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Regularity theory, exact controllability, and optimal quadratic cost problem for spherical shells with physical boundary controls

by

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Abstract. We consider an established model of a thin, shallow spherical shell. Complementing Lasiecka and Triggiani (1996a), Lasiecka, Triggiani and Valente (1996) — which referred to well-posedness and uniform stabilization of the closed-loop problem with boundary dissipation — in this paper we consider the corresponding open-loop problem, with physical boundary controls as forces, moments and shear forces. We prove an optimal (boundary and, by duality) interior regularity result, as well as exact controllability on $[0, T]$, T sufficiently large. That is, the input-solution operator (with zero initial conditions) is both continuous and surjective from the space of controls to the state space, at time $t = T$. As a consequence, the abstract theory from Flandoli, Lasiecka and Triggiani (1988) of optimal control with quadratic cost over an infinite time horizon and corresponding algebraic Riccati equation is applicable to such spherical shells.

Keywords: spherical shells, exact controllability, partial differential equations

1. Introduction

1.1. The model

We consider a thin elastic spherical shell whose reference configuration in spherical coordinates (r, θ, ϕ) is the region: $r \in [R-h, R+h]$, $\theta \in [0, \theta_0]$, $\phi \in [0, 2\pi)$, where the half-thickness h of the shell, the middle surface ray R, and the opening angle $\theta_0 < \pi$ are given. A spherical shell has two characteristic parameters, *thinness* and *shallowness* defined respectively by: $\eta = h/R\sin\theta_0$; and $\beta = (R - R\cos\theta_0)/R\sin\theta_0$. We restrict our interest to a *thin* and *shallow* spherical shell, so that for some fixed $\ell > 0$, we take $\ell = R\sin\theta_0$ and suppose $\eta = h/\ell << 1$ and $\beta << 1$. This latter condition implies θ_0 sufficiently small. Starting from the Koiter linear shell model as in Geymonat, Loreti and Valente

(1993), Koiter (1970), Love (1927) we arrive at the shallow shell approximation by introducing the coordinate $\rho = R\theta$ and by replacing $\cot\theta$ by $\frac{1}{\theta}$. Then, the axially symmetric vibration for the meridional and radial middle surface displacements (x, y) can be written in the following form on $(0,T) \times (0, \rho_0)$:

$$\begin{cases} x_{tt} + \frac{e}{R} z_{tt} - L(x) - \frac{e}{R} L(z) + \frac{(1+\nu)}{R} y' = 0; & (1.1a) \\ y_{tt} - \frac{e}{\rho}[z_{tt}\rho]' + \frac{e}{\rho}[L(z)\rho]' - \frac{(1+\nu)}{\rho R}(x\rho)' + \frac{2(1+\nu)}{R^2} y = 0; & (1.1b) \end{cases}$$

$$z \equiv \frac{x}{R} + y'; \; L(x) \equiv x'' + \frac{x'}{\rho} - \frac{x}{\rho^2}; \; \rho_0 = R\theta_0; \; e = \frac{h^2}{3}; \; 0 < \nu < 1; \tag{1.1c}$$

$\nu =$ Poisson's ratio, where the prime symbol $'$ denotes differentiation with respect to ρ, along with the initial conditions

$$x(0, \cdot\,) = x_0, \; x_t(0, \cdot\,) = x_1; \; y(0, \cdot\,) = y_0. \; y_t(0, \cdot\,) = y_1, \tag{1.1d}$$

the following boundary conditions at $\rho = 0$:

$$x = y' = L(z) = 0, \qquad \rho = 0, \; t > 0, \tag{1.1e}$$

and the following non-homogeneous boundary conditions at $\rho = \rho_0$:

$$\begin{cases} x' - \frac{(1+\nu)}{R} y + \frac{\nu}{\rho_0} x + x = g_1; & (1.1f) \\ ez' = g_2; \qquad \rho = \rho_0, \; t > 0, & (1.1g) \\ eL(z) - ez_{tt} = g_3; & (1.1h) \end{cases}$$

where $g_i \in L_2(0,T)$ are the boundary control functions. To the mixed problem (1.1) we shall associate the corresponding boundary homogeneous problem on $(0,T) \times (0, \rho_0)$:

$$\begin{cases} u_{tt} + \frac{e}{R} v_{tt} - L(u) - \frac{e}{R} L(v) + \frac{(1+\nu)}{R} w' = 0; & (1.2a) \\ w_{tt} - \frac{e}{\rho}[v_{tt}\rho]' + \frac{e}{\rho}[L(v)\rho]' - \frac{(1+\nu)}{\rho R}(u\rho)' + \frac{2(1+\nu)}{R^2} w = 0; & (1.2b) \end{cases}$$

$$v \equiv \frac{u}{R} + w'; \; L(u) \equiv u'' + \frac{u'}{\rho} - \frac{u}{\rho^2}; \; \rho_0 = R\theta_0, \tag{1.2c}$$

where the prime symbol $'$ denotes differentiation with respect to ρ, along with the initial conditions at $t = T$:

$$u(T, \cdot\,) = u_0, \; u_t(T, \cdot\,) = u_1; \; w(T, \cdot\,) = w_0, \; w_t(T, \cdot\,) = w_1, \tag{1.2d}$$

and the following homogeneous boundary conditions at $\rho = 0$:

$$u = w' = L(v) = 0, \qquad \rho = 0,\ t > 0, \tag{1.2e}$$

and the following dissipative boundary conditions at $\rho = \rho_0$:

$$\begin{cases} u' - \dfrac{(1+\nu)}{R} w + \nu \dfrac{u}{\rho_0} + u &= 0; & (1.2f) \\ ev' &= 0; \qquad \rho = \rho_0,\ t > 0, & (1.2g) \\ eL(v) - ev_{tt} &= 0; & (1.2h) \end{cases}$$

which is, in fact, the adjoint problem corresponding to (1.1). (Since the equations are time-reversible, one may equivalently consider the initial conditions at $t = 0$, without affecting regularity properties.)

1.2. The homogeneous problem (1.2) is conservative

With system (1.2) we associate the energy functional

$$E(t) = E_k(t) + E_p(t) = E(t; u, w); \tag{1.3}$$

$$2E_k(t) \equiv \int_0^{\rho_0} [u_t^2 + w_t^2 + ev_t^2]\rho\, d\rho \tag{1.4}$$

$$\begin{aligned} 2E_p(t) \equiv{}& e \int_0^{\rho_0} \left[(v')^2 \rho + \frac{v^2}{\rho} \right] d\rho \\ &+ (1-\nu) \int_0^{\rho_0} \left[\left(u' - \frac{w}{R} \right)^2 \rho + \left(\frac{u}{\rho} - \frac{w}{R} \right)^2 \rho \right] d\rho \\ &+ \int_0^{\rho_0} \left[\left(u' - \frac{w}{R} \right) \sqrt{\rho} + \left(\frac{u}{\rho} - \frac{w}{R} \right) \sqrt{\rho} \right]^2 d\rho + u^2(\rho_0)\rho_0. \end{aligned} \tag{1.5}$$

We then have problem (1.2) is conservative:

$$E(t) \equiv E(0), \qquad t \in R. \tag{1.6}$$

The proof of (1.6) is given in Lasiecka, Triggiani and Valente (1996), Theorem 1.3(iii); Section 6.

1.3. Regularity and exact controllability of problem (1.1a-h)

References Lasiecka and Triggiani (1996a), Lasiecka, Triggiani and Valente (1996) studied (i) the well-posedness and (ii) the uniform stabilization problem of the corresponding (closed loop) boundary feedback dissipative system, which is obtained from (1.1a-h) by setting

$$g_1 = -x_t;\ g_2 = -z_t;\ g_3 = y_t,\ \rho = \rho_0,\ t > 0, \tag{1.7}$$

is a natural finite energy space. This turns out to be the space $\mathcal{E} = [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2] \times \mathcal{V}_\rho^1$ recalled below in (1.15), which is norm equivalent to the finite energy space, see (1.18). It was explicitly noted in Lasiecka, Triggiani and Valente (1996), Corollary 1.5 and Remark 1.1, that the obtained uniform stabilization of the closed-loop problem (1.1a-h), (1.7) implies *a-fortiori* a corresponding exact controllability result for the open-loop problem (1.1a-h), on the same state space $\mathcal{E}$ and with $L_2(0,T)$-controls g_i, $i = 1,2,3$. Here T is larger than a suitable $T_0 > 0$ which depends on the finite speed of propagation of the two hyperbolic equations (1.1a) and (1.1b). Indeed, such implication may go in two ways: (i) either by invoking a well-known (constructive) general result in Russel (1974), according to which uniform stabilization of conservative problems implies exact controllability on the same state space on a sufficiently large time-interval; or else (ii) by specializing the energy estimates analysis, carried out in Lasiecka, Triggiani and Valente (1996) for the closed-loop feedback problem (1.1a-h), (1.7), to the corresponding homogeneous problem (1.2a-h).

1.4. Goal

We can now state the three-fold goal of the present note:

(a) to carry out the strategy in (ii) above and provide a *direct* proof of the exact controllability problem of (1.1a-h), without passing through uniform stabilization as in Russel (1974);

(b) to provide the complementary result of *optimal* regularity of the open-loop problem (1.1a-h) on the same space $\mathcal{E}$, with controls $g_i \in L_2(0,T)$;

(c) to extend to problem (1.1a-h) the optimal control theory with quadratic cost over an infinite time horizon, and corresponding algebraic Riccati equation, as developed abstractly in Flandoli, Lasiecka and Triggiani (1988), see also Lasiecka and Triggiani (1991), Lasiecka and Triggiani (1996c), as a consequence of exact controllability and regularity on the same (optimal) space $\mathcal{E}$.

In synthesis: the mixed problem (1.1a-h) for the thin, shallow spherical shell can be rewritten abstractly as the operator equation (see Section 3 below),

$$\dot{\xi} = \mathcal{A}\xi + \mathcal{B}g \in [\mathcal{D}(\mathcal{A}^*)]', \qquad \xi(0) = \xi_0 \in \mathcal{E}, \tag{1.8}$$

$\mathcal{A}^{-1}\mathcal{B} \in \mathcal{L}(U,\mathcal{E})$, $U = R^3_{\rho_0}$, the Euclidean 3-space with weight ρ_0, where $\xi = [x, y, x_t, y_t]$ and $g = [g_1, g_2, g_3]$, and $\mathcal{A}^*$ is the $\mathcal{E}$-adjoint of $\mathcal{A}$. It is established directly in the present paper that the following two inequalities hold true for the model (1.8) of the spherical shell (1.1):

(i) (abstract trace regularity) for any $0 < T < \infty$, there exists $C_T = CT > 0$, such that

$$\int_0^T \left\|\mathcal{B}^* e^{\mathcal{A}^* t}\xi\right\|_U^2 dt \le C_T \|\xi\|_{\mathcal{E}}^2, \qquad \xi \in \mathcal{E}; \tag{1.9}$$

(ii) (abstract continuous observability inequality) for all $T >$ some $T_0 > 0$, there is $c > 0$ such that

$$\int_0^T \left\| \mathcal{B}^* e^{\mathcal{A}^* t} \xi \right\|_U^2 dt \leq C(T - T_0) \|\xi\|_{\mathcal{E}}^2, \qquad \xi \in \mathcal{E}. \tag{1.10}$$

According to Section 3 below, inequalities (1.9) and (1.10) are the abstract versions of the P.D.E. inequalities (1.19) and (1.20) below, respectively, concerning the homogeneous problem (1.2) (possibly, with initial conditions at $t = 0$). This is analyzed in Sections 3 and 4. Thus, (1.9) and (1.10) fulfill for the shell problem (1.1) the two basic assumptions (regularity and Finite Cost Condition) of the abstract treatment of the optimal quadratic cost problem on an infinite time horizon, as developed in Flandoli, Lasiecka and Triggiani (1988), see also Lasiecka and Triggiani (1991), Lasiecka and Triggiani (1996c).

1.5. Function spaces

Well-posedness, regularity, exact controllability, and stability results pertaining to the shell model (1.1) require the introduction of the following weighted spaces (Lasiecka, Triggiani and Valente, 1996)

(i) $$L_2^\rho(0, \rho_0) \equiv \{u : \; u\sqrt{\rho} \in L_2(0, \rho_0)\}, \tag{1.11a}$$

with norm

$$\|u\|_{L_2^\rho} = \left\{ \int_0^{\rho_0} u^2 \rho \, d\rho \right\}^{\frac{1}{2}}; \tag{1.11b}$$

(ii) $$\mathcal{U}_\rho^1(0, \rho_0) = \left\{ u : \; \frac{u}{\sqrt{\rho}}, \; u'\sqrt{\rho} \in L_2(0, \rho_0), \; u(0) = 0 \right\}, \tag{1.12a}$$

with norm

$$\|u\|_{\mathcal{U}_\rho^1} = \left\{ \int_0^{\rho_0} \left[\frac{u^2}{\rho} + (u')^2 \rho \right] d\rho \right\}^{\frac{1}{2}}, \tag{1.12b}$$

where we note that $\frac{u}{\sqrt{\rho}}$, $u'\sqrt{\rho} \in L_2(0, \rho_0)$, hence $uu' \in L_1(0, \rho_0)$, makes the function

$$u^2(\rho) = u^2(\rho_0) + \int_{\rho_0}^{\rho} \frac{d}{dr}(u^2) dr = u^2(\rho_0) + 2 \int_{\rho_0}^{\rho} uu' dr \tag{1.12c}$$

absolutely continuous, so that the condition $u(0)$ in (1.12a) is well-defined;

(iii) $$\mathcal{W}_\rho^2(0, \rho_0) = \{w : \; w\sqrt{\rho} \in L_2(0, \rho_0), \; w' \in \mathcal{U}_\rho^1(0, \rho_0)\} \tag{1.13a}$$

with norm

$$\|w\|_{\mathcal{W}_\rho^2} = \left\{ \int_0^{\rho_0} w^2 \rho d\rho + \|w'\|_{\mathcal{U}_\rho^1}^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_0^{\rho_0} \left[w^2 \rho + \frac{(w')^2}{\rho} + (w'')^2 \rho \right] d\rho \right\}^{\frac{1}{2}}, \tag{1.13b}$$

where $w\sqrt{\rho}$, $\frac{w'}{\sqrt{\rho}} \in L_2(0, \rho_0)$, as in the definition of $\mathcal{W}_\rho^2(0, \rho_0)$, hence $ww' \in L_1(0, \rho_0)$, makes—as in (1.12c)— $w^2(\rho)$ absolutely continuous;

(iv) $$\mathcal{V}^1_\rho(0,\rho_0) = \Big\{ (u,w) \in L^\rho_2(0,\rho_0) \times L^\rho_2(0,\rho_0) :$$

$$v = \frac{u}{R} + w' \in L^\rho_2(0,\rho_0) \text{ or, equivalently, } w' \in L^\rho_2(0,\rho_0) \Big\} \tag{1.14a}$$

with norm

$$\|(u,w)\|_{\mathcal{V}^1_\rho} = \left\{ \|u\|^2_{L^\rho_2} + \|w\|^2_{L^\rho_2} + e\|v\|^2_{L^\rho_2} \right\}^{\frac{1}{2}}; \tag{1.14b}$$

(v) the state space will be shown to be

$$\mathcal{E} = [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho;$$

$$\|\{u,w,u_1,w_1\}\|^2_{\mathcal{E}} = \|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho} + \|[u_1,w_1]\|^2_{\mathcal{V}^1_\rho}. \tag{1.15}$$

1.6. Statement of main results for problem (1.1)

Regarding the mixed problem (1.1), the main results of the present paper are as follows.

THEOREM 1.1 (WELL-POSEDNESS OF (1.1)) *Let the initial and boundary data satisfy*

$$\{x_0, y_0, [x_1, y_1]\} \in \mathcal{E} = [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho; \quad g_i \in L_2(0,T),\ i = 1,2,3, \tag{1.16}$$

for any $0 < T < \infty$. Then, the unique solution of problem (1.1a-h) satisfies

$$\{x(t), y(t), x_t(t), y_t(t)\} \in C([0,T];\mathcal{E}). \tag{1.17}$$

We shall also provide a *direct* proof of the following result.

THEOREM 1.2 (EXACT CONTROLLABILITY) *The mixed problem (1.1a-h) is exactly controllable on the space $\mathcal{E}$ defined in (1.15) within the class of boundary controls as in (1.16), for all $T >$ some sufficiently large $T_0 > 0$: i.e., given*

$$\{x_0, y_0, x_1, y_1\} \in \mathcal{E} = [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho,$$

and $T > T_0$, there exist boundary controls $g_i \in L_2(0,T)$, $i = 1,2,3$, such that the corresponding solution of problem (1.1a-h) satisfies:

$$x(T,\cdot) = y(T,\cdot) = x_t(T,\cdot) = y_t(T,\cdot) = 0.$$

As a consequence of Theorems 1.1 and 1.2, we obtain

THEOREM 1.3 *The theory of the optimal control problem with quadratic cost over an infinite time horizon, as developed abstractly in Flandoli, Lasiecka and Triggiani (1988), Lasiecka and Triggiani (1991), see also Lasiecka and Triggiani (1996c) holds true and is applicable to the mixed problem (1.1a-h).*

1.7. Statement of main results for problem (1.2)

As usual, the most direct proof of Theorems 1.1 and 1.2 is by duality, and hence it hinges on the following equivalent results for problem (1.2a-h). Before stating them, we recall that the finite energy space defined via (1.3) coincides with the space $\mathcal{E}$ in (1.15) under equivalent norms; more precisely (Lasiecka, Triggiani and Valente, 1996, Proposition 2.1, Eqn. (2.4)):

$$E(u, w) \text{ is equivalent to } \|\{[u, w], [u_t, w_t]\}\|^2_{\mathcal{E}}. \tag{1.18}$$

THEOREM 1.4 (TRACE REGULARITY INEQUALITY OF (1.2)) *With reference to problem (1.2) (possibly, with initial data at $t = 0$), the following inequality holds true for any $0 < T < \infty$: there is a constant $C_T = CT > 0$, such that*

$$\int_0^T [u_t^2(t, \rho_0) + w_t^2(t, \rho_0) + ev_t^2(t, \rho_0)]dt \leq C_T E(0). \tag{1.19}$$

Conversely, we have

THEOREM 1.5 (CONTINUOUS OBSERVABILITY INEQUALITY) *(a) Problem (1.1) is exactly controllable on $[0, T]$ in the sense of Theorem 1.2 if and only if problem (1.2) (with possibly initial data at $t = 0$) satisfies the following inequality: there is $C_T > 0$ such that*

$$\int_0^T [u_t^2(t, \rho_0) + w_t^2(t, \rho_0) + ev_t^2(t, \rho_0)]dt \geq C_T E(0). \tag{1.20}$$

(b) There exists $T_0 > 0$ such that, for all $T > T_0$, inequality (1.20) holds true with $C_T = C(T - T_0)$, and so problem (1.1) is indeed exactly controllable in the sense of Theorem 1.2. □

Putting together Theorems 1.4 and 1.5 and with reference to problem (1.2), we obtain that for $T > T_0$, the following equivalences hold true:

$$\left\{\int_0^T \left[u_t^2(t, \rho_0) + w_t^2(t, \rho_0) + ev_t^2(t, \rho_0)\right] dt\right\}^{\frac{1}{2}} \text{ equivalent to } E^{\frac{1}{2}}(0),$$

$$\text{in turn, equivalent to } \|\{[u_0, w_0], [u_1, w_1]\}\|_{\mathcal{E}}\,. \tag{1.21}$$

REMARK 1.1 *In the case of the wave equation (or, more generally, of a second-order hyperbolic equation) with $L_2(0, T; L_2(\Gamma))$-control function in the Neumann boundary condition, the space of exact controllability is $H^1(\Omega) \times L_2(\Omega)$, while the space of regularity is $H^1(\Omega) \times L_2(\Omega)$ only for dim $\Omega = 1$ (indeed, for dim $\Omega \geq 2$, the space of sharp regularity is $H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)$, $\alpha \leq \frac{3}{4}$ and depending on Ω (Lasiecka and Triggiani, 1990, Lasiecka and Triggiani, 1994). A similar situation arises for Kirchoff equations with control functions in high boundary conditions, of the second- and third-order (moments and shear forces): in dim*

$\Omega \geq 2$, the space of regularity is larger (with weaker topology) than the space $H^2(\Omega) \times H^1(\Omega)$ of exact controllability (Lagnese, 1989). On the other hand, the spherical shell model (1.1) consists of the wave equation (1.1a) in x with Neumann control in (1.1f), highly coupled with the Kirchoff equation (1.1b) in y with control in moments and shear forces, (1.1g) and (1.1h), both in one dimension. Because of the one-dimensionality, the results of Theorem 1.1 and 1.2, where regularity and exact controllability are achieved on the same state space *$\mathcal{E}$ are to be expected (i.e., the map*

$$\{g_1, g_2, g_3\} \Rightarrow \{x(T), y(T), x_t(T), y_t(T)\} \tag{1.22}$$

where $x(0) = y(0) = x_1(0) = y_1(0) = 0$ is continuous and surjective (onto): $[L_2(0,T)]^3 \Rightarrow \mathcal{E}$.)

In the sections below we shall provide proofs of the main results by analyzing and complementing the technical development and energy estimates in Lasiecka and Triggiani (1996a), Lasiecka, Triggiani and Valente (1996), which were carried out for the closed loop feedback problem (1.1a-h), (1.7). In order to follow the present paper, it is essential to have Lasiecka and Triggiani (1996a), Lasiecka, Triggiani and Valente (1996) at hand.

As to the literature, we note that Geymonat, Loreti and Valente (1993) give an exact controllability result (i) for a simplified model with no rotational inertia terms (i.e., no term z_{tt} in (1.1a) and (1.1h)); (ii) with different boundary conditions (of Dirichlet type); (iii) and, above all, for R sufficiently large (or the opening angle θ_0 suitably small) as to have an easier case no lower order terms to be absorbed (so that arguments such as the ones in Lasiecka, Triggiani and Valente, 1996, Section 8 or Lasiecka and Triggiani, 1996b, are dispensed with); (iv) finally, with no regularity provided on the steering controls. Moreover, Geymonat, Loreti and Valente (1993) consider the hemispherical shell ($\theta_0 = \pi/2$) and give an exact controllability result via harmonic analysis, by using the explicitly computed eigenvalues and eigenfunctions in this case.

2. (Direct) Proof of Theorem 1.4 and Theorem 1.5(b)

We shall heavily rely on the computations in Lasiecka, Triggiani and Valente (1996) for the closed loop feedback problem (1.1a-h), (1.7), which we will specialize to the *homogeneous* problem (1.2). In order to follow the analysis below, it is essential to have paper Lasiecka, Triggiani and Valente (1996) at hand.

Proof of Theorem 1.5(b). Step 1. Theorem 7.1, Eqn. (7.1), in Lasiecka, Triggiani and Valente (1996) simplifies to the following identity

$$\frac{1}{2}\int_0^T \left[\|u\|^2_{\mathcal{U}^1_\rho} + e\|v\|^2_{\mathcal{U}^1_\rho} + \|u_t\|^2_{L^\rho_2} + 3\|w_t\|^2_{L^\rho_2} + \|v_t\|^2_{L^\rho_2}\right] dt$$

$$+(E_n T)^T_0 + IT + BT \equiv 0, \tag{2.1}$$

where the end terms $(E_nT)_0^T$ and the interior terms IT are as in Lasiecka, Triggiani and Valente (1996, Eqns. (7.2) and (7.3)), and satisfy respectively,

$$\left|(E_nT)_0^T\right| \le c[E(0)+E(T)]; \tag{2.2}$$

$$|IT| \le \epsilon \int_0^T \|u\|^2_{\mathcal{U}^1_\rho} dt + c_\epsilon \int_0^T \int_0^{\rho_0} w^2 \rho \, d\rho \, dt, \tag{2.3}$$

while the boundary terms BT in Lasiecka, Triggiani and Valente (1996, Eqn. (7.4)) now simplify to the following expression,

$$\begin{aligned} BT \;=\; & -\frac{1}{2}\int_0^T \left[u_t^2(t,\rho_0)+w_t^2(t,\rho_0)+ev_t^2(t,\rho_0)\right]\rho_0^2 dt \\ & -\frac{1}{2}\int_0^T \left[(u'(t,\rho_0))^2\rho_0^2 + u'(t,\rho_0)u(t,\rho_0)\rho_0\right] dt \\ & +\frac{1}{2}\int_0^T \left[u^2(t,\rho_0)+ev^2(t,\rho_0)\right] dt \\ & -\frac{(1+\nu)}{2R}\int_0^T u(t,\rho_0)w(t,\rho_0)\rho_0 \, dt + \frac{(1+\nu)}{R^2}\int_0^T w^2(t,\rho_0)\rho_0^2 dt, \end{aligned} \tag{2.4}$$

for problem (1.2a-e) after the use of the B.C. (1.2g) and (1.2h) (and with no use of the B.C. (1.2f) yet). Four terms within the brackets { } in Lasiecka, Triggiani and Valente (1996, Eqn. (7.17)) vanish now.

Step 2. We next use the B.C. (1.2f) in (2.4), and thus express u' in terms of w and u at $\rho = \rho_0$, and estimate readily to obtain the counterpart inequality of Eqn. (7.24) from Lasiecka, Triggiani and Valente (1996):

$$\begin{aligned} & \left\| -\frac{1}{2}\int_0^T \left[(u'(t,\rho_0))^2\rho_0^2 + u'(t,\rho_0)u(t,\rho_0)\rho_0\right] dt \right\| \\ & \le c\int_0^T [u^2(t,\rho_0)+w^2(t,\rho_0)]dt. \end{aligned} \tag{2.5}$$

The remaining argument in Lasiecka, Triggiani and Valente (1996, Eqn. (7.25) through (7.32)) is unchanged, and the counterpart of Eqn. (2.3) from Lasiecka, Triggiani and Valente (1996) is now

$$\begin{aligned} |BT| \;\le\; & \epsilon_0 \int_0^T E(t)dt + \frac{\rho_0^2}{2}\int_0^T [u_t^2(t,\rho_0)+w_t^2(t,\rho_0)+ev_t^2(t,\rho_0)]dt \\ & + c\int_0^T [u^2(t,\rho_0)+w^2(t,\rho_0)]dt + \int_0^T\int_0^{\rho_0} w^2\rho \, d\rho \, dt. \end{aligned} \tag{2.6}$$

Step 3. We return to identity (2.1). On its left-hand side, we use property (p.5) in Eq. (1.27) of Lasiecka, Triggiani and Valente (1996)

$$c\left[\|w\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\right] - \alpha^2\|w\|^2_{L^\rho_2} \le \|u\|^2_{\mathcal{U}^1_\rho} + e\|v\|^2_{\mathcal{U}^1_\rho} \le C\left[\|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\right], \tag{2.7}$$

and (1.14b)

$$\|u_t\|^2_{L_2^\rho} + \|w_t\|^2_{L_2^\rho} + e\|v_t\|^2_{L_2^\rho} = \|\{u_t, w_t\}\|^2_{\mathcal{V}_\rho^1}, \tag{2.8}$$

as well as the norm equivalence (1.18), to obtain

$$cE(t) - \alpha^2\|w\|^2_{L_2^\rho} \le \frac{1}{2}\left[\|u\|^2_{\mathcal{U}_\rho^1} + e\|v\|^2_{\mathcal{U}_\rho^1} + \|u_t\|^2_{L_2^\rho} + 3\|w_t\|^2_{L_2^\rho} + e\|v_t\|^2_{L_2^\rho}\right] \le cE(t). \tag{2.9}$$

From here, we obtain the same inequality as in Lasiecka, Triggiani and Valente (1996, Eqn. (7.35)),

$$\int_0^T E(t)dt \le c[E(0) + E(T)] + C\left\{\int_0^T [u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho)]dt + \ell.o.t.\right\} \tag{2.10}$$

$$|\ell.o.t.| \le C\left\{\int_0^T [u^2(t,\rho_0) + w^2(t,\rho_0)]dt + \int_0^T\int_0^{\rho_0} w^2\rho\, d\rho\, dt\right\}, \tag{2.11}$$

where now, however, $E(t) \equiv E(0)$ by (1.6). Using this constant energy in (2.10) yields

$$(T-c)E(0) \le C\left\{\int_0^T [u_t^2(t,\rho_0) + w_t^2(t,\rho_0)] + ev_t^2(t,\rho_0)] + \ell.o.t.\right\}, \tag{2.12}$$

which is the desired continuous observability inequality (1.20), save for the lower-order terms. These can then be absorbed by compactness/uniqueness, where the delicate uniqueness result (Lasiecka, Triggiani and Valente, 1996, Theorem 8.1, Lasiecka and Triggiani, 1996b), based on Carleman estimates is used. This way we obtain the desired inequality (1.20) from (2.12). Theorem 1.5(b), Eqn. (1.20) is proved. □

Proof of Theorem 1.4. We rewrite identity (2.1) explicitly by (2.2) as

$$\frac{1}{2}\int_0^T \left[\|u\|^2_{\mathcal{U}_\rho^1} + e\|v\|^2_{\mathcal{U}_\rho^1} + \|u_t\|^2_{L_2^\rho} + 3\|w_t\|^2_{L_2^\rho} + e\|v_t\|^2_{L_2^\rho}\right] dt$$

$$+(E_nT)_0^T + IT$$

$$+\frac{1}{2}\int_0^T \left[(u'(t,\rho_0))^2\rho_0^2 + u'(t,\rho_0)u(t,\rho_0)\rho_0\right] dt$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T [u^2(t,\rho_0)+ev^2(t,\rho_0)]dt \\
&+\frac{(1+\nu)}{2R}\int_0^T u(t,\rho_0)w(t,\rho_0)\rho_0 dt-\frac{(1+\nu)}{R^2}\int_0^T w^2(t,\rho_0)\rho_0^2 dt \\
=\;& \frac{1}{2}\int_0^T [u_t^2(t,\rho_0)+w_t^2(t,\rho_0)+ev_t^2(t,\rho_0)]\rho_0^2 dt. \qquad (2.13)
\end{aligned}
$$

Recalling (2.7), as well as the estimates (2.2) and (2.3) for $|(E_nT)_0^T|$ and $|IT|$, we readily obtain inequality (1.19) from (2.13). Thus, Theorem 1.4 is proved. □

3. The adjoint of the input-solution operator of problem (1.1a-h)

The following issue will be relevant, in Section 4, in the proof of Theorems 1.1 and 1.2: with reference to the mixed problem (1.1a-h), with zero initial conditions $x_0 = y_0 = x_1 = y_1 = 0$, let $\mathcal{L}_T$ be the operator defined by

$$
\begin{bmatrix} x(T) \\ y(T) \\ x_t(T) \\ y_t(T) \end{bmatrix} = \mathcal{L}_T g : \; R^3_{\rho_0} \to \mathcal{E}, \quad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}. \qquad (3.1)
$$

Then, find $\mathcal{L}_T^* : \mathcal{E} \to R^3_{\rho_0}$. In this section we identify the adjoint of $\mathcal{L}_T$ more precisely with respect to a (natural) topology, $\mathbf{E}$ below in (3.4), equivalent to $\mathcal{E}$. To this end, we shall rely heavily on the abstract operator model for problem (1.1a-h), as developed in Lasiecka and Triggiani (1996a, Section 2) to the original model (1.1a-h). Actually, in such a reference, it was sufficient to consider a (mathematically) simplified version of model (1.1a-h), which left out a few lower-order terms: $\frac{(1+\nu)}{R} y'$ from the x-equation (1.1a); $-\frac{(1+\nu)}{\rho R}(x\rho)'$ from the y-equation (1.1b); and (essentially) $-\frac{(1+\nu)}{R} y$ from the boundary condition (1.1f). Perturbation theory then allows one to handle these lower-order terms and to transfer the desired well-posedness result. Indeed, these additional terms, viewed on the abstract second-order (in time) model, act as a continuous perturbation: $\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \to L_2^\rho \times L_2^\rho$; consequently, these terms, viewed on the first-order (in time) model, act as a bounded perturbation $\mathbf{E} \to \mathbf{E}$. This will be seen in Propositions 3.1 and 3.2 below.

Step 1. Let M, A, B be the operators introduced and studied in Lasiecka and Triggiani (1996a, Section 2); in connection with the simplified model:

M : isomorphism $\mathcal{V}^1_\rho$ onto $[\mathcal{V}^1_\rho]'$, positive self-adjoint on $L_2^\rho \times L_2^\rho$

$$\text{(duality with respect to pivot space } L_2^\rho \times L_2^\rho); \tag{3.2}$$

$$A \ : \ \text{positive self-adjoint on } L_2^\rho \times L_2^\rho, \quad \text{bounded } \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \to [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2]'. \tag{3.3}$$

$$\mathbf{E} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}}) \text{ norm equivalent to } \mathcal{E} = [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2] \times \mathcal{V}_\rho^1; \tag{3.4}$$

$$B^* \begin{bmatrix} \bar{u} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} -\bar{u}(\rho_0) \\ -\bar{v}(\rho_0) \\ \bar{w}(\rho_0) \end{bmatrix} : \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \to R^3_{\rho_0}. \tag{3.5}$$

The following results are needed in the sequel. They are a minor variation of those in Lasiecka and Triggiani (1996a, Section 2), which refer to the case of $P = 0$.

Proposition 3.1 *The abstract model of problem (1.1a-h) is given by*

$$M \begin{bmatrix} x_{tt} \\ y_{tt} \end{bmatrix} + (A+P) \begin{bmatrix} x \\ y \end{bmatrix} - Bg = 0, \tag{3.6a}$$

where P is a self-adjoint perturbation on $L_2^\rho \times L_2^\rho$, which satisfies

$$P: \ \textit{continuous } \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \to L_2^\rho \times L_2^\rho.$$

(ii) The corresponding first-order model is

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ x_t \\ y_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} x \\ y \\ x_t \\ y_t \end{bmatrix} + \mathcal{B}g; \tag{3.7}$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -M^{-1}(A+P) & 0 \end{bmatrix} : \ \mathbf{E} \supset \mathcal{D}(\mathcal{A}) \to \mathbf{E} \tag{3.8a}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} [[u_0, w_0], [u_1, w_1]] \in [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2] \times \mathcal{V}_\rho^1 \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}}) \ s.c. \\ \text{(i)} \quad [u_1, w_1] \in \mathcal{D}(A^{\frac{1}{2}}) \equiv \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2; \\ \text{(ii)} \quad M^{-1}(A+P) \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} \in \mathcal{V}_\rho^1 \equiv \mathcal{D}(M^{\frac{1}{2}}). \end{array} \right\} \tag{3.8b}$$

$$\mathcal{B}g = \begin{bmatrix} 0 \\ M^{-1}Bg \end{bmatrix} : \ R^3_{\rho_0} \to [\mathcal{D}(\mathcal{A}^*)]'. \tag{3.9}$$

(iii) The perturbation operator,

$$\mathcal{P}\begin{bmatrix} u_0 \\ w_0 \\ u_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -M^{-1}P & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \\ u_1 \\ w_1 \end{bmatrix} = -M^{-1}P\begin{bmatrix} u_0 \\ w_0 \end{bmatrix}, \tag{3.10}$$

satisfies

$$\mathcal{P}: \text{ continuous } \mathbf{E} \to \mathbf{E}. \tag{3.10a}$$

Accordingly, since the operator $\mathcal{A}_0$ obtained from $\mathcal{A}$ in (3.8) by setting $P = 0$ is the generator of a unitary group on $\mathbf{E}$ [L-T, Section 2], it follows that $\mathcal{A} = \mathcal{A}_0 + \mathcal{P}$ in (3.8) is the generator of a s.c. group on $\mathbf{E}$.

Step 2. It follows from Proposition 3.1 that an explicit representation formula for the operator $\mathcal{L}_T$ defined by (3.1) is given by

$$\begin{bmatrix} x(T) \\ y(T) \\ x_t(T) \\ y_t(T) \end{bmatrix} = \mathcal{L}_T g = \int_0^T e^{\mathcal{A}(T-t)}\mathcal{B}g(t)dt: \ R^3_\rho \to \mathbf{E}. \tag{3.11}$$

Thus, the dual operator $\mathcal{L}_T^*: \ \mathbf{E} \to R^3_{\rho_0}$, $(\mathcal{L}_T g, h)_{\mathbf{E}} = (g, \mathcal{L}_T^* h)_{R^3_{\rho_0}}$ is given by

$$\{\mathcal{L}_T^* h\}(t) = \mathcal{B}^* e^{\mathcal{A}^*(T-t)}h, \ 0 \le t \le T, \ h \in \mathbf{E}, \tag{3.12}$$

where $\mathcal{B}^*$ is readily computed to be

$$\mathcal{B}^* v = \mathcal{B}^*\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -B^* v_2 \in R^3_{\rho_0}, \ v_2 \in \mathcal{D}(A^{\frac{1}{2}}), \tag{3.13}$$

where we recall (3.5). In fact, for $g \in R^3$, $v \in \mathbf{E}$, we compute via (3.9) and (3.4),

$$\begin{aligned} (\mathcal{B}g, v)_{\mathbf{E}} &= -(M^{-1}Bg, v_2)_{\mathcal{D}(M^{\frac{1}{2}})} = -(Bg, v_2)_{L_2^\rho \times L_2^\rho} \\ &= (g, -B^* v_2)_{R^3_{\rho_0}} = (g, \mathcal{B}^* v)_{R^3_{\rho_0}}, \end{aligned} \tag{3.14}$$

and (3.13) follows from (3.14).

Step 3.

PROPOSITION 3.2 *In P.D.E. terms, the dual operator $\mathcal{L}_T^*$ expressed by (3.12) is defined by*

$$\{\mathcal{L}_T^* h\}(t) = \begin{bmatrix} u_t(t, \rho_0) \\ v_t(t, \rho_0) \\ -w_t(t, \rho_0) \end{bmatrix}, \quad h = [h_1, h_2] \in \mathbf{E}, \quad v = \frac{u}{R} + w', \tag{3.15}$$

where $\{u,w\}$ solves problem (1.2a-h), with initial conditions (1.2d) at $t=T$, given by

$$h_1 = \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho; \tag{3.16a}$$

$$h_2 = \begin{bmatrix} u_1 \\ w_1 \end{bmatrix} \in \mathcal{D}(M^{\frac{1}{2}}) = \mathcal{V}^1_\rho, \tag{3.16b}$$

so that, if $\sim$ denotes equivalence as usual, we obtain

$$\|h\|^2_{\mathbf{E}} \sim \|\{[u_0,w_0],[u_1,w_1]\}\|^2_{\mathcal{E}} \sim E(0;u,w), \tag{3.17}$$

where $E(0)=E(0;w,u)$ is defined in (1.3).

Proof. The group $e^{\mathcal{A}^*(T-t)}$ can be expressed, as usual, in terms of the corresponding cosine and sine operators $C^*(t)=C(t)$ and $S^*(t)=S(t)$ by

$$e^{\mathcal{A}^*(T-t)}h = \begin{bmatrix} C(T-t) & S(T-t) \\ \mathring{A}S(T-t) & -C(T-t) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad h \in \mathbf{E}, \tag{3.18}$$

where

$$\begin{bmatrix} u(t) \\ w(t) \end{bmatrix} = \phi(t;\phi_0,\phi_1) = C(t-T)\phi_0 + S(t-T)\phi_1 \tag{3.19}$$

corresponds to the free dynamics $m\dot{\phi}+(A+P)\phi=0$ or

$$M\begin{bmatrix} u_{tt} \\ w_{tt} \end{bmatrix} + (A+P)\begin{bmatrix} u \\ w \end{bmatrix} = 0, \qquad \begin{bmatrix} u(T) \\ w(T) \end{bmatrix} = \phi_0; \begin{bmatrix} u_t(T) \\ w_t(T) \end{bmatrix} = \phi_1, \tag{3.20}$$

$A+P$ being self-adjoint, where $\mathring{A}=-M^{-1}(A+P)$ is the generator of the cosine operator $C(t)$, so that (3.20) is the abstract version of the homogeneous problem (1.2a-h), with $\phi_0=[u_0,w_0]$, $\phi_1=[u_1,w_1]$. This justifies the notation $[u,w]$ in (3.19). From (3.19) we obtain

$$\begin{bmatrix} u_t(t) \\ w_t(t) \end{bmatrix} = \phi_t(t;\phi_0,\phi_1) = C(t-T)\phi_1 + S(t-T)(-\mathring{A}\phi_0). \tag{3.21}$$

On the other hand, for $h\in\mathbf{E}$, (3.13) and (3.18) yield, since $C(\cdot)$ is even and $S(\cdot)$ is odd:

$$\mathcal{B}^* e^{\mathcal{A}^*(T-t)}h = -B^*[C(t-T)(-h_2) + S(t-T)(-\mathring{A}h_1)]$$

and by (3.21)

$$\mathcal{B}^* e^{\mathcal{A}^*(T-t)}h = -B^*\begin{bmatrix} u_t(t) \\ w_t(t) \end{bmatrix},$$

provided we take, comparing (3.22) with (3.21),

$$\phi_0 = \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} = h_1 = \mathcal{D}(A^{\frac{1}{2}}); \qquad \phi_1 = \begin{bmatrix} u_1 \\ w_1 \end{bmatrix} = -h_2 = \mathcal{D}(M^{\frac{1}{2}}). \quad (3.22)$$

Thus, recalling (3.4) and (1.18), we obtain from (3.24),

$$\|h\|^2_{\mathbf{E}} = \|\{h_1, h_2\}\|^2_{\mathcal{D}(A^{\frac{1}{2}})\times\mathcal{D}(M^{\frac{1}{2}})} \sim \|\{[u_0, w_0], [u_1, w_1]\}\|^2_{\mathcal{E}} \sim E(0; u, w). \quad (3.23)$$

Moreover, (3.5) used in (3.23) yields via (3.12)

$$\{\mathcal{L}_T^* h\}(t) = \mathcal{B}^* e^{\mathcal{A}^*(T-t)} h = \begin{bmatrix} u_t(t, \rho_0) \\ v_t(t, \rho_0) \\ -w_t(t, \rho_0) \end{bmatrix}. \quad (3.24)$$

Thus, (3.25) and (3.26) prove (3.17) and (3.15), as desired. □

4. Completion of the proofs of Theorem 1.1, Theorem 1.5a, and Theorem 1.2

Proof of Theorem 1.1. In light of Eqn. (3.15) of Proposition 3.2, with $\mathcal{L}_T^*$ given by (3.12), the trace regularity inequality (1.19) already proved in Section 2 can be reformulated as

$$\begin{aligned} &\mathcal{B}^* e^{\mathcal{A}^* t} \ : \text{continuous} \\ &\mathcal{E} \equiv [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2] \times \mathcal{V}_\rho^1 = \mathbf{E} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}}) \to [L_2(0,T)]^3, \end{aligned} \quad (4.1)$$

which by duality (Flandoli, Lasiecka and Triggiani, 1988, Appendix) is equivalent to the regularity

$$\begin{aligned} (\mathcal{L}g)(t) \quad &\equiv \quad \int_0^t e^{\mathcal{A}(t-\tau)} \mathcal{B} g(\tau) d\tau = [x(t), y(t), x_t(t), y_t(t)] \\ &: \quad \text{continuous } [L_2(0,T)]^3 \to C([0,T]; \mathcal{E}), \end{aligned} \quad (4.2)$$

for problem (1.1a-h) with initial conditions $x_0 = y_0 = x_1 = y_1 = 0$, as desired.

Proof of Theorems 1.5(a) and 1.2. In light of Proposition 3.2, Eqn. (3.15) and (3.17), the continuous observability inequality (1.20) already proved in Section 2 can be reformulated as

$$\|\mathcal{L}_T^* h\|_{[L_2(0,T)]^3} \geq C_T \|h\|_{\mathbf{E}}, \quad (4.3)$$

a condition which is then equivalent (Taylor and Lay, 1980, p. 235) to the surjectivity condition of $\mathcal{L}_T$:

$$\mathcal{L}_T : [L_2(0,T)]^3 \to \text{ onto } \mathbf{E} = \mathcal{E}. \quad (4.4)$$

The latter is a restatement of exact controllability (from the origin). Theorems 1.5(a) and 1.2 are proved.

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