$J_\tau(y, x, u, w) = \phi_0(y) \tag{7}$$

subject to the constraints

$$\phi(y) = 0, \tag{8}$$

$$y = \int_0^1 \alpha_\tau \circ (g(x(z), w(z), z)\, dz + \alpha_\tau \circ (h(u)), \tag{9}$$

$$x(z) = T(z)(Bu + b) + \int_0^z T(z-\zeta) f(x(\zeta), w(\zeta), \zeta) d\zeta, \quad z \in I, \tag{10}$$

$$\big(u(t), w(z)(t)\big) \in U \times W, \quad z \in I, \ t \in I_\tau \tag{11}$$

where $\tau$ is the operation period treated as a process parameter,

$$x \in \mathcal{X}_\tau = C(I; C_\tau^{0,n})$$

is the abstract trajectory of the process state,

$$u \in \mathcal{U}_\tau = C_\tau^{0,l}, \quad w \in \mathcal{W}_\tau = L^\infty(I; C_\tau^{0,m})$$

are the inlet and spatial controls, respectively,

$$\phi_0 : R^r \to R, \quad \phi : R^r \to R^q, \quad h : C_\tau^{0,l} \to C_\tau^{0,r}$$
$$g : C_\tau^{0,n} \times C_\tau^{0,m} \times R \to C_\tau^{0,r}, \quad f : C_\tau^{0,n} \times C_\tau^{0,m} \times R \to C_\tau^{0,n},$$

$B$ is an $n \times m$ real matrix, $b \in R^n$, $U$ and $W$ are parallelepipeds in $R^l$ and $R^m$, respectively, and $T(z)$ is the strongly continuous trigonometric semi-group of bounded linear operators in the space $C_\tau^{0,n}$ (Butzer, Berens, 1967) generated by the linear unbounded operator of differentiation

$$A = -\frac{1}{v} D_t : C_\tau^{0,n} \to C_\tau^{0,n}, \ \mathbf{D}(A) = C_\tau^{1,n},$$

and defined by the formula

$$[T(z)\chi](t) = \sum_{\kappa=-\infty}^{\infty} \exp(-j\kappa \frac{\omega}{v} z) \chi_\kappa^\wedge \exp(j\kappa\omega t) \ \text{ for } \ \chi \in C_\tau^{0,n}, \tag{12}$$

which may also be characterized as the semi-group of right translations

$$[T(z)\chi](t) = \chi(t - \frac{z}{v}). \tag{13}$$

The objective function (7) and the constraints (8) are defined on the process characteristics (9) written with the help of the functional $\alpha_\tau$ averaging function-valued mappings $g$ and $h$. The equation (10) is the semi-group version of the initial value abstract differential equation (5),(6). The inclusion (11) depicts the pointwise constraints imposed on the control variables.

The choice $\mathcal{B} = C_\tau^{0,n}$ involves the property necessary for further analysis, namely F-differentiability of non-linear functions $f$, $g$, and $h$ as set against the more general case $\mathcal{B} = L_\tau^{2,n}$, for which this property may be violated (Krasnoselskii et al., 1966). This choice is suitable for our aim, because we intend to work with time-harmonic variations of the control variables.

The optimal time-steady control (OTSC) problem consists here in minimizing the objective function

$$J(y, x, u, w) = \phi_0(y) \tag{14}$$

subject to

$$\phi(y) = 0, \tag{15}$$

$$y = \int_0^1 g(x(z), w(z), z)dz + h(u), \tag{16}$$

$$x(z) = Bu + b + \int_0^z f(z(\zeta), w(\zeta), \zeta)d\zeta, \quad z \in I, \tag{17}$$

$$(u, w(z)) \in U \times W, \quad z \in I, \tag{18}$$

where

$$x \in \mathcal{X} = C(I; R^n), \quad u \in \mathcal{U} = R^l, \quad w \in \mathcal{W} = L^\infty(I; R^m)$$

are the time-steady state and controls, respectively. The process equation (10) reduces to the form (17), since the semi-group $T(z)$ does not change time-constant functions.

Let us denote by $S_\tau(S)$ the set of all admissible solutions of the OTPC (OTSC) problem, i.e. quadruples $s = (y, x, u, w) \in R^r \times \mathcal{X}_\tau \times \mathcal{U}_\tau \times \mathcal{W}_\tau$ ($R^r \times \mathcal{X} \times \mathcal{U} \times \mathcal{W}$) satisfying the constraints (8)–(11) ((15)–(18)). Let $\bar{s}$ be a locally optimal solution of the OTSC problem, i.e.

$$J(\bar{s}) \leq J(s) \text{ for all } s \in S \cup N(\bar{s}),$$

where $N(\bar{s})$ is a certain neighbourhood of $\bar{s}$ induced by the norm topology.

Speaking about improving time-periodic processes, called traditionally proper, we use the following alternative terminology.

DEFINITION 1 *A time-variable solution $s \in S_\tau$ admissible for the OTPC problem is called dominating over a time-steady solution $\bar{s} \in S$ locally optimal for the OTSC problem iff*

$$J(s) < J(\bar{s}).$$

DEFINITION 2 *The OTPC problem is called locally dominating over the OTSC problem iff it possesses a time-variable solution $s \in S_\tau$ dominating over a locally optimal time-steady solution of the OTSC problem.*

Looking for conditions guaranteeing the dominance of the OTPC problem over the OTSC problem we assume that the functions $\phi_0, \phi, h, g$ and $f$ are twice continuously F-differentiable in the vicinity of the solution $\bar{s}$. We denote the above functions and their derivatives evaluated at $\bar{s}$ with the help of an upper bar, i.e. $\bar{\phi}_0 = \phi_0(\bar{y})$, $\bar{\phi}'_0 = \phi'(\bar{y})$, $\bar{f}(z) = f(\bar{x}(z), \bar{w}(z), z)$, $\bar{f}_x(z) = f_x(\bar{x}(z), \bar{w}(z), z)$, etc.

To obtain a simple form of optimality conditions for the solution $\bar{s}$ we reduce the OTPC problem in the vicinity of this solution to the space of controls denoted by

$$\mathcal{E}_\tau = \mathcal{U}_\tau \times \mathcal{W}_\tau.$$

To this end we rewrite the process equation (10) as

$$P(x, e) = 0, \quad P : \mathcal{X}_\tau \times \mathcal{E}_\tau \to \mathcal{X}_\tau,$$

where $e = (u, w)$ and

$$P(x, e)(z) = x(z) - T(z)(Bu + b) - \int_0^z T(z - \zeta) f(x(\zeta), w(\zeta), \zeta) d\zeta, \quad z \in I.$$

The partial F-differential of the mapping P with respect to $x$ at $(\bar{x}, \bar{e})$ takes the form

$$(\bar{P}_x \delta x)(z) = \delta x(z) - \int_0^z T(z - \zeta) \bar{f}_x(\zeta) \delta x(\zeta) d\zeta, \quad z \in I.$$

The operator $\bar{P}_x$ is boundedly invertible in the space $\mathcal{X}_\tau$ (Curtain, Pritchard, 1978; Pazy, 1983), which means that the process equation is, by virtue of the implicit function theorem, locally resolvable with respect to $x$ as a function of $e$, i.e. the functions $x(e)$ and $y(x(e), e)$ are defined in the neighbourhood of $\bar{s}$.

Thus the OTPC problem may be locally reduced to the problem

$$\text{minimize } \psi_0(e) \text{ s.t. } \psi(e) = 0, \ e \in E \subset \mathcal{E}_\tau \tag{19}$$

where

$$\psi_0(e) = \phi_0(y(x(e), e)), \quad \psi(e) = \phi(y(x(e), e)),$$
$$E = \{e \in \mathcal{E}_\tau : (u(t), w(z)(t)) \in U \times W, \ z \in I, \ t \in I_\tau\}.$$

The reduced problem (19) has, as compared with the OTPC problem, finite dimensional space of equality constraints, which will be advantageously exploited in further considerations.

To compute in a convenient way the F-differentials of the functions $\psi_p(e)$, $p = 0, 1, ..., q$, we define for the problem (19) the following inner Lagrange functionals:

$$\mathcal{L}_p(y, x, e, \mu, \eta) = \phi_p(y) + \mu_p^T \Big( \int_0^1 \alpha_\tau \circ \big(g(x(z), w(z), z)\big) dz + \alpha_\tau \circ \big(h(u)\big) - y \Big)$$
$$+ \int_0^1 \alpha_\tau \circ \Big( \eta_p^T(z) \Big( T(z)(Bu + b) + \int_0^z T(z - \zeta) f(x(\zeta), w(\zeta), \zeta) d\zeta - x(z) \Big) \Big) dz \tag{20}$$

where $\mu_p \in R^r$ and $\eta_p \in C(I; R^n)$, $p = 0, 1, ..., q$. For the partial F-differentials of $\mathcal{L}_p$ at $\bar{s}$ we obtain the following expressions:

$$\bar{\mathcal{L}}_{py}\delta y = \bar{\phi}'_p \delta y - \mu_p^T \delta y, \tag{21}$$

$$\bar{\mathcal{L}}_{px}\delta x = \mu_p^T \int_0^1 \alpha_\tau \circ \big(\bar{g}_x(z)\delta x(z)\big) dz$$
$$+ \int_0^1 \alpha_\tau \circ \Big(\eta_p^T(z)\Big(\int_0^z T(z-\zeta)\bar{f}_x(\zeta)\delta x(\zeta)d\zeta - \delta x(z)\Big)\Big) dz$$
$$= \int_0^1 \Big(\mu_p^T \bar{g}_x(z) + \int_z^1 \eta_p^T(\zeta)d\zeta \bar{f}_x(z) - \eta_p^T(z)\Big)\delta\bar{x}(z)dz, \tag{22}$$

$$\bar{\mathcal{L}}_{pu}\delta u = \big(\mu_p^T \bar{h}' + \int_0^1 \eta_p^T(z)Bdz\big)\delta\bar{u}, \tag{23}$$

$$\bar{\mathcal{L}}_{pw}\delta w = \mu_p^T \circ \int_0^1 \alpha_\tau \circ \big(\bar{g}_w(z)\delta w(z)\big) dz$$
$$+ \int_0^1 \alpha_\tau \circ \Big(\eta_p^T(z) \int_0^z T(z-\zeta)\bar{f}_w(\zeta)\delta w(\zeta)d\zeta\Big) dz$$
$$= \int_0^1 \Big(\mu_p^T \bar{g}_w(z) + \int_z^1 \eta_p^T(\zeta)d\zeta \bar{f}_w(z)\Big)\delta\bar{w}(z)dz. \tag{24}$$

We have used in the above formulas the independence of the mean value of a periodic function on its translation along the real axis (Butzer, Nessel, 1971), and the Dirichlet formula (Yosida, 1980). The bar over the symbols $\delta\bar{x}, \delta\bar{u}, \delta\bar{w}$ means that we deal with time-steady variations of the solution $\bar{s}$.

Introducing the adjoint equations

$$\mu_p = \bar{\phi}'^T_p, \quad \eta_p(z) = f_x^T(z)\int_z^1 \eta_p(\zeta)d\zeta + \bar{g}_x^T(z)\mu_p, \quad p = 0, 1, ..., q, \tag{25}$$

we obtain

$$\bar{\psi}'_p \delta e = \bar{\mathcal{L}}_{pu}\delta u + \bar{\mathcal{L}}_{pw}\delta w, \quad p = 0, 1, ..., q. \tag{26}$$

Let us consider the set of time-harmonic control variations

$$M_\tau = \{\delta e = (\delta u, \delta w) \in \mathcal{E}_\tau : \delta u(t) = \sum_{\kappa=0,\pm 1} u_\kappa^\wedge \exp(j\kappa\omega t),$$
$$\bar{u} + \delta u(t) \subset U, \quad \delta w(z)(t) = \sum_{\kappa=0,\pm 1} w_\kappa^\wedge(z)\exp(j\kappa\omega t),$$
$$\bar{w}(z) + \delta w(z)(t) \subset W\},$$

and its time-steady variant

$$M = \{\delta\bar{e} = (\delta\bar{u}, \delta\bar{w}) \in \mathcal{E} : \bar{u} + u_0^\wedge \subset U, \quad \bar{w}(z) + w_0^\wedge(z) \subset W\},$$

and let us denote

$$a_p(u_0^\wedge, w_0^\wedge) = (\mu_p^T \bar{h}' + \int_0^1 \eta_p^T(z) B dz) u_0^\wedge$$
$$+ \int_0^1 \Big(\mu_0^T \bar{g}_w(z) + \int_z^1 \eta_0^T(\zeta) d\zeta \bar{f}_w(z)\Big) w_0^\wedge(z) dz, \quad p = 0, 1, ..., q.$$

Let us define the outer Lagrange functional for the problem (17) as

$$\mathcal{L}(e, \lambda) = \sum_{p=0}^{q} \lambda_p \psi_p(e), \tag{27}$$

where $\lambda^T = (\lambda_0, \lambda_1, ..., \lambda_q) \in R^{1+q}$ is the vector of Lagrange multipliers. Since (26) implies

$$\bar{\psi}_p' \delta e = a_p(u_0^\wedge, w_0^\wedge), \quad p = 0, 1, ..., q; \ \delta e \in M \tag{28}$$

we see that the set of normalized multipliers associated with the solution $\bar{e}$ coincides for time-variable variations $M_\tau$ and time-steady variations $M$ with the set

$$\Lambda = \{\lambda \in R^{1+q} : \lambda_0 \geq 0, |\ \lambda\ |= 1, \ \sum_{p=0}^{q} \lambda_p a_p(u_0^\wedge, w_0^\wedge) \geq 0, \ \delta \bar{e} \in M\}.$$

Thus the first-order necessary optimality condition $\Lambda \neq \emptyset$ cannot discern time-periodic solutions dominating over the time-steady solution $\bar{s}$. We hope that the second-order necessary optimality condition will be helpful. To use it we introduce the set of critical directions at $\bar{s}$ coinciding on the basis of (28) for the OTPC problem with the set

$$\mathcal{D}_\tau = \{\delta e \in M : a_0(u_0^\wedge, w_0^\wedge) \leq 0, \ a_p(u_0^\wedge, w_0^\wedge) = 0, \ p = 1, ..., q\}.$$

Denoting

$$\nu_p(z) = \int_z^1 \eta_p(\zeta) d\zeta \tag{29}$$

and

$$H_p(x(z), w(z), \mu, \eta(z), z) = \mu_p^T g(x(z), w(z), z) + \nu_p^T(z) f(x(z), w(z), z), \tag{30}$$

and using the theory of optimization problems with resolvable constraints (Wierzbicki, 1984), we can represent the second F-differentials of the functions $\psi_p(e)$ at $\bar{e}$ as follows

$$\begin{aligned} &2\bar{\psi}_p''(\delta e, \delta e) \\ &= \bar{\mathcal{L}}_{pyy}(\delta y, \delta y) + \bar{\mathcal{L}}_{puu}(\delta u, \delta u) + \bar{\mathcal{L}}_{pxx}(\delta x, \delta x) + 2\bar{\mathcal{L}}_{pxw}(\delta x, \delta w) + \bar{\mathcal{L}}_{pww}(\delta w, \delta w) \\ &\quad = \delta y^T \bar{\phi}_p'' \delta y + \alpha_\tau \circ (\delta u^T \mu_p^T \bar{h}'' \delta u) + \int_0^1 \alpha_\tau \circ \Big(\delta x^T(z) \bar{H}_{pxx}(z) \delta x(z) \\ &\quad + 2\delta x^T(z) \bar{H}_{pxw}(z) \delta w(z) + \delta w^T(z) \bar{H}_{pww}(z) \delta w(z)\Big) dz \end{aligned} \tag{31}$$

where $\delta y, \delta x$ solve the linearized equations

$$\delta y = \int_0^1 \bar{g}_x(z)\delta\bar{x}(z)dz + \bar{h}'\delta\bar{u}, \tag{32}$$

$$\delta x(z) + T(z)B\delta u + \int_0^z T(z-\zeta)\Big(\bar{f}_x(\zeta)\delta x(\zeta) + \bar{f}_w(\zeta)\delta w(\zeta)\Big)d\zeta. \tag{33}$$

Setting

$$\delta x(z) = \sum_{k=0,\pm 1} x_\kappa^\wedge(z)\exp(j\kappa\omega t)$$

and substituting the trigonometric semi-group (12) to (33) we obtain the following equations for components of the state variation $\delta x$:

$$x_\kappa^\wedge(z) = \exp(-j\kappa\omega z)Bu_\kappa^\wedge$$
$$+\int_0^z \exp(-j\kappa\omega(z-\zeta))(\bar{f}_x(\zeta)x_\kappa^\wedge(\zeta) + \bar{f}_w(\zeta)w_\kappa^\wedge(\zeta))d\zeta, \quad z\in I,\ \kappa = 0,\pm 1,$$

which are equivalent to the initial value differential equations

$$x_\kappa^{\wedge\prime}(z) = (\bar{f}_x(z) - j\kappa\omega z I_n)x_\kappa^\wedge(z) + \bar{f}_w(z)w_\kappa^\wedge(z), \tag{34}$$

$$x_\kappa^\wedge(0) = Bu_\kappa^\wedge, \quad \kappa = 0,\pm 1. \tag{35}$$

Let us denote by $\Phi(z, j\kappa\omega)$ the normalized fundamental matrix for the system (34),(35), and let us define for $\kappa = 0,\pm 1$

$$G^0(z, j\kappa\omega, u_\kappa^\wedge) = \Phi(z, j\kappa\omega)Bu_\kappa^\wedge,$$
$$G^1(z, j\kappa\omega, \omega_k^\wedge) = \int_0^z \Phi(z, j\kappa\omega)\Phi^{-1}(\zeta, j\kappa\omega)\bar{f}_w(\zeta)w_k^\wedge(\zeta)d\zeta,$$
$$\vartheta_{i,\kappa} = u_\kappa^\wedge \ \text{if}\ i=0, \ \text{and}\ \vartheta_{i,\kappa} = w_\kappa^\wedge \ \text{if}\ i=1.$$

THEOREM 1 *The OTPC problem is locally dominating over the OTSC problem if for a certain $\delta e \in \mathcal{D}_\tau$ the following condition is satisfied:*

$$\max_{\lambda\in\Lambda}\sum_{p=0}^{q}\lambda_p\Big(\frac{1}{2}\delta y^T\bar{\phi}_p''\delta y + \frac{1}{2}u_0^{\wedge^T}\mu_p^T\bar{h}''u_0^\wedge + u_{-1}^{\wedge^T}\mu_p^T\bar{h}''u_1^\wedge$$
$$+\int_0^1\Big(\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{\kappa=0}^{1}G^{i^T}(z,-j\kappa\omega,\vartheta_{i,-\kappa})\bar{H}_{pxx}(z)G^j(z,j\kappa\omega,\vartheta_{j,\kappa})$$
$$+\sum_{i=0}^{1}\sum_{\kappa=0}^{1}G^{i^T}(z,-j\kappa\omega,\vartheta_{i,-\kappa})\bar{H}_{pxw}(z)w_k^\wedge(z)$$
$$+\sum_{j=0}^{1}\sum_{\kappa=0}^{1}w_{-\kappa}^{\wedge^T}(z)\bar{H}_{pwx}(z)G^j(z,j\kappa\omega,\vartheta_{j,k})$$
$$+\sum_{\kappa=0}^{1}w_{-\kappa}^{\wedge^T}(z)\bar{H}_{pww}(z)w_\kappa^\wedge(z)\Big)dz < 0 \tag{36}$$

*where*

$$\delta y = \int_0^1 \bar{g}_x(z)\Big(G^0(z,0,u_0^\wedge) + G^1(z,0,w_0^\wedge)\Big)dz + \bar{h}' u_0^\wedge.$$

The proof follows from the above considerations, because the condition (36) contradicts the second-order necessary optimality condition (Bernstein, 1984) for $\bar{e}$ as the solution of the OTPC problem.

We note that the substitution (29) allows us to solve the adjoint equations (25) as equivalent final value differential equations for variables $\nu_p$:

$$\nu_p'(z) = -\bar{f}_x^T(z)\nu_p(z) - \bar{g}_x^T(z)\mu_p$$
$$\nu_p(1) = 0, \quad p = 0, 1, ..., q.$$

The time-periodic solution dominating over the steady-state solution $\bar{s}$ may be interpreted as weak or strong solution of the abstract differential equation (5),(6) depending upon properties of the control $\bar{w}$ such as its differentiability with respect to $z$ (Balakrishnan, 1980; Pazy, 1983).

## 3. Proper quasi-periodic control

The question of existence of advantageous time-variable processes for systems under discussion can be extended to the case of time-quasi-periodic processes having a finite frequency basis and possessing important practical meaning (Haken, 1983; Samoilenko, 1987). The optimal time-quasi-periodic control (OTQC) problem for semi-linear hyperbolic systems may be viewed as the following generalization of the OTPC problem (7)–(11): minimize the objective function

$$J_\omega(y, x, u, w) = \phi_0(y) \tag{37}$$

subject to the constraints

$$\phi(y) = 0, \tag{38}$$

$$y = \int_0^1 \alpha \circ \big(g(x(z), w(z), z)\big)dz + \alpha \circ \big(h(u)\big), \tag{39}$$

$$x(z) = T(z)(Bu + b) + \int_0^z T(z-\zeta)f(x(\zeta), w(\zeta), \zeta)d\zeta, \quad z \in I, \tag{40}$$

$$(u(t), w(z)(t)) \in U \times W, \quad z \in I, \quad t \in I_\infty, \tag{41}$$

where $\omega = (\omega_1, ..., \omega_\gamma) \in R_+^\gamma$ is the frequency basis of the state and control with respect to the time coordinate,

$$x \in \mathcal{X}_\omega = C(I; C_\omega^{0,n}), \quad u \in \mathcal{U}_\omega = C_\omega^{0,l}, \quad w \in \mathcal{W}_\omega = L^\infty(I; C_\omega^{0,m})$$

are the process state and controls with values in appropriate spaces of time-quasi-periodic functions,

$$h : C_\omega^{0,l} \to C_\omega^{0,r}, \quad g : C_\omega^{0,n} \times C_\omega^{0,m} \times R \to C_\omega^{0,r}, \quad f : C_\omega^{0,n} \times C_\omega^{0,m} \times R \to C_\omega^{0,n},$$

and $T(z)$ is the semi-group of right translations

$$\big(T(z)\chi\big)(t) = \chi(t - \frac{z}{v}) \quad \text{for} \quad \chi \in C_\omega^{0,n} \tag{42}$$

generated by the differentiation operator

$$A = \frac{1}{v} D_t : C_\omega^{0,n} \to C_\omega^{0,n}, \quad \mathbf{D}(A) = C_\omega^{1,n}.$$

Let us denote by $S_\omega$ the set of all admissible solutions of the OTQC problem, i.e. quadruples $(y, x, u, w) \in R^r \times \mathcal{X}_\omega \times \mathcal{U}_\omega \times \mathcal{W}_\omega$ satisfying the constraints (38)–(41). Let $\tilde{s}$ be a locally optimal solution of the OTPC problem, i.e.

$$J(\tilde{s}) \leq J(s) \quad \text{for all} \quad s \in S_\tau \cup N(\tilde{s})$$

where $N(\tilde{s})$ is a certain neighbourhood of $\tilde{s}$ induced by the norm topology of the space $R^r \times \mathcal{X}_\omega \times \mathcal{U}_\omega \times \mathcal{W}_\omega$.

DEFINITION 3 *A non-periodic solution $s \in S_\omega$ admissible for the OTQC problem is called dominating over a time-periodic solution $\tilde{s} \in S_\tau$ locally optimal for the OTPC problem iff*

$$J_\omega(s) < J_\tau(\tilde{s}).$$

DEFINITION 4 *The OTQC problem is called locally dominating over the OTPC problem if it possesses a non-periodic solution $s \in S_\omega$ dominating over a locally optimal solution of the OTPC problem.*

Let us assume $\tilde{s} = \bar{s}$, which means that the locally optimal solution $\bar{s}$ of the OTSC problem is also locally optimal for the OTPC problem. Denoting

$$\mathcal{E}_\omega = \mathcal{U}_\omega \times \mathcal{W}_\omega$$

we reduce the OTQC problem to the above space by means of the approach presented in Section 2.

Let us introduce the set of time-quasi-periodic control variations

$$M_\omega = \{\delta e = (\delta u, \delta w) \in \mathcal{E}_\omega : \delta u(t) = \sum_{\kappa=0,\pm1,\pm2} u_\kappa^\wedge \exp(j\omega_\kappa t),$$

$$\bar{u} + \delta u(t) \subset U, \; \delta w(z)(t) = \sum_{\kappa=0,\pm1,\pm2} w_\kappa^\wedge(z) \exp(j\omega_\kappa t),$$

$$\bar{w}(z) + \delta w(z)(t) \subset W, \;\; \omega_0 = 0, \omega_1 \text{ and } \omega_2 \text{ are incommensurable frequencies}\},$$

and the set of critical directions for the OTQC problem at $\bar{s}$

$$\mathcal{D}_\omega = \{\delta e \in M_\omega : a_0(u_0^\wedge, w_0^\wedge) \leq 0, \; a_p(u_0^\wedge, w_0^\wedge) = 0,$$
$$p = 1, ..., q\}$$

THEOREM 2 *The OTQC problem is locally dominating over the OTPC problem for the case $\tilde{s} = \bar{s}$ if the following condition is satisfied:*

$$\max_{\lambda\in\Lambda}\sum_{p=0}^{1}\lambda_p\{\frac{1}{2}\Big(\delta y^T\bar{\phi}_p''\delta y+\alpha\circ\left(\delta u^T\mu_p^T\bar{h}''\delta u\right)$$
$$+\int_0^1\alpha\circ\Big(\delta x^T(z)\bar{H}_{pxx}(z)\delta x(z)+2\delta x^T(z)\bar{H}_{pxw}(z)\delta w(z)$$
$$+\delta w^T(z)\bar{H}_{pww}(z)\delta w(z)\Big)dz\Big)\}<0 \tag{43}$$

*for a certain $\delta e = (\delta u, \delta w) \in \mathcal{D}_\omega$, where $\delta y, \delta x$ solve the linearized equations (32), (33) for the above $\delta e$.*

The proof follows from the considerations of Section 2. The condition (43) may be reduced to a frequency dependent form similar to the form (36).

Essentially the same approach is applicable for the case $\tilde{s} \neq \bar{s}$, when the locally optimal solution of the OTPC problem is time-variable.

For this case time-dependent adjoint variables must be considered and such devices as the dual trigonometric semigroup and the Parseval theorem may be useful to derive the adjoint equations and to characterize the sets $\mathcal{D}_\omega$ and $\Lambda_\omega$.

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