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Suboptimal control of systems with distributed parameters: minimizing sequences, value function, regularity, normality[1]

by

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Abstract. Some new results on suboptimal control theory of systems with distributed parameters are presented. These results are connected with necessary and sufficient conditions for elements of minimizing sequences, with differential properties of value functions and with conditions of regularity, normality, abnormality. The paper contains illustrative examples.

Keywords: suboptimal control, minimizing sequence, value function, regularity, normality

1. Introduction

It is known that the traditional optimal control theory (first of all, the theory of necessary conditions) supposes existence of optimal controls in a class of usual (i.e. Lebesgue measurable) or relaxed in the sense of Young (1969), Filippov (1959), Gamkrelidze (1962), Warga (1971) and Fattorini (1991). It is known also that realization of the existence conditions requires sufficiently severe assumptions, see Filippov (1959), Gamkrelidze (1962), Warga (1971), Fattorini (1991), for the initial data of optimal control problems connected, for example, with various compactness properties of the sets of solutions of controlled distributed systems. We assert that many theoretical "difficulties" may be surmounted to a great extent if we consider a minimizing sequence (m.s.) as a "main element" of the theory instead of optimal control (usual or relaxed). Such transition permits to develop useful suboptimal control theory. Above all here we keep in mind necessary and sufficient conditions for elements of m.s., regularity and normality, differential properties of value functions, sensitivity, Kuhn-Tucker vectors, nonsmooth problems, numerical methods, etc. We can interpret the transition to m.s. in a certain sense as a "maximal" extension of the initial optimal control

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problem. It corresponds well to Hilbert's dictum (Young, (1969), p.123) that *Every problem of the calculus of variations has a solution, provided the word "solution" is suitably understood.* Here we consider a m.s. as a solution of the optimal control problem. At the same time, we show that the theory, based on this concept, generalizes the traditional one and gives a new useful information about the optimal control problem. Such situation corresponds to the transition from usual optimal controls to relaxed controls in the sense of Young (1969), Filippov (1959), Gamkrelidze (1962), Warga (1971), Fattorini (1991).

In this paper we consider some new results of suboptimal control theory, see for example Plotnikov and Sumin (1982), Sumin (1987, 1990, 1995, 1996a, 1996b). These results are connected with necessary and sufficient conditions for elements of m.s., with value functions and with conditions of regularity, as well as normality and abnormality. Some of these results generalize corresponding results of Clarke (1983) expressed in terms of usual optimal controls for controlled ordinary differential systems. According to Sumin (1987,1990,1995,1996a,1996b) we consider as m.s. the so-called minimizing approximate solutions in the sense of Warga (1971). Firstly, precisely this use of the minimizing approximate solutions gives us the possibility to write down the results in terms of the extended (relaxed), Young (1969), Filippov (1959), Gamkrelidze (1962), Warga (1971), Fattorini (1991), optimization problem if the problem admits such an extension. Secondly, the application of m.s. in the mentioned sense permits to develop many of the results of Clarke (1983) for investigation of differential properties of corresponding value functions of optimization problems as functions of their parameters (Sumin 1990,1995,1996a,1996b). Finally, the minimizing approximate solutions permit also to establish more close connection of optimal control theory with numerical methods (Sumin (1990,1996b)), since the concept of suboptimality is the central one in numerical methods of optimization.

In the paper we consider a concrete optimal control problem for a linear controlled parabolic equation. But all results remain true for essentially more general optimal control problems for various equations of mathematical physics with boundary controls, fixed and free time, and various constraints. Full proofs of the results can be found in Sumin (1996a,1996b). The paper contains illustrative examples. Other results connected with the suboptimal control theory can be found e.g., in Fattorini (1990), Fattorini and Frankowska (1990).

2. Optimal control problem statement

Consider the controlled first boundary-value problem for a linear parabolic equation in divergence form

$$z_t - \frac{\partial}{\partial x_i} a_{i,j}(x,t) z_{x_j} + b_i(x,t,u(x,t)) z_{x_i} + a(x,t,u(x,t)) z + f(x,t,u(x,t)) = 0,$$
$$z(x,0) = v(x),\ x \in \Omega; \quad z(x,t) = 0,\ (x,t) \in S_T. \tag{2.1}$$

The controls $u : Q_T \to R^m$, $v : \Omega \to R^1$ are elements of the sets $\mathcal{W}_1 \equiv \{u \in L_\infty(Q_T) : u(x,t) \in U$ for a.e. $(x,t) \in Q_T\}$, $\mathcal{W}_2 \equiv \{v \in L_\infty(\Omega) : v(x) \in V$ for a.e. $x \in \Omega\}$, respectively, $U \subset R^m, V \subset R^1$ are compacts, $\mathcal{W} \equiv \{w \equiv (u,v) : w \in \mathcal{W}_1 \times \mathcal{W}_2\}$, Ω is a bounded domain in R^n.

The object of our studies is the following minimization problem

$$(P) \qquad I_0(\pi) \to \inf, \quad I^{\kappa_1}(\pi) \le 0, \quad I^{\kappa_2}(\pi) = 0, \quad \pi \in \mathcal{D}, \quad \kappa_2 \equiv \kappa - \kappa_1,$$

where: $I^{\kappa_1}(\pi) \equiv (I_1(\pi), \dots, I_{\kappa_1}(\pi))$, $I^{\kappa_2}(\pi) \equiv (I_{\kappa_1+1}(\pi), \dots, I_\kappa(\pi))$,

$$I_i(\pi) \equiv \int_\Omega G_i(x, z[w](x,T), b, v(x))\, dx,$$

$\mathcal{D} \equiv \{\pi \equiv (u,v,b) : \pi \in \mathcal{W}_1 \times \mathcal{W}_2 \times B\}$ is a set of triples of controls, B is a convex compact in R^l, $\mathcal{D} \equiv \mathcal{W} \times B$, $\mathcal{W} \equiv \{w \equiv (u,v) : w \in \mathcal{W}_1 \times \mathcal{W}_2\}$, $z[w] \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ is the weak solution in the sense of Ladyzhenskaya, Solonnikov and Ural'tseva (1967), Ch.III, of the first boundary-value problem (2.1) corresponding to the pair $w \in \mathcal{W}$.

Assume that the following conditions on the initial data of problem (P) are fulfilled:

(i) the functions G_k, $\partial G_k/\partial z$, $\partial G_k/\partial b_s$ $: \Omega \times R^1 \times B \times R^1 \to R^1$, $k = 0,1,\dots,\kappa$, $s = 1,\dots,l$, $a_{i,j} : Q_T \to R^1$, $b_i, a, f : Q_T \times R^m \to R^1, i,j = 1,\dots,n$, are Lebesgue measurable in (x,t,z,b,u,v) and continuous in (z,b,u,v) for a.e. (x,t) ;

(ii) the coefficients of the boundary-value problem (2.1) satisfy the inequalities $\nu|\xi|^2 \le a_{i,j}(x,t)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall (x,t) \in Q_T, \quad \nu, \mu > 0,$
$|b_i(x,t,u)| \le K_0(x,t)$, $|a(x,t,u)| \le K_1(x,t)$, $|f(x,t,u)| \le K_2(x,t)$
$\quad \forall (x,t) \in Q_T, u \in U,$
where $K_0 \in L_{2q,2r}(Q_T)$, $K_1, K_2 \in L_{q,r}(Q_T)$ and the pair of numbers (q,r) satisfies certain conditions (7.1), (7.2) in Ladyzhenskaya, Solonnikov and Ural'tseva (1967), Ch.III;

(iii) the integrands G_k, $k = 0,1,\dots,\kappa$, satisfy the inequalities
$|G_k(x,z,b,v)|$, $|\partial G_k(x,z,b,v)/\partial z|$, $|\partial G_k(x,z,b,v)/\partial b_s| \le N(M)$
$\quad \forall\, (x,z,b,v) \in \Omega \times S^1_M \times B \times V,$
where $S^n_M \equiv \{x \in R^n : |x| < M\}$, $N(\cdot)$ is a positive nondecreasing function of $M > 0$.

REMARK 2.1 *In view of the conditions (i) - (iii) and Theorem 4.2 in Ladyzhenskaya, Solonnikov and Ural'tseva (1967), Ch.III, the primal problem (2.1) has a unique solution* $z[w] \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ *for any pair* $w = (u,v) \in \mathcal{W}$. *For the same reason, the adjoint problem*

$$-\eta_t - \frac{\partial}{\partial x_j}(a_{i,j}(x,t)\eta_{x_i} + b_j(x,t,u(x,t))\eta) + a(x,t,u(x,t))\eta = 0,$$

$$\eta(x,T) = -d_i[\pi](x),\ x \in \Omega;\ \eta(x,t) = 0,\ (x,t) \in S_T,\ \pi \equiv (w,b) \equiv (u,v,b) \quad (2.2)$$

has a unique solution $\eta_i[\pi] \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ *for any triple* $\pi \equiv (w,b) \equiv (u,v,b) \in \mathcal{D}$, $i = 0,1,\dots,\kappa$, *where*

$$d_i[\pi](x) \equiv \partial G_i(x, z[w](x,T), b, v(x))/\partial z.$$

Moreover, by Theorems 7.1, 10.1 in Ladyzhenskaya, Solonnikov and Ural'tseva (1967), Ch.III, the solutions $z[w]$, $\eta_i[\pi]$ *are bounded in* $L_\infty(Q_T)$ *uniformly with respect to* $\pi \in \mathcal{D}$ *and* $z[w]$, $\eta_i[\pi] \in H^{\alpha,\alpha/2}(Q_T)$ *for some* $\alpha > 0$.

3. Necessary and sufficient conditions for minimizing sequences

Let us define

$$\beta_\epsilon \equiv \inf_{\mathcal{D}^\epsilon} I_0(\pi), \quad \epsilon \geq 0,$$

where $\mathcal{D}^\epsilon \equiv \{\pi \in \mathcal{D} : I_i(\pi) \leq \epsilon, i = 1,\dots,\kappa_1; |I_i(\pi)| \leq \epsilon, i = \kappa_1+1,\dots,\kappa\}$, $\beta_\epsilon = +\infty$, if $\mathcal{D}_\epsilon = \emptyset$. Obviously, $\beta_{\epsilon_1} \geq \beta_{\epsilon_2}$ for $\epsilon_1 \leq \epsilon_2$. Consequently, there exists the finite or infinite limit (the value of problem (P))

$$\lim_{\epsilon \to +0} \beta_\epsilon \equiv \beta_{+0} \equiv \beta \leq \beta_0.$$

Just as in Sumin (1987, 1990, 1995, 1996a), we are interested in deriving necessary and sufficient conditions for elements of m.s. of triples $\pi^i \in \mathcal{D}, i = 1,2,\dots$, for problem (P) such that

$$I_0(\pi^i) \leq \beta + \epsilon_i, \quad \pi^i \in \mathcal{D}^{\epsilon_i}, \quad \epsilon_i \geq 0, \quad \epsilon_i \to 0, \quad i \to \infty. \tag{3.1}$$

REMARK 3.1 *The concept of m.s. in the sense of (3.1) for problem (P) coincides with the well-known concept of minimizing approximate solution in the sense of Warga (1971), Ch.II.*

Introduce the notations:

$$H(x,t,z,p,u,\eta) \equiv -\eta\Big(\sum_{i=1}^n b_i(x,t,u)p_i + a(x,t,u)z + f(x,t,u)\Big),$$
$$H_k(x,z,b,v,\eta) \equiv \eta v - G_k(x,z,b,v), \quad k = 0,1,\dots,\kappa,$$
$$\xi \equiv (z,p_1,\dots,p_n), \quad \xi[w](x,t) \equiv \big(z[w](x,t), z_{x_1}[w](x,t),\dots,z_{x_n}[w](x,t)\big).$$

The following theorem gives necessary conditions for elements of m.s. in the sense of (3.1). We omit the proof of this theorem due to the lack of space. The details may be found in Sumin (1996a).

THEOREM 3.1 *Let* $\pi^i \in \mathcal{D}, i = 1,2,\dots$, *be a m.s. in the sense of (3.1) for problem (P). Then there exist a sequence of numbers*

$$\gamma^i \geq 0, \ i = 1,2,\dots, \ \gamma^i \to 0, \ i \to \infty, \tag{3.2}$$

and a sequence of vectors $\mu^i \in R^{\kappa+1}$,

$$|\mu^i| = 1,\ \mu^i_k \geq 0,\ k = 0,1,\ldots,\kappa_1,\ \ \mu^i_k I_k(\pi^i) \geq -\gamma^i,\ k = 1,\ldots,\kappa_1, \tag{3.3}$$

such that

$$\int_{Q_T} \max_{v\in U} \sum_{k=0}^{\kappa} \mu^i_k (H(x,t,\xi[w^i](x,t),v,\eta_k[\pi^i](x,t)) - \\ H(x,t,\xi[w^i](x,t),u^i(x,t),\eta_k[\pi^i](x,t)))\, dxdt \leq \gamma^i, \tag{3.4}$$

$$\int_{\Omega} \max_{v\in V} \sum_{k=0}^{\kappa} \mu^i_k (H_k(x,z[w^i](x,T),b^i,v,\eta_k[\pi^i](x,0)) - \\ H_k(x,z[w^i](x,T),b^i,v^i(x),\eta_k[\pi^i](x,0)))\, dx \leq \gamma^i, \tag{3.5}$$

$$\max_{b\in B} \sum_{k=0}^{\kappa} \mu^i_k \langle \int_{\Omega} \nabla_b G_k(x,z[w^i](x,T),b^i,v^i(x))\, dx, b^i - b\rangle \leq \gamma^i, \tag{3.6}$$

where $\eta_k[\pi^i]$, $k = 0,1,\ldots,\kappa$, *are the solutions of the adjoint problem (2.2) for* $\pi = \pi^i$.

REMARK 3.2 *If the cost functional in problem (P) has the form*

$$\phi(I_0(\pi)) + \psi(b), \quad \phi \in C^1(R^1),\ \ \psi \in C^1(R^l),$$

then we must replace the value μ^i_0 *in (3.4),(3.5) by* $\phi'(I_0(\pi^i))\mu^i_0$ *and rewrite (3.6) in the form*

$$\max_{b\in B}\{\mu^i_0\langle \nabla\psi(b^i) + \phi'(I_0(\pi^i)) \int_{\Omega} \nabla_b G_0(x,z[w^i](x,T),b^i,v^i(x))\, dx, b^i - b\rangle + \\ \sum_{k=1}^{\kappa} \mu^i_k \langle \int_{\Omega} \nabla_b G_k(x,z[w^i](x,T),b^i,v^i(x))\, dx, b^i - b\rangle\} \leq \gamma^i.$$

Further, we define the so-called E-functions of Weierstrass-Plotnikov (see for example Plotnikov (1972), Sumin (1985,1987,1996a), Novozhenov, Sumin V. and Sumin M. (1986)) to formulate sufficient conditions for elements of m.s. in the sense of (3.1): $E_{\overline{a}} : Q_T \times R^1 \times R^n \times R^1 \times R^n \times R^m \times R^m \to R^1$, $E_{G_j} : \Omega \times R^1 \times R^1 \times R^l \times R^l \times R^1 \times R^1$, $j = 0,1,\ldots,\kappa$,

$$E_{\overline{a}}(x,t,z^2,p^2,z^1,p^1,u^2,u^1) \equiv \overline{a}(x,t,z^2,p^2,u^2) - \overline{a}(x,t,z^1,p^1,u^2) - \\ \langle \nabla_p \overline{a}(x,t,z^1,p^1,u^1), p^2 - p^1\rangle - \nabla_z \overline{a}(x,t,z^1,p^1,u^1)(z^2 - z^1),$$

$$\overline{a}(x,t,z,p,u) \equiv \sum_{i=1}^{n} b_i(x,t,u)p_i + a(x,t,u)z + f(x,t,u),$$

$$E_{G_j}(x,z^2,z^1,b^2,b^1,v^2,v^1) \equiv G_j(x,z^2,b^2,v^2) - G_j(x,z^1,b^1,v^2) - \\ \nabla_z G_j(x,z^1,b^1,v^1)(z^2 - z^1) - \langle \nabla_b G_j(x,z^1,b^1,v^1), b^2 - b^1\rangle.$$

Denote also:

$$E_{\overline{a}}[\pi^2,\pi^1](x,t) \equiv E_{\overline{a}}(x,t,\xi[w^2](x,t),\xi[w^1](x,t),u^2(x,t),u^1(x,t)),$$
$$E_{G_j}[\pi^2,\pi^1](x) \equiv E_{G_j}(x,z[w^2](x,T),z[w^1](x,T),b^2,b^1,v^2(x),v^1(x)).$$

In order to prove sufficient conditions for elements of m.s. we consider the following two auxiliary lemmas. The proof of the first lemma is omitted, since it may be found for example in Novozhenov, Sumin V. and Sumin M. (1986).

LEMMA 3.1 *Let us consider the first boundary-value problem for the following linear parabolic equation*

$$z_t - \frac{\partial}{\partial x_i} a_{i,j}(x,t) z_{x_j} + b_i(x,t) z_{x_i} + a(x,t) z + f(x,t) = 0,$$
$$z(x,0) = \psi(x),\ x \in \Omega; \quad z(x,t) = 0,\ (x,t) \in S_T, \tag{3.7}$$

where coefficients $a_{i,j}$, b_i, a, f, ψ satisfy the following assumptions:

$$\nu|\xi|^2 \le a_{i,j}(x,t)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall (x,t) \in Q_T, \quad \nu,\mu > 0,$$
$$b_i \in L_{2q,2r}(Q_T),\ a, f \in L_{q,r}(Q_T),\ \psi \in L_2(\Omega),\ i,j = 1,2,\ldots,n.$$

Here the pair of numbers (q,r) is the same as in the assumption (ii) of Section 2. If a function $z \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ is a weak solution of the problem (3.7), then for any functions $c \in L_{q,r}(Q_T)$, $d \in L_2(\Omega)$ we have

$$\int_{Q_T} c(x,t) z(x,t)\,dxdt - \int_\Omega d(x) z(x,T)\,dx$$
$$= \int_{Q_T} f(x,t)\eta(x,t)\,dxdt - \int_\Omega \psi(x)\eta(x,0)\,dx,$$

where the function $\eta \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ is a weak solution of the adjoint problem

$$-\eta_t - \frac{\partial}{\partial x_j}(a_{i,j}(x,t)\eta_{x_i} + b_j(x,t)\eta) + a(x,t)\eta + c(x.t) = 0,$$
$$\eta(x,T) = d(x),\ x \in \Omega; \quad \eta(x,t) = 0,\ (x,t) \in S_T.$$

LEMMA 3.2 *The increment $\Delta I_i \equiv I_i(\pi^2) - I_i(\pi^1)$, $i = 0,1,\ldots,\kappa$, $\pi^1,\pi^2 \in \mathcal{D}$, is equal to*

$$\Delta I_i = \int_{Q_T} (\sum_{i=1}^n \Delta_u b_i(u^2(x,t),u^1(x,t)) z_{x_i}[w^1](x,t) +$$
$$\Delta_u a(u^2(x,t),u^1(x,t)) z[w^1](x,t) + \Delta_u f(u^2(x,t),u^1(x,t)))\eta_i[\pi^1](x,t)\,dxdt -$$
$$\int_\Omega (v^2(x) - v^1(x))\eta_i[\pi^1](x,0)\,dx + \int_\Omega (G_i(x,z[w^1](x,T),b^1,v^2(x))$$
$$-G_i(x,z[w^1](x,T),b^1,v^1(x)))\,dx +$$

$$\langle \int_\Omega \nabla_b G_i(x, z[w^1](x,T), b^1, v^1(x))\, dx, b^2 - b^1\rangle$$
$$+ \int_{Q_T} E_{\overline{a}}[\pi^2, \pi^1](x,t)\eta_i[\pi^1](x,t)\, dxdt + \int_\Omega E_{G_i}[\pi^2, \pi^1](x)\, dx, \tag{3.8}$$

where: $\Delta_u b_i(u^2(\cdot,\cdot), u^1(\cdot,\cdot)) \equiv b_i(\cdot,\cdot,u^2(\cdot,\cdot)) - b_i(\cdot,\cdot,u^1(\cdot,\cdot))$, *etc.*

Proof. Since $z[\pi^i]$, $i = 1,2$, are the solutions of the boundary-value problem (2.1), it is easy to see that the increment $\Delta z \equiv z[\pi^2] - z[\pi^1]$ satisfies the following boundary-value problem

$$\Delta z_t - \frac{\partial}{\partial x_i} a_{i,j}(x,t)\Delta z_{x_j} + b_i(x,t,u^1(x,t))\Delta z_{x_i} + a(x,t,u^1(x,t))\Delta z$$
$$+E_{\overline{a}}[\pi^2,\pi^1](x,t) + \Delta_u b_i(u^2(x,t), u^1(x,t)) z_{x_i}[\pi^1](x,t)$$
$$+\Delta_u a(u^2(x,t), u^1(x,t)) z[\pi^1](x,t) + \Delta_u f(u^2(x,t), u^1 x, t) = 0$$
$$\Delta z(x,0) = v^2(x) - v^1(x),\ x \in \Omega; \quad \Delta z(x,t) = 0,\ (x,t) \in S_T. \tag{3.9}$$

On the other hand

$$I_i(\pi^2) - I_i(\pi^1) = \int_\Omega \nabla_z G_i(x, z[\pi^1](x,T), b^1, v^1(x))\Delta z(x,T)\, dx +$$
$$\langle \int_\Omega \nabla_b G_i(x, z[w^1](x,T), b^1, v^1(x))\, dx, b^2 - b^1\rangle + \int_\Omega E_{G_i}[\pi^2,\pi^1](x)\, dx +$$
$$\int_\Omega (G_i(x, z[\pi^1](x,T), b^1, v^2(x)) - G_i(x, z[\pi^1](x,T), b^1, v^1(x)))\, dx.$$

In view of (3.9) we can apply Lemma 3.1 to rearrange the first term on the right-hand side of the last equality. As a result, we get (3.8). □

THEOREM 3.2 *A sequence $\pi^i \in \mathcal{D}$, $i = 1,2,\ldots$, is a m.s. in the sense of (3.1) for problem (P), if for some sequence of numbers (3.2) we have $\pi^i \in \mathcal{D}^{\gamma^i}$ and there exists a sequence of vectors $\mu^i \in R^{\kappa+1}$, $i = 1,2,\ldots$,*

$$|\mu^i| \le C,\ |\mu^i| \ne 0, \quad \mu_0^i \ge \gamma,\ \mu_k^i \ge 0,\ \mu_k^i I_k(\pi^i) \ge -\gamma^i,\ k = 1,\ldots,\kappa_1, \tag{3.10}$$

where C, $\gamma > 0$ are some positive constants, such that

$$\sum_{k=0}^{\kappa} \mu_k^i (I_k(\pi) - I_k(\pi^i)) \ge -\gamma^i \quad \forall \pi \in \mathcal{D}^{\gamma^i}. \tag{3.11}$$

Moreover, the following expressions hold for the left-hand side of (3.11) and for the increment $I_k(\pi) - I_k(\pi^i)$:

$$\sum_{k=0}^{\kappa} \mu_k^i (I_k(\pi) - I_k(\pi^i)) =$$

$$\{\sum_{k=0}^{\kappa}\mu_k^i\mathcal{H}_k[\pi,\pi^i]\}+\{\sum_{k=0}^{\kappa}\mu_k^i\mathcal{E}_k[\pi,\pi^i]\}\equiv\{\mathcal{H}[\pi,\pi^i]\}+\{\mathcal{E}[\pi,\pi^i]\}, \tag{3.12}$$

$$\begin{aligned}I_k(\pi)-I_k(\pi^i)=\{&\int_{Q_T}(H(x,t,\xi[w^i](x,t),u^i(x,t),\eta_k[\pi^i](x,t))-\\&H(x,t,\xi[w^i](x,t),u(x,t),\eta_k[\pi^i](x,t)))\,dxdt+\\&\int_\Omega(H_k(x,z[w^i](x,T),b^i,v^i(x),\eta_k[\pi^i](x,0))-\\&H_k(x,z[w^i](x,T),b^i,v(x),\eta_k[\pi^i](x,0)))\,dx+\\&\langle\int_\Omega\nabla_bG_k(x,z[w^i](x,T),b^i,v^i(x))\,dx,b-b^i\rangle\}+\\&\{\int_{Q_T}E_{\overline{a}}[\pi,\pi^i](x,t)\eta_k[\pi^i](x,t)\,dxdt+\int_\Omega E_{G_k}[\pi,\pi^i](x)\,dx\}\\&\equiv\{\mathcal{H}_k[\pi,\pi^i]\}+\{\mathcal{E}_k[\pi,\pi^i]\}.\end{aligned} \tag{3.13}$$

Proof. In view of (3.10),(3.11) we can write

$$\begin{aligned}&\mu_0^i(I_0(\pi)-I_0(\pi^i))\geq\\&-\sum_{k=1}^{\kappa_1}\mu_k^i(I_k(\pi)-I_k(\pi^i))-\sum_{k=\kappa_1+1}^{\kappa}\mu_k^i(I_k(\pi)-I_k(\pi^i))-\gamma^i\geq\\&-2\gamma^i\sum_{k=1}^{\kappa}|\mu_k^i|-\gamma^i\equiv\alpha_i\ \ \forall\pi\in\mathcal{D}^{\gamma^i}.\end{aligned}$$

Obviously, by (3.10) and by the inclusion $\pi^i\in\mathcal{D}^{\gamma^i}$ we have: $\alpha_i\to 0,\ i\to\infty$. From here and (3.10) we conclude that the first assertion of the lemma is correct. Equalities (3.12),(3.13) follow from equality (3.8) of Lemma 3.2. □

REMARK 3.3 *By virtue of (3.12), inequality (3.11) holds if the following inequalities are fulfilled:*

$$\mathcal{H}[\pi,\pi^i]\geq-\gamma^i,\quad \mathcal{E}[\pi,\pi^i]\geq 0\quad\forall\pi\in\mathcal{D}^{\gamma^i}.$$

It is easy to see that the first of these inequalities is directly connected with the conditions of suboptimality of Theorem 3.1. We distinguish here the case of those initial data of problem (P) for which the summary E-function of Weierstrass-Plotnikov $\mathcal{E}[\pi,\pi^i]$ is nonnegative. Exactly in this case the corresponding "perturbed" maximum principle (3.2)-(3.6) constitutes a sufficient condition of suboptimality under additional assumptions of regularity of problem (P). Note that the inequality $\mathcal{E}[\pi,\pi^i]\geq 0$ certainly holds for all $\pi\in\mathcal{D}$ in the case of the so-called "linear-convex" problem (P): $b_i(x,t,u)=b_i(x,t)$, $i=1,\dots,n$, $a(x,t,u)=a(x,t)$, $G_i(x,z,b,v)=G_i^1(x,z,b)+G_i^2(x,v)$, the functions G_i^1 are convex in (z,b), $i=0,1,\dots,\kappa_1$, $G_i(x,z,b,v)=G_i^1(x)z+\sum_{j=1}^{l}G_{i,j}^2(x)b_j+G_i^3(x,v)$, $i=\kappa_1+1,\dots,\kappa$.

4. The value function of optimal control problem

In this section we consider the problem of minimization

$(P_0) \qquad I_0(w) \to \inf, \quad I^{\kappa_1}(w) \le 0, \ I^{\kappa_2}(w) = 0.$

Problem (P_0) is the same as problem (P), but integrands $G_i, i = 0, 1, \dots, \kappa$, and, consequently, functionals I_i, $i = 0, 1, \dots, \kappa$, in problem (P_0) do not depend on the vector parameter b. We assume that the data of problem (P_0) satisfy all conditions (i) - (iii) of Section 2; certainly those that do not refer to parameter b.

We embed problem (P_0) into a family of problems $(P_{p,q})$

$(P_{p,q}) \qquad I_0(w) \to \inf, \quad I^{\kappa_1}(w) \le p, \quad I^{\kappa_2}(w) = q, \quad w \in \mathcal{W},$

where $p \equiv (p_1, \dots, p_{\kappa_1})$, $q \equiv (q_{\kappa_1+1}, \dots, q_\kappa)$.

As in Section 3 we denote: $\mathcal{W}^\epsilon_{p,q} \equiv \{w \in \mathcal{W} : I_i(w) \le p_i + \epsilon, i = 1, \dots, \kappa_1, |I_i(w) - q_i| \le \epsilon, i = \kappa_1 + 1, \dots, \kappa\}$, $\epsilon \ge 0$, $\beta_\epsilon(p, q) \equiv \inf\{I_0(w) : w \in \mathcal{W}^\epsilon_{p,q}\}$, $\beta_\epsilon(p, q) \equiv +\infty$, if $\mathcal{W}^\epsilon_{p,q} = \emptyset$.

Obviously, there exists the limit (finite or infinite)

$$\lim_{\epsilon \to 0} \beta_\epsilon(p, q) \equiv \beta_{+0}(p, q) \equiv \beta(p, q) \le \beta_0(p, q), \tag{4.1}$$

usually called the value function of the problem (P_0).

According to the concept of m.s. in Section 3, a sequence of pairs $w^k \in \mathcal{W}$, $k = 1, 2, \dots$, is called a m.s. for problem $(P_{p,q})$ if

$$\begin{aligned} &I_0(w^k) \le \beta_{p,q} + \epsilon_k, \ I_j(w^k) \le p_j + \epsilon_k, \ j = 1, \dots, \kappa_1, \\ &\quad |I_j(w^k) - q_j| \le \epsilon_k, \ j = \kappa_1 + 1, \dots, \kappa, \end{aligned} \tag{4.2}$$

for some sequence of numbers $\epsilon_k \ge 0$, $k = 1, 2, \dots$, $\epsilon_k \to 0$, $k \to \infty$.

Just as in Sumin (1990,1995,1996a), we are interested in differential properties of the value function $\beta(p, q)$. In this section we shall derive an expression for the Clarke's generalized gradient, Clarke (1983), of the function $\beta(p, q)$. To this end, following Sumin (1990, 1995, 1996a) we use the general approach of Clarke (see for example Clarke, 1983, Clarke and Loewen, 1986).

The following simple but important lemma permits to differentiate formally, in Clarke's sense, the value function $\beta(p, q)$ but not $\beta_0(p, q)$. See Example 6.1 below for illustration of this assertion.

LEMMA 4.1 *The value function $\beta : R^\kappa \to R^1 \cup \{+\infty\}$ is bounded from below and lower semicontinuous.*

Proof. The boundedness from below of the function β follows from the boundedness of the solutions $z[w]$, $w \in \mathcal{W}$ (see Remark 2.1). Let us take an arbitrary sequence

$(p^i, q^i), \ i = 1, 2, \dots, \ (p^i, q^i) \to (p, q), \ i \to \infty.$

By virtue of (4.1) we have

$$\beta(p^i, q^i) = \lim_{k \to \infty} \beta_{\epsilon_k}(p^i, q^i), \quad \epsilon_k > 0, \ \epsilon_k \to 0, \ k \to \infty.$$

Without loss of generality we can assume that $\beta(p^i, q^i) \to \overline{\beta}(p, q)$, where $\overline{\beta}(p, q)$ is finite or $+\infty$. Let k_i, $i = 1, 2, \ldots$, be a subsequence of the sequence $k = 1, 2, \ldots$, such that the sequence $\beta_{\epsilon_{k_i}}(p^i, q^i)$, $i = 1, 2, \ldots$, has a limit and the following equality holds

$$\lim_{i \to \infty} \beta_{\epsilon_{k_i}}(p^i, q^i) = \overline{\beta}(p, q), \quad \epsilon_{k_i} > 0, \ \epsilon_{k_i} \to 0, \ i \to \infty.$$

Then for all $i = 1, 2, \ldots$ and for some sequence $\overline{\epsilon}_i$, $i = 1, 2, \ldots$, the following inclusion holds

$$\mathcal{W}^{\epsilon_{k_i}}_{p^i, q^i} \subset \mathcal{W}^{\overline{\epsilon}_i}_{p,q}, \quad \overline{\epsilon}_i > 0, \ \overline{\epsilon}_i \to 0, \ i \to \infty.$$

Consequently, we can write

$$\beta_{\epsilon_{k_i}}(p^i, q^i) \geq \beta_{\overline{\epsilon}_i}(p, q), \ i = 1, 2, \ldots.$$

Whence it follows that

$$\beta(p, q) = \lim_{i \to \infty} \beta_{\overline{\epsilon}_i}(p, q) \leq \overline{\beta}(p, q).$$

The last inequality means that the lemma is proved. □

Further, we derive an expression for Clarke's generalized gradient $\partial\beta(p, q)$. Let $(p, q) \in R^\kappa$ be a point such that $\beta(p, q) < +\infty$. Let also $(p^i, q^i) \in R^\kappa$, $v^i \in R^1$, $\zeta^i \in R^\kappa$, $\eta^i \in R^1$, $i = 1, 2, \ldots$, be sequences such that

$$\begin{aligned} &(p^i, q^i) \to (p, q), \ v^i \to \beta(p, q), \ v^i \geq \beta(p^i, q^i), \\ &\eta^i \geq 0, \ \zeta^i \to 0, \ \eta^i \to 0, \ i \to \infty, \end{aligned} \tag{4.3}$$

and, moreover, (see the definition of perpendicular or proximal normal to a set in Clarke, 1983)

$$(\zeta^i, -\eta^i) \perp \mathrm{epi}\beta \ \text{ at } \ ((p^i, q^i), v^i). \tag{4.4}$$

The existence of such sequences follows from the closedness of the set epiβ. Thus the existence is a corollary of Lemma 4.1 (see Clarke, 1983).

By the condition of orthogonality (4.4) and by Proposition 2.5.5 in Clarke (1983), it follows that

$$\begin{aligned} &\langle (\zeta^i, -\eta^i), ((p', q'), I + v^i - \beta(p^i, q^i)) - ((p^i, q^i), v^i) \rangle \leq \\ &\quad \frac{1}{2} |((p', q'), I + v^i - \beta(p^i, q^i)) - ((p^i, q^i), v^i)|^2 \quad \forall (p', q') \in R^\kappa, \ \ I \geq \beta(p', q'). \end{aligned}$$

Whence

$$\begin{aligned} &\eta^i \beta(p^i, q^i) - \langle \zeta^i, (p^i, q^i) \rangle \leq \eta^i I - \langle \zeta^i, (p', q') \rangle + \\ &\quad \frac{1}{2} |((p', q') - (p^i, q^i), I - \beta(p^i, q^i))|^2 \quad \forall (p', q') \in R^\kappa, \ \ I \geq \beta(p', q'). \end{aligned} \tag{4.5}$$

In turn, it follows from the last inequality that any m.s. $w^{i,k} \in \mathcal{W}$, $k = 1, 2, \ldots$, in the sense of (4.2) for problem (P_{p^i,q^i}) is also a m.s. in the same sense for the following problem

$$I^i(w,(p',q')) \equiv \eta^i I_0(w) - \langle \zeta^i, (p',q')\rangle + \frac{1}{2}|((p',q') - (p^i,q^i), I_0(w) - \beta(p^i,q^i))|^2 \to \inf,$$
$$I^{\kappa_1}(w) \le p', \quad I^{\kappa_2}(w) = q', \quad (p',q') \in R^\kappa, \quad w \in \mathcal{W}, \tag{4.6}$$

since the sequence $(w^{i,k}, (p^i,q^i))$, $i = 1, 2, \ldots$, satisfies the inequalities

$$\eta^i I_0(w^{i,k}) - \langle \zeta^i, (p^i,q^i)\rangle + \frac{1}{2}(I_0(w^{i,k}) - \beta(p^i,q^i))^2 \le$$
$$\eta^i \beta(p^i,q^i) - \langle \zeta^i, (p^i,q^i)\rangle + \eta^i \epsilon_k + \frac{1}{2}\epsilon_k^2,$$
$$I_j(w^{i,k}) \le p_j^i + \epsilon_k, \ j = 1, \ldots, \kappa_1; \ |I_j(w^{i,k}) - q_j^i| \le \epsilon_k, \ j = \kappa_1 + 1, \ldots, \kappa. \tag{4.7}$$

At the same time, the lower bound $\widetilde{\beta}^i$ in the problem (4.6),

$$\widetilde{\beta}^i \equiv \lim_{\epsilon \to 0} \widetilde{\beta}^i_\epsilon, \ \ \widetilde{\beta}^i_\epsilon \equiv \inf_{\widetilde{\mathcal{W}}_\epsilon} I^i(w,(p',q')),$$
$$\widetilde{\mathcal{W}}_\epsilon \equiv \{(w,(p',q')) \in \widetilde{\mathcal{W}} \equiv \mathcal{W} \times R^\kappa : I_j(w) - p'_j \le \epsilon, \ j = 1, \ldots, \kappa_1,$$
$$|I_j(w) - q'_j| \le \epsilon, \ j = \kappa_1 + 1, \ldots, \kappa\},$$

satisfies the equality

$$\widetilde{\beta}^i = \eta^i \beta(p^i,q^i) - \langle \zeta^i, (p^i,q^i)\rangle \equiv \alpha^i. \tag{4.8}$$

We will show that (4.8) really holds. Indeed, suppose that it is not satisfied. Then $\widetilde{\beta}^i < \alpha^i$ and there exists a sequence $(w^s, (p^s,q^s)) \in \widetilde{\mathcal{W}}$, $s = 1, 2, \ldots$, such that

$$I^i(w^s,(p^s,q^s)) \le \widetilde{\beta}^i + \delta \le \alpha^i, \quad , \ s = 1, 2, \ldots, \tag{4.9}$$
$$I_j(w^s) - p_j^s \le \epsilon_s, \ j = 1, \ldots, \kappa_1, \ \ |I_j(w^s) - q_j^s| \le \epsilon_s, \ j = \kappa_1 + 1, \ldots, \kappa,$$

for some sequence $\epsilon_s \ge 0$, $s = 1, 2, \ldots$, $\epsilon_s \to 0$, $s \to \infty$. By virtue of (4.9) and by the definition of the value $\beta_{\epsilon_s}(p^s,q^s)$ we have

$$I_0(w^s) \ge \beta_{\epsilon_s}(p^s,q^s), \ \ s = 1, 2, \ldots.$$

On the other hand, by (4.9), the boundedness of functional I_0 (see Remark 2.1) and the construction of functional I^i, the sequence (p^s,q^s), $s = 1, 2, \ldots$ is bounded. Without loss of generality we can assume that this sequence converges:

$$(p^s,q^s) \to (\overline{p},\overline{q}), \ \ s \to \infty.$$

It follows from this limit relation that

$$\beta_{\epsilon_s}(p^s,q^s) \ge \beta_{\overline{\epsilon}_s}(\overline{p},\overline{q}), \ \ s = 1, 2, \ldots,$$

for some sequence $\overline{\epsilon}_s > 0,\ s = 1, 2, \ldots,\ \overline{\epsilon}_s \to 0,\ s \to \infty$. Then, in view of the previous inequality and the definition of the value $\beta(\overline{p}, \overline{q})$ we obtain

$$I_0(w^s) \geq \beta(\overline{p}, \overline{q}) - \widetilde{\epsilon}_s,\ \ \widetilde{\epsilon}_s \geq 0,\ \ \widetilde{\epsilon}_s \to 0,\ \ s \to \infty.$$

Thus, for $s \to \infty$ the points $((p^s, q^s), I_0(w^s))$ converge to the set epiβ. Hence, by (4.5) and by continuity of the function

$$\eta^i I - \langle \zeta^i, (p', q') \rangle + \frac{1}{2}|((p', q') - (p^i, q^i), I - \beta(p^i, q^i))|^2$$

with respect to $((p', q'), I)$, the value $I^i(w^s, (p^s, q^s))$ can not satisfy the strict inequality (4.9). This contradiction completes the proof of (4.8).

Consider the following restriction of the problem (4.6):

$$I^i(w, (p', q')) \to \inf,\ \ I^{\kappa_1}(w) \leq p',\ \ I^{\kappa_2}(w) = q',\ \ (p', q') \in \overline{S^{\kappa}_M},\ \ w \in \mathcal{W}, \tag{4.10}$$

where S^{κ}_M is a ball containing the points (p, q), (p^i, q^i), $i = 1, 2, \ldots$. The sequence $(w^{i,k}, (p^i,\ q^i))$, $i = 1, 2, \ldots$, is m.s. also for problem (4.10). Moreover, problem (4.10) has the form of problem (P) with $\mathcal{D} \equiv \mathcal{W} \times \overline{S^{\kappa}_M}$, $B \equiv \overline{S^{\kappa}_M}$, $b \equiv (p',\ q')$, $l \equiv \kappa$, $G_i(x, z, b, v) \equiv G_i(x, z, v) - p'_i/\text{meas}\ \Omega$, $i = 1, \ldots, \kappa_1$, $G_i(x, z, b, v) \equiv G_i(x, z, v) - q'_i/\text{meas}\ \Omega$, $i = \kappa_1 + 1, \ldots, \kappa$, and with the cost functional $\eta^i I_0(w) - \langle \zeta^i, (p', q') \rangle + \frac{1}{2}(I_0(w) - \beta(p^i, q^i))^2 + \frac{1}{2}|p' - p^i|^2 + \frac{1}{2}|q' - q^i|^2$ (see Remark 3.2 for $\phi(I) \equiv \eta^i I + \frac{1}{2}(I - \beta(p^i, q^i))^2$, $\psi(p', q') \equiv -\langle \zeta^i, (p', q') \rangle) + \frac{1}{2}|p' - p^i|^2 + \frac{1}{2}|q' - q^i|^2$). Thus, we can apply Theorem 3.1 to problem (4.10). Taking into account relations (4.7) and Remark 3.2 we obtain the following lemma.

LEMMA 4.2 *Let $w^{i,k}$, $k = 1, 2, \ldots$, be an arbitrary m.s. in the sense of (4.2) ($\epsilon_k = \epsilon_{i,k}$) for problem $(P_{p,q})$ with $(p, q) = (p^i, q^i)$. Then, there exist a sequence of numbers*

$$\gamma^{i,k} \geq 0,\ k = 1, 2, \ldots,\ \gamma^{i,k} \to 0,\ k \to \infty, \tag{4.11}$$

and a sequence of vectors $\mu^{i,k} \in R^{\kappa+1}$,

$$|\mu^{i,k}| = 1,\quad \mu^{i,k}_j \geq 0,\ j = 0, 1, \ldots, \kappa_1,\ \mu^{i,k}_j (I_j(w^{i,k}) - p^i_j) \geq -\gamma^{i,k},$$
$$j = 1, \ldots, \kappa_1, \tag{4.12}$$

such that

$$\int_{Q_T} \max_{v \in U} \{\mu^{i,k}_0 (\eta^i + (I_0(w^{i,k}) - \beta(p^i, q^i)))H(x, t, \xi[w^{i,k}](x, t), v, \eta_0[w^{i,k}](x, t)) +$$
$$\sum_{j=1}^{\kappa} \mu^{i,k}_j H(x, t, \xi[w^{i,k}](x, t), v, \eta_j[w^{i,k}](x, t)) -$$
$$\mu^{i,k}_0 (\eta^i + (I_0(w^{i,k}) - \beta(p^i, q^i)))H(x, t, \xi[w^{i,k}](x, t), u^{i,k}(x, t), \eta_0[w^{i,k}](x, t)) -$$

$$\sum_{j=1}^{\kappa} \mu_j^{i,k} H(x,t,\xi[w^{i,k}](x,t),u^{i,k}(x,t),\eta_j[w^{i,k}](x,t))\}\, dxdt \le \gamma^{i,k},$$

$$\int_\Omega \max_{v\in V}\{\mu_0^{i,k}(\eta^i + (I_0(w^{i,k}) - \beta(p^i,q^i)))H_0(x,z[w^{i,k}](x,T),v,\eta_0[w^{i,k}](x,0))+$$
$$\sum_{j=1}^{\kappa} \mu_j^{i,k} H_j(x,z[w^{i,k}](x,T),v,\eta_j[w^{i,k}](x,0)) -$$
$$\mu_0^{i,k}(\eta^i + (I_0(w^{i,k}) - \beta(p^i,q^i)))H_0(x,z[w^{i,k}](x,T),v^{i,k}(x),\eta_0[w^{i,k}](x,0)) -$$
$$\sum_{j=1}^{\kappa} \mu_j^{i,k} H_j(x,z[w^{i,k}](x,T),v^{i,k}(x),\eta_j[w^{i,k}](x,0))\}\, dx \le \gamma^{i,k},$$

$$\max_{(p',q')\in \overline{S_M^\kappa}} \langle -\mu_0^{i,k}\zeta^i - \sum_{j=1}^{\kappa}\mu_j^{i,k} e^j, (p^i,q^i)-(p',q')\rangle \le \gamma^{i,k},$$

$$e^j \equiv (\overbrace{0,\dots,0}^{j-1},1,0,\dots,0). \tag{4.13}$$

We have $I_0(w^{i,k}) \to \beta(p^i,q^i)$, $k\to\infty$, $(p^i,q^i)\to(p,q)$, $i\to\infty$. Hence, in view of (4.11) there exists a subsequence k_i, $i=1,2,\dots$, of the sequence $k=1,2,\dots$, such that

$$I_j(w^i) \le p_j + (p_j^i - p_j) + \epsilon_{i,k_i},\ \ p_j^i - p_j + \epsilon_{i,k_i} \to 0,\ \ j=1,\dots,\kappa_1, \tag{4.14}$$
$$|I_j(w^i) - q_j| \le |I_j(w^i) - q_j^i| + |q_j^i - q_j| \le \epsilon_{i,k_i} + |q_j^i - q_j|,$$
$$\epsilon_{i,k_i} + |q_j^i - q_j| \to 0,\ \ j=\kappa_1+1,\dots,\kappa,\ \ \epsilon_{i,k_i}\to 0,$$
$$(I_0(w^i) - \beta(p^i,q^i))/|(\zeta^i,-\eta^i)| \to 0,\ \ \gamma^{i,k_i}/|(\zeta^i,-\eta^i)| \to 0,\ \ i\to\infty,\quad w^i \equiv w^{i,k_i}.$$

Moreover, by virtue of (4.3),(4.12),(4.13) and of Proposition 2.5.7 in Clarke (1983) we can assume without loss of generality that $N_C(x)$ is the cone of normals to the set C at x (see Clarke, 1983)

$$\frac{(\zeta^i,-\eta^i)}{|(\zeta^i,-\eta^i)|} \to (\zeta^0,-\eta^0) \in N_{\mathrm{epi}\ \beta}((p,q),\beta(p,q)),\ \ i\to\infty, \tag{4.15}$$
$$\mu_0^{i,k_i} \to 1,\ \ \mu_j^{i,k_i} \to 0,$$
$$\mu_j^{i,k_i}/|(\zeta^i,-\eta^i)| \to \mu_j,\ \ j=1,\dots,\kappa,\ i\to\infty,\qquad (\eta^0,\mu_1,\dots,\mu_\kappa)\ne 0. \tag{4.16}$$

Relations (4.14)-(4.16) allow to obtain the following corollary of Lemma 4.2.

LEMMA 4.3 *Let (p,q) be a point such that $\beta(p,q) < +\infty$. Then there exist a sequence of numbers γ^i, $i=1,2,\dots$,*

$$\gamma^i \ge 0,\quad \gamma^i \to 0,\ i\to\infty, \tag{4.17}$$

a sequence of pairs

$$w^i \in \mathcal{W}_{p,q}^{\gamma^i},\ i=1,2,\dots, \tag{4.18}$$

and a sequence of vectors $\mu^i \in R^{\kappa+1}$, $i = 1, 2, \ldots$,

$$|\mu^i| \neq 0, \quad \mu^i_k \geq 0,\ k = 0, 1, \ldots, \kappa_1, \quad \mu^i_k(I_k(w^i) - p_k) \geq -\gamma^i,$$
$$k = 1, 2, \ldots, \kappa_1, \tag{4.19}$$

such that

$$\int_{Q_T} \max_{v \in U} \sum_{k=0}^{\kappa} \mu^i_k (H(x, t, \xi[w^i](x, t), v, \eta_k[w^i](x, t)) -$$
$$H(x, t, \xi[w^i](x, t), u^i(x, t), \eta_k[w^i](x, t)))\, dxdt \leq \gamma^i, \tag{4.20}$$

$$\int_{\Omega} \max_{v \in V} \sum_{k=0}^{\kappa} \mu^i_k (H_k(x, z[w^i](x, T), v, \eta_k[w^i](x, 0)) -$$
$$H_k(x, z[w^i](x, T), v^i(x), \eta_k[w^i](x, 0)))\, dx \leq \gamma^i, \tag{4.21}$$

$$\zeta^0 + \sum_{j=1}^{\kappa} \mu_j e^j = 0, \tag{4.22}$$

where $\mu \equiv (\eta^0, \mu_1, \ldots, \mu_\kappa) \neq 0$ *is an arbitrary accumulation point of the sequence* μ^i, $i = 1, 2, \ldots$, *and* $(\zeta^0, -\eta^0)$ *is a vector satisfying the relations*

$$|(\zeta^0, -\eta^0)| = 1, \quad (\zeta^0, -\eta^0) \in N_{\text{epi}\ \beta}((p, q), \beta(p, q)). \tag{4.23}$$

DEFINITION 4.1 *A sequence of pairs* $w^i \in \mathcal{W}$, $i = 1, 2, \ldots$, *is called stationary for problem* $(P_{p,q})$ *if there exists a sequence of numbers (4.17), for which all relations (4.18)-(4.21) hold and for the corresponding sequence* μ^i, $i = 1, 2, \ldots$ *all accumulation points are non-zero.*

REMARK 4.1 *In view of Lemma 4.3 and Definition 4.1, there exist stationary sequences for problem* $(P_{p,q})$ *which satisfy also relations (4.22), (4.23).*

Further, we consider the only two cases possible here: $\eta^0 = 0$, $\eta^0 > 0$. To this end define the following sets:

$L^\lambda_{p,q} \equiv \{-\sum_{j=1}^{\kappa} \mu_j e^j \in R^\kappa : \mu \equiv (\eta^0, \mu_1, \ldots, \mu_\kappa) \in R^{\kappa+1}, \mu \neq 0, \eta^0 = \lambda,$ there exists a stationary sequence for problem $(P_{p,q})$ such that the vector μ is an accumulation point of the corresponding sequence of vectors μ^i, $i = 1, 2, \ldots\}$; $\lambda = 0, 1$;

$$M^0_{p,q} \equiv L^0_{p,q} \bigcup \{0\}, \quad M^1_{p,q} \equiv L^1_{p,q}.$$

If $\eta^0 = 0$, then by Lemma 4.3 and by the definition of the asymptotic generalized gradient of Clarke (1983) we have

$$\zeta^0 \in \partial^\infty \beta(p, q) \cap M^0_{p,q}. \tag{4.24}$$

If $\eta^0 > 0$, then by (4.22)

$$\frac{\zeta^0}{\eta^0} = -\sum_{j=1}^{\kappa} \frac{\mu_j}{\eta^0} e^j.$$

Since we have also $\zeta^0/\eta^0 \in \partial\beta(p,q)$, then

$$\frac{\zeta^0}{\eta^0} \in \partial\beta(p,q) \cap M^1_{p,q}. \tag{4.25}$$

Define the sets:

$$N_1 \equiv \{r(\zeta,-1) : r > 0,\ \zeta \in \partial\beta(p,q) \cap M^1_{p,q}\}$$
$$N_2 \equiv \{(\zeta,0) : \zeta \in \partial^\infty\beta(p,q) \cap M^0_{p,q}\}.$$

Obviously, by the definitions of the sets $\partial\beta(p,q)$, $\partial^\infty\beta(p,q)$ we have the inclusion – see Proposition 2.9.6 in Clarke (1983):

$$\overline{\text{conv}}(N_1 \cup N_2) \subset N_{\text{epi } \beta}((p,q),\beta(p,q)).$$

On the other hand, since sequences (p^i,q^i), v^i, ζ^i, η^i, $i = 1,2,\ldots$ in (4.3), (4.4) are arbitrary, by inclusion (4.24), (4.25) and by the limit relation (4.15) we obtain – see Proposition 2.5.7 in Clarke (1983):

$$N_{\text{epi } \beta}((p,q),\beta(p,q)) \subset \overline{\text{conv}}(N_1 \cup N_2).$$

The last inclusion together with the previous one gives us

$$N_{\text{epi } \beta}((p,q),\beta(p,q)) = \overline{\text{conv}}(N_1 \cup N_2). \tag{4.26}$$

LEMMA 4.4 *The set $M^1_{p,q}$ is closed, $0^+M^1_{p,q} \subset M^0_{p,q}$. In the case, where $M^0_{p,q} = \{0\}$ the set $M^1_{p,q}$ is bounded. The set $M^0_{p,q}$ is a closed cone.*

Proof. Since all four assertions can be proved using similar arguments, we confine ourselves to proving the second one.

Recall that the recession cone 0^+C of a set $C \subset R^n$ is given by $0^+C = \{\lim_{i\to\infty} \delta_i y_i : y_i \in C,\ \delta_i > 0,\ \delta_i \to 0,\ i \to \infty\}$.

Let $\lambda \in 0^+M^1_{p,q}$. Then for some sequence of the vectors $\lambda^s \in M^1_{p,q}$, $s = 1,2,\ldots$, we have

$$\delta_s\lambda^s \to \lambda,\ \ \delta_s > 0,\ \ \delta_s \to 0,\ \ s \to \infty.$$

Denoting $\mu^s \equiv (1,\lambda^s)$, we see that for each $s = 1,2,\ldots$ relations (4.17)-(4.21) hold with $\mu^{s,i}$, μ^s, $\gamma^{s,i}$, $w^{s,i}$ substituted by μ^i, μ, γ^i, w^i, respectively. At the same time, $\gamma^{s,i} \to 0$, $w^{s,i} \in \mathcal{W}^{\gamma^{s,i}}_{p,q}$, $i \to \infty$. Without loss of generality assume that $\delta_s \leq 1$, $s = 1,2,\ldots$. Then

$$\delta_s\mu^{s,i} \neq 0,\ \ \delta_s\mu_0^{s,i} \to \delta_s,\ \ \delta_s\mu_j^{s,i} \to \delta_s\lambda_j^s,\ j = 1,\ldots,\kappa,\ i \to \infty,$$
$$\delta_s\lambda_j^s \geq 0,\ \ \delta_s\mu_j^{s,i}(I_j(w^{s,i}) - p_j) \geq -\gamma^{s,i},\ j = 1,\ldots,\kappa_1$$
$$\int_{Q_T} \max_{v\in U} \sum_{j=0}^{\kappa} \delta_s\mu_j^{s,i}(H(x,t,\xi[w^{s,i}](x,t),v,\eta_j[w^{s,i}](x,t)) -$$
$$H(x,t,\xi[w^{s,i}](x,t),u^{s,i}(x,t),\eta_j[w^{s,i}](x,t)))\,dxdt \leq \gamma^{s,i},$$

$$\int_{\Omega} \max_{v \in V} \sum_{j=0}^{\kappa} \delta_s \mu_j^{s,i} (H_j(x, z[w^{s,i}](x,T), v, \eta_j[w^{s,i}](x,0)) -$$
$$H_j(x, z[w^{s,i}](x,T), v^{s,i}(x), \eta_j[w^{s,i}](x,0))) \, dx \leq \gamma^{s,i}.$$

Choosing a subsequence i_s, $s = 1, 2, \ldots$, of the sequence $i = 1, 2, \ldots$, such that $\gamma^{s,i_s} \to 0$, $\delta_s \mu_0^{s,i_s} \to 0$, $\delta_s \mu_j^{s,i_s} \to \lambda_j$, $j = 1, \ldots, \kappa$, $s \to \infty$, we verify easily that $\lambda \in M_{p,q}^0$. Thus the lemma is proved. □

Further, we note that the closed cone $D^\infty \equiv \partial^\infty \beta(p,q) \cap M_{p,q}^0$ contains always zero and the recession cone $0^+ D$ of the closed set $D \equiv \partial \beta(p,q) \cap M_{p,q}^1$. Indeed, we have $0^+ D \subset 0^+ \partial \beta(p,q) \cap 0^+ M_{p,q}^1$. Moreover, it is known (see for example Clarke, 1983) that the inclusion $0^+ \partial \beta(p,q) \subset \partial^\infty \beta(p,q)$ always holds. Besides that, according to Lemma 4.4 we have $0^+ M_{p,q}^1 \subset M_{p,q}^0$. Thus, $0^+ D \subset D^\infty$. The last relation, together with equality (4.26) permits to apply Proposition 15 from Rockafellar (1982) (see also Proposition 3.5 in Clarke and Loewen, 1986) and to obtain the following theorem.

THEOREM 4.1 *If $\beta(p,q) < +\infty$, then $M_{p,q}^1 \bigcup M_{p,q}^0 \setminus \{0\} \neq \emptyset$ and the Clarke's generalized gradient $\partial \beta(p,q)$ of the value function β at (p,q) is equal to*

$$\partial \beta(p,q) = \overline{conv} \{ \partial \beta(p,q) \bigcap M_{p,q}^1 + \partial^\infty \beta(p,q) \bigcap M_{p,q}^0 \}, \tag{4.27}$$

where $\partial^\infty \beta(p,q)$ is the Clarke's asymptotic generalized gradient of β at (p,q).

5. Conditions of regularity and normality in suboptimal control theory

In this section we consider various conditions of regularity and normality of problem $(P_{p,q})$. The following definition (Sumin, 1995, 1996b) generalizes the well-known classical concepts (see for example Warga (1971), Clarke (1983)).

DEFINITION 5.1 *A stationary sequence $w^i \in \mathcal{W}_{p,q}^{\gamma^i}$, $i = 1, 2, \ldots$, $\gamma^i \geq 0$, $\gamma^i \to 0$, $i \to \infty$, for problem $(P_{p,q})$ is called normal (regular, abnormal), if all (there exist, do not exist) sequences μ^i, $i = 1, 2, \ldots$, have (having, having) accumulation points μ with the component $\mu_0 \neq 0$ only (with the component $\mu_0 \neq 0$ only, with the component $\mu_0 \neq 0$). The problem $(P_{p,q})$ is called normal (regular, abnormal) if all its stationary sequences are normal (regular, abnormal).*

LEMMA 5.1 *Let $\beta(p,q) < +\infty$ and $M_{p,q}^0 = \{0\}$ (i.e., problem $(P_{p,q})$ is normal). Then the function β is Lipschitz in a neighborhood of (p,q).*

Proof. Since $M_{p,q}^0 = \{0\}$, then all sets $M_{p',q'}^1$ are bounded uniformly with respect to $(p', q') \in O_{p,q}$, where $O_{p,q}$ is a neighborhood of (p,q). The proof of this fact is similar to the proof of Lemma 4.4. By this remark the assertion of the lemma is a simple corollary of Propositions 2.3.7, 2.9.7 in Clarke (1983) and of equality (4.27). □

This lemma is one of many important corollaries of Theorem 4.1. It gives a necessary condition of normality of problem $(P_{p,q})$. Note that Theorem 4.1 and Lemma 5.1 are suboptimal analogues of certain corresponding results of Clarke (1983), expressed in terms of usual optimal controls for controlled systems of ordinary differential equations.

On the other hand, we can get the following sufficient condition of regularity for problem $(P_{p,q})$ expressed in terms of perpendiculars to epi β. Its proof is similar to the proof of Lemma 4.2.

LEMMA 5.2 *If $(\zeta, -\eta) \perp \operatorname{epi} \beta$ at $((p,q), \beta(p,q))$ and $\eta > 0$ then problem $(P_{p,q})$ is regular.*

Other sufficient condition of regularity of problem $(P_{p,q})$ is connected with the existence of a Kuhn-Tucker vector. To this end we generalize the classical concept of Kuhn-Tucker vector.

DEFINITION 5.2 *A vector $\mu \in \Lambda \equiv \{(\lambda_1, \ldots, \lambda_\kappa) \in R^\kappa : \lambda_1 \geq 0, \ldots, \lambda_{\kappa_1} \geq 0\}$ is called a Kuhn-Tucker vector of problem $(P_{p,q})$, if it satisfies the inequality*

$$\beta(p,q) \leq I_0(w) + \sum_{i=1}^{\kappa_1} \mu_i (I_i(w) - p_i) + \sum_{i=\kappa_1+1}^{\kappa} \mu_i (I_i(w) - q_i) \quad \forall w \in \mathcal{W}. \tag{5.1}$$

To prove the above mentioned sufficient condition of regularity, we shall use the following criterion of perpendicularity.

LEMMA 5.3 *A nonzero vector v is perpendicular to a set C at a point $x \in \operatorname{cl} C$ if and only if*

$$\langle v, c - x \rangle < |x - c|^2/2 \quad \forall c \in \operatorname{cl} C, \ c \neq x.$$

Proof. The necessity follows from Proposition 2.5.5 in Clarke (1983). We shall prove the sufficiency. Let $x' = v + x$. Then, it follows from the inequality of the lemma that

$$\langle x' - x, x' - x + c - x' \rangle < |x' - c - (x' - x)|^2/2 \quad \forall c \in \operatorname{cl} C, \ c \neq x,$$

and hence

$$0 < |x' - x| < |x' - c| \quad \forall c \in \operatorname{cl} C, \ c \neq x, \quad x' \notin \operatorname{cl} C.$$

Thus, the point $x' \notin \operatorname{cl} C$ has a unique nearest point x in $\operatorname{cl} C$. It means that the vector $v = x' - x$ is perpendicular to $\operatorname{cl} C$ at x. □

LEMMA 5.4 *If $\mu \in \Lambda$ is a Kuhn-Tucker vector of problem $(P_{p,q})$, then $-\mu \in M^1_{p,q}$ and $(-\mu, -1) \perp \operatorname{epi} \beta$ at $((p,q), \beta(p,q))$.*

Proof. It follows from (5.1) that

$$\beta(p,q) + \sum_{i=1}^{\kappa_1} \mu_i p_i + \sum_{i=\kappa_1+1}^{\kappa} \mu_i q_i \le I_0(w) + \sum_{i=1}^{\kappa} \mu_i I_i(w) \quad \forall w \in \mathcal{W},$$

and, consequently,

$$\beta(p,q) - \langle -\mu, (p,q) \rangle \le I_0 - \langle -\mu, (p',q') \rangle \quad \forall (p',q') \in \mathrm{dom}\, \beta,\ I_0 \ge \beta(p',q')$$

or

$$\beta(p,q) - \langle -\mu, (p,q) \rangle < I_0 - \langle -\mu, (p',q') \rangle + \frac{1}{2} |((p',q') - (p,q), I_0 - \beta(p,q))|^2$$
$$\forall (p',q') \in \mathrm{dom}\, \beta,\ I_0 \ge \beta(p',q'),\ ((p',q'), I_0) \ne ((p,q), \beta(p,q)).$$

From here we conclude that

$$\langle (-\mu, -1), ((p',q'), I_0) - ((p,q), \beta(p,q)) \rangle < \frac{1}{2} |((p',q'), I_0) - ((p,q), \beta(p,q))|^2$$
$$\forall (p',q') \in \mathrm{dom}\, \beta,\ I_0 \ge \beta(p',q'),\ ((p',q'), I_0) \ne ((p,q), \beta(p,q)).$$

From the last strict inequality and from Lemma 5.3 it follows that the vector $(-\mu, -1)$ is perpendicular to the set epi β at the point $((p,q), \beta(p,q))$. Therefore, by the argument of the proof of Lemma 4.2, we get $-\mu \in M^1_{p,q}$. The lemma is proved. □

The following lemma connects conditions of normality of problem $(P_{p,q})$ with the magnitude of the gap $\beta_0(p,q) - \beta(p,q)$ and generalizes to the considered situation the corresponding result in Warga (1971), Theorem V.3.4. The proof of the lemma is omitted due to the lack of the space. It can be found in Sumin (1995,1996b).

LEMMA 5.5 *If the strict inequality $\beta(p,q) < \beta_0(p,q)$ holds for problem $(P_{p,q})$, then any sequence $w^i \in \mathcal{W}$, $i = 1, 2, \ldots$ satisfying the relations*

$$I_0(w^i) \to \overline{\beta} \in [\beta(p,q), \beta_0(p,q)],\ I_0(w^i) \le \beta_0(p,q) + \epsilon_i,\ w^i \in \mathcal{W}^{\epsilon_i}_{p,q},$$
$$\epsilon_i \ge 0,\ \epsilon_i \to 0,\ i \to \infty$$

is a stationary one. At the same time, it is not a normal stationary sequence if $\overline{\beta} \in [\beta(p,q),\ \beta_0(p,q))$.

COROLLARY 5.1 *The strict inequality $\beta(p,q) < \beta_0(p,q)$ for problem $(P_{p,q})$ does not hold at least in the following two cases, where: 1) all stationary sequences are normal ones; 2) there exists a normal minimizing sequence in the sense of (4.2).*

Further, we shall prove two lemmas that constitute sufficient conditions of normality for problem $(P_{p,q})$ and generalize to the case of suboptimal theory the classic conditions of normality in mathematical programming (condition of Slater and condition of linearity).

LEMMA 5.6 *Let in problem* $(P_{p,q})$ *the equality constraints be absent, i.e.,* $\kappa = \kappa_1$, *and the initial data have the form:* $b_i(x,t,u) = b_i(x,t)$, $i = 1,\dots,n$, $a(x,t,u) = a(x,t)$. *Moreover the functions* G_i *are convex in* z, $i = 1,\dots,\kappa$. *If there exists a pair* $w^0 \in \mathcal{W}$ *such that* $I_i(w^0) < p_i$, $i = 1,\dots,\kappa$, *then problem* $(P_{p,q}) = (P_p)$ *is normal.*

Proof. Assume that the assertion of the lemma is not true. Let $w^i \in \mathcal{W}$, $i = 1,2,\dots$, be a stationary sequence for problem (P_p) such that the corresponding sequence of vectors $\mu^i \in R^{\kappa+1}$, $i = 1,2,\dots$, has an accumulation point μ with the component $\mu_0 = 0$. Then, by virtue of (3.12), (3.13) and the assumptions of the lemma we have

$$\mu_0^i(I_0(w^0) - I_0(w^i)) + \sum_{k=1}^{\kappa} \mu_k^i(I_k(w^0) - I_k(w^i)) =$$

$$\mu_0^i(\mathcal{H}_0[w^0,w^i] + \mathcal{E}_0[w^0,w^i]) + \sum_{k=1}^{\kappa} \mu_k^i(\mathcal{H}_k[w^0,w^i] + \mathcal{E}_k[w^0,w^i]) \geq$$

$$\sum_{k=0}^{\kappa} \mu_k^i \mathcal{H}_k[w^0,w^i] + \mu_0^i \mathcal{E}_0[w^0,w^i],$$

where

$$\begin{aligned}\mathcal{H}_k[w^2,w^1] = \int_{Q_T} &(H(x,t,\xi[w^1](x,t),u^1(x,t),\eta_k[w^1](x,t)) - \\ &H(x,t,\xi[w^1](x,t),u^2(x,t),\eta_k[w^1](x,t)))\,dxdt + \\ &\int_\Omega (H_k(x,z[w^1](x,T),v^1(x),\eta_k[w^1](x,0)) - \\ &H_k(x,z[w^1](x,T),v^2(x),\eta_k[w^1](x,0)))\,dx,\end{aligned} \tag{5.2}$$

$$\mathcal{E}_k[w^2,w^1] = \int_{Q_T} E_{\overline{a}}[w^2,w^1](x,t)\eta_k[w^1](x,t)\,dxdt + \int_\Omega E_{G_k}[w^2,w^1](x)\,dx,$$

$E_{\overline{a}}$, E_{G_k} are corresponding E-functions of Weierstrass-Plotnikov (see Theorem 3.2). From here due to the assumptions of the lemma, stationarity of the sequence w^i and the convergence $\mu_0^i \to 0$, $i \to \infty$, for all i sufficiently large we have

$$\sum_{k=0}^{\kappa} \mu_k^i \mathcal{H}_k[w^0,w^i] \leq$$

$$-\mu_0^i \mathcal{E}_0[w^0,w^i] + \mu_0^i(I_0(w^0) - I_0(w^i)) + \kappa\gamma^i + \sum_{k=1}^{\kappa} \mu_k^i(I_k(w^0) - p_k) < -\alpha,$$

where $\alpha > 0$ is some number. The last inequality contradicts the stationarity of the sequence w^i. The lemma is proved. □

LEMMA 5.7 *Let the functions $b_i(x,t,u)$, $i=1,\dots,n$, $a(x,t,u)$ in the problem $(P_{p,q})$ be the same as in Lemma 5.6, $G_i(x,z,v) = G_i^1(x)z + G_i^2(x,v)$, $i = 1,\dots,\kappa$, and there exists a nonstationary sequence $w^i \in \mathcal{W}_{p,q}^{\gamma^i}$, $i=1,2,\dots$, $\gamma^i \geq 0$, $\gamma^i \to 0$, $i \to \infty$. Then problem $(P_{p,q})$ is normal.*

Proof. Assume that the assertion of the lemma is not true. Let $w^i \in \mathcal{W}_{p,q}^{\gamma^i}$, $i = 1,2,\dots$, $\gamma^i \geq 0$, $\gamma^i \to 0$, $i \to \infty$, be a stationary sequence for problem $(P_{p,q})$ such that the corresponding sequence of vectors $\mu^i \in R^{\kappa+1}$, $i=1,2,\dots$, satisfies the relations

$$\mu^i \to \mu,\ \mu \neq 0,\ \mu_0^i \to 0,\ \mu_k^i(I_k(w^i) - p_k) \geq -\gamma^i,\ k=1,\dots,\kappa_1.$$

Let also $\overline{w}^i \in \mathcal{W}_{p,q}^{\overline{\gamma}^i}$, $i=1,2,\dots$, $\overline{\gamma}^i \geq 0$, $\overline{\gamma}^i \to 0$, $i \to \infty$, be a given sequence. By virtue of (3.12),(3.13) and the assumptions of the lemma we get (see also (5.2))

$$\mu_0^i(I_0(\overline{w}^i) - I_0(w^i)) + \sum_{k=1}^{\kappa} \mu_k^i(I_k(\overline{w}^i) - I_k(w^i)) =$$

$$\mu_0^i(\mathcal{H}_0[\overline{w}^i, w^i] + \mathcal{E}_0[\overline{w}^i, w^i]) + \sum_{k=1}^{\kappa} \mu_k^i(\mathcal{H}_k[\overline{w}^i, w^i] + \mathcal{E}_k[\overline{w}^i, w^i]) =$$

$$\mu_0^i(\mathcal{H}_0[\overline{w}^i, w^i] + \mathcal{E}_0[\overline{w}^i, w^i]) + \sum_{k=1}^{\kappa} \mu_k^i\mathcal{H}_k[\overline{w}^i, w^i]. \tag{5.3}$$

From here by the convergence $\mu_0^i \to 0$, $i \to \infty$, and stationarity of the sequence w^i, it follows

$$-\beta^i \leq \sum_{k=0}^{\kappa} \mu_k^i\mathcal{H}_k[\overline{w}^i, w^i] = -\mu_0^i\mathcal{E}_0[\overline{w}^i, w^i] + \sum_{k=0}^{\kappa} \mu_k^i(I_k(\overline{w}^i) - I_k(w^i)) \leq$$

$$-\mu_0^i\mathcal{E}_0[\overline{w}^i, w^i] + \mu_0^i(I_0(\overline{w}^i) - I_0(w^i)) + \sum_{k=1}^{\kappa_1} \mu_k^i\overline{\gamma}^i + \kappa_1\gamma^i +$$

$$\sum_{k=\kappa_1+1}^{\kappa} \mu_k^i(I_k(\overline{w}^i) - I_k(w^i)), \tag{5.4}$$

where $\beta^i \geq 0$, $\beta^i \to 0$, $i \to \infty$, is a sequence of numbers. Consequently, we have

$$\sum_{k=0}^{\kappa} \mu_k^i\mathcal{H}_k[\overline{w}^i, w^i] \equiv \{\sum_{k=0}^{\kappa} \mu_k^i \int_{Q_T} (H(x,t,\xi[w^i](x,t), u^i(x,t), \eta_k[w^i](x,t)) -$$

$$H(x,t,\xi[w^i](x,t), \overline{u}^i(x,t), \eta_k[\pi^i](x,t)))\, dxdt\} +$$

$$\{\sum_{k=0}^{\kappa} \mu_k^i \int_{\Omega} (H_k(x, z[w^i](x,T), v^i(x), \eta_k[w^i](x,0)) -$$

$$H_k(x, z[w^i](x,T), \overline{v}^i(x), \eta_k[w^i](x,0)))\, dx\} \equiv \{\alpha_i^1\} + \{\alpha_i^2\} \to 0,\ i \to \infty.$$

From here, in view of the stationarity of the sequence w^i, it follows

$\alpha_i^1, \alpha_i^2 \to 0, \ i \to \infty.$

In turn, due to the last limit relation, independence of the adjoint functions $\eta_k[w]$ of $w \in \mathcal{W}$ for $k = 1, \ldots, \kappa$ (this follows from the assumptions of the lemma) and the convergence $\mu_0^i \to 0, \ i \to \infty$ we conclude that the sequence $\overline{w}^i, \ i = 1, 2, \ldots$, is also a stationary one, since due to (5.3),(5.4), we can write

$$\mu_k^i(I_k(\overline{w}^i) - p_k) \geq \mu_k^i(I_k(\overline{w}^i) - p_k - \overline{\gamma}^i) \geq \sum_{k=1}^{\kappa_1} \mu_k^i(I_k(\overline{w}^i) - p_k - \overline{\gamma}^i) =$$

$$-\mu_0^i(I_0(\overline{w}^i) - I_0(w^i)) + \sum_{k=1}^{\kappa_1} \mu_k^i(I_k(w^i) - p_k) - \sum_{k=\kappa_1+1}^{\kappa} \mu_k^i(I_k(\overline{w}^i) - I_k(w^i)) +$$

$$\sum_{k=0}^{\kappa} \mu_k^i \mathcal{H}_k[\overline{w}^i, w^i] + \mu_0^i \mathcal{E}_0[\overline{w}^i, w^i] - \sum_{k=1}^{\kappa_1} \mu_k^i \overline{\gamma}^i \geq -\overline{\alpha}_i, \ \ k = 1, \ldots, \kappa_1,$$

for some sequence of numbers $\overline{\alpha}_i \geq 0, \ \overline{\alpha}_i \to 0, \ i \to \infty$. The last inequality contradicts the assumptions of the lemma. The lemma is proved. □

Finally, we formulate the following lemma improving Theorem 4.1 for "linear-convex" case of problem $(P_{p,q})$. The lemma is a corollary of Theorem 3.2 and of Lemmas 5.4, 5.6, 5.7. The proof is omitted for the same reason as that of Lemma 5.5.

LEMMA 5.8 *Let problem $(P_{p,q})$ be normal and the initial data be the same as in Theorem 3.2. Then the equality*

$\partial\beta(p, q) = -M_{p,q}^1 = M_{p,q}$

holds, where $M_{p,q}$ is the set of all Kuhn-Tucker vectors of problem $(P_{p,q})$.

6. Illustrative examples

Example 6.1 To illustrate Lemma 4.1 we consider the following optimal control problem

$$I_0(v) \equiv \int_0^l (z^2(x,T) - v^2(x))\,dx \to \inf, \quad I_1(v) \equiv \int_0^l z^2(x,T)\,dx = q, \ l, T > 0,$$

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \ z(x,0) = v(x), \ x \in (0,l), \ z(0,t) = z(l,t) = 0, \ t \in (0,T), \ V = [-1,1].$$

It is easy to see that obvious equality $\beta_0(0) = 0$ holds. But on the other hand the limit relations $I_0(v^i) \to -l, \ I_1(v^i) \to 0$ for $i \to \infty$ are fulfilled for example for the sequence $v^i \in \mathcal{W}_0^{\epsilon_i}$ of the following elements

$$v^i(x) \equiv \begin{cases} 1 & 2k/2i \leq x < (2k+1)/2i, \\ -1 & (2k+1)/2i \leq x < (2k+2)/2i, \ \ k = 0, 1, \ldots, i-1, \end{cases} \tag{6.1}$$

where $\epsilon_i \to 0$ is a suitable sequence of positive numbers. Hence, for $q \to +0$ the function $\beta_0(q)$ takes on values approaching $-l$, so it is not lower semicontinuous.

Example 6.2 Consider the optimal control problem $(P_{p,q})$ in the following form

$$I_0(v) \equiv \int_0^l (z^2(x,T) - v^2(x))\,dx \to \inf, \quad I_1(v) \equiv \int_0^l z(x,T)\,dx = q \in [-1,1],$$

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2},\ z(x,0) = v(x),\ x \in (0,l),\ z(0,t) = z(l,t) = 0,\ t \in (0,T),\ V = [-1,1].$$

The relations of Theorem 3.1 have the form

$$\begin{aligned}\int_0^l & \max_{v\in V}\{\mu_0^i(\eta_0[v^i](x,0)(v - v^i(x)) + v^2 - (v^i(x))^2) + \\ &\mu_1^i\eta_1[v^i](x,0)(v - v^i(x)\}\,dx \le \gamma^i, \\ &(\mu_0^i,\mu_1^i) \ne 0,\ \mu_0^i \ge 0,\ |I_1(v^i) - q| \le \gamma^i,\ \gamma^i \to 0,\ i \to \infty,\end{aligned} \tag{6.2}$$

where the adjoint functions $\eta_0[v^i]$, $\eta_1[v^i]$ satisfy the boundary-value problem

$$\frac{\partial \eta}{\partial t} + \frac{\partial^2 \eta}{\partial x^2} = 0,\ \eta(x,T) = \psi(x),\ x \in (0,l),\ \eta(0,t) = \eta(l,t) = 0,\ t \in (0,T)$$

for $\psi(x) = -2z[v](x,T)$, $v = v^i$ and $\psi(x) = -1$, $x \in (0,l)$ respectively.

Due to elementary properties of solutions of these primal and adjoint boundary-value problems

$$\|\eta_0[v^i](\cdot,0)\|_{2,(0,l)},\ \|z[v^i](\cdot,T)\|_{2,(0,l)} \to 0, \tag{6.3}$$

if $v^i \to 0$, $i \to \infty$ weakly in $L_2(0,l)$. Therefore, in the case that $q = 0$, the sequence (6.1) satisfies the relations (6.2) if $\mu_1^i \to 0$, $i \to \infty$. For this reason if $q = 0$, then (6.1) is a stationary sequence for our problem. At the same time, by Theorem 3.2 this sequence is a minimizing one.

Calculate $\partial\beta(0)$. First note that the control $v(x) = 0$ satisfies the equality $I_1(0) = 0$ but it is not a stationary one for the considered problem at $q = 0$ (this fact can be easily verified). Then due to Lemma 5.7 our problem is normal for $q = 0$ and we have $M_0^0 = \{0\}$.

If $\mu_0^i \to 1$, $i \to \infty$ then an elementary analysis shows that only a sequence of controls v^i, $i = 1,2,\dots$ such that $|v^i(x)| \to 1$ for a.e. $x \in (0,l)$ can satisfy the relations (6.2). Let $A[v](x) \equiv z[v](x,T)$. Since $A : L_2(0,l) \to L_2(0,l)$ is a linear bounded operator and $\eta_0[v^i](x,0) = -2A^*[z[v^i](\cdot,T)](x)$, $\eta_1[v^i](x,0) = -A^*[1](x)$, then in view of (6.2) we conclude that

$$\begin{aligned}\int_0^l & \max_{v\in V}\{\mu_0^i(\eta_0[v^i](x,0)(v - v^i(x)) + v^2 - (v^i(x))^2) + \\ &\mu_1^i\eta_1[v^i](x,0)(v - v^i(x)\}\,dx \ge \\ &\int_0^l (-2(\mu_0^i\eta_0[v^i](x,0) + \mu_1^i\eta_1[v^i](x,0))v^i(x))\,dx =\end{aligned} \tag{6.4}$$

$$\int_0^l (2(2\mu_0^i A^*[z[v^i](\cdot,T)](x) + \mu_1^i A^*[1](x))v^i(x))\,dx =$$
$$2\int_0^l (2\mu_0^i z^2[v^i](x,T) + \mu_1^i z[v^i](x,T))\,dx.$$

Therefore, since $I_1(v^i) \to 0,\ i \to \infty$, then in view of (6.2) and (6.4), we find that $\|z[v^i](\cdot,T)\|_{2,(0,l)} \to 0,\ i \to \infty$. On the other hand, the first limit relation (6.3) obviously holds. This means that

$$\int_0^l \max_{v\in V}\{\mu_0^i(v^2 - (v^i(x))^2) + \mu_1^i \eta_1[v^i](x,0)(v - v^i(x)\}\,dx \le \gamma^i,$$

and, consequently, (since $|v^i(x)| \to 1,\ i \to \infty$)

$$\int_0^l \max_{v\in V}\{\mu_1^i \eta_1[v^i](x,0)(v - v^i(x)\}\,dx \le \gamma^i, \tag{6.5}$$

where $\gamma^i \ge 0,\ \gamma^i \to 0,\ i \to \infty$. At the same time, since our problem for $q = 0$ is normal, in the case $\mu_1^i \to a \ne 0,\ i \to \infty$, for the same sequence of the controls v^i we have

$$\int_0^l \max_{v\in V}\{\mu_1^i \eta_1[v^i](x,0)(v - v^i(x)\}\,dx \ge \alpha > 0,\ i = 1,2,\ldots$$

The last relation contradicts (6.5). Thus, in the case $\mu_0^i \to 1,\ i \to \infty$, we must have the limit relation $\mu_1^i \to 0,\ i \to \infty$ which means that $M_0^1 = \{0\}$. Then, by virtue of Lemma 5.8, we find that $\partial\beta(0) = M_0^1 = M_0 = \{0\}$, where M_0 is the set of Kuhn-Tucker vectors of our problem.

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