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Qualitative and numerical study of the nonequilibrium high–rate processes in relaxing media[1]

by

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Abstract. The dynamic equation of state, aimed at describing high-rate processes in relaxing media is proposed. Together with balance of mass and momentum equations it forms a closed system of nonlinear PDE. Invariant travelling wave solutions of this system are studied by means of qualitative theory methods, allowing to state the conditions that guarantee the existence of periodic, quasiperiodic and soliton-like solutions. The results of qualitative investigations are confirmed by the numerical simulation.

Keywords: multicomponent relaxing media models, nonlinear waves, bifurcations, patterns formation

1. Introduction

In recent decades a new approach, called synergetics, has been developed, dealing with the spontaneous creation of spatial and (or) temporal ordering in distributed parameter nonlinear systems. Starting with pioneering works by Glansdorff and Prigogine (1971), this approach has been successfully applied to the description of self organization phenomena in nonlinear media, simulated by parabolic–type equations (as given in Samarskij et al., 1987). These equations describe well enough processes of moderate intensity but lose their applicability in the cases where gradients are changing during correlation time and inside correlation length. In this case essential dispersive length effects take place and the nonlocality and memory effects have to be taken into account.

The simplest equation of this type is the hyperbolic–type nonlinear telegraph equation (see Danylenko, Kudinov and Makarenko, 1984) which is a natural generalization of the mass–heat transport equation, taking into account the finite propagation velocity of perturbation. The telegraph equation was shown to possess families of quasiperiodic solutions, as well as solutions describing blow-up regimes. So it inherits the main features of nonlinear heat equations

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(Samarskij et al., 1987), but, in contrast to them, predicts a finite value for the characteristic velocity. Another example of hyperbolic – type system, aimed at describing explosion mechanics problems for multicomponent relaxing media was studied by Danylenko, Sorokina and Vladimirov (1993).

In this work the investigations of the high–rate processes in multicomponent relaxing media are continued. In Section 2 we derive the dynamic equation of state, describing the medium with two relaxing processes and employ this equation, containing higher–derivative terms, to close the balance of mass and momentum system. In Section 3 we consider the system of ODE, obtained by the group–theoretical reduction (Ovsyannikov, 1982) of the PDE system, derived in previous section. This ODE system is investigated by means of qualitative theory methods that allow for obtaining critical sets in the space of parameters, corresponding to the loss of stability of the self–similar solutions. The analysis of the ODE system allows to classify the regimes created in vicinity of bifurcation set and to state the conditions, leading to the existence of limit cycles, toroidal attractors and other localized regimes. The qualitative investigations are confirmed by direct numerical simulations.

2. Governing equation for the media with two relaxing processes

The first model of viscoelastic continuum, based on the concept of relaxation has been proposed by J.C.Maxwell (1890) over one hundred years ago. Since then many sophisticated models have been constructed to describe the reacting and relaxing materials. Nevertheless the problem of constructing multicomponent condensed media model adequately describing the structural relaxation, manifested in high–rate processes is actually far from being solved. The difficulties arising in construction of a governing (constitutive) equation are mainly associated with ignorance of the mechanism controlling the process of relaxation.

In this situation the deviation from the state of thermodynamical equilibrium is expedient to be described by internal variables $\lambda_1,...,\lambda_n$ (Vladimirov, Danylenko, Korolevich, 1990, and Danylenko, Sorokina, Vladimirov, 1993), satisfying chemical kinetics equations

$$d\lambda_i/dt = W_i(p, T, \lambda), \qquad i = 1, ...n$$

with unknown mechanism. Generally speaking, this approach does not give any advantage, yet it becomes very helpful if the processes under consideration are not far from equilibrium, for in this case the entropy growth law $dS > 0$, which is valid for any irreversible thermodynamical process can be written as $dS = d_eS + d_iS > 0$, where d_eS is the entropy change due to reversible process, while $d_iS > 0$ is the entropy production due to the process of relaxation. In consistence with the nonequilibrium thermodynamics principles (Glansdorff and Prigogine, 1971), entropy production is a multilinear function of generalized thermodynamical forces X_k and fluxes J_k: $dS_i = \sum \Delta S_i = \sum_{k=1}^{n} X_k J_k$, while X_k and J_k are coupled by means of the well known Onsager reciprocity principle: $J_i = \sum_k L_{ik} X_k, \qquad L_{ik} = L_{ki}$.

In this work we restrict ourselves to the processes described by two independent internal variables (λ, η). If we identify $d\lambda/dt, d\eta/dt$ with thermodynamical fluxes then the corresponding thermodynamical forces will be given by the affinity of the relaxing processes A and B, which may be introduced by means of the second law of thermodynamics (De Groot and Mazur, 1962, Glansdorff and Prigogine, 1971):

$$TdS = dE + pdV + Ad\lambda + Bd\eta, \tag{1}$$

where V is the specific volume, p denotes pressure, S is the entropy, T is the temperature and $E = E(p, V, \lambda, \eta)$ is the internal energy per unit mass. For weakly nonequilibrium processes the governing equations for λ and η can be written in the form (De Groot and Mazur, 1962):

$$d\lambda/dt = aA(p, T, \lambda), \qquad d\eta/dt = bB(p, T, \eta). \tag{2}$$

Equations (1) and (2) can be chosen to close the system of balance equations for mass and momentum that are valid regardless of the detailed mechanism of relaxing processes.

Let us assume that equation (1) (with function $E(p, V, \lambda, \eta)$ defined in an explicit form) possesses the first integral that can be expressed in the following form:

$$V = v(p, T, \lambda, \eta; C). \tag{3}$$

It is possible to expand functions A, B and v near the equilibrium state $A(p_0, T_0, \lambda_0) = B(p_0, T_0, \eta_0) = 0$ into the power series:

$$A = (A_T)_{p\lambda}(T - T_0) + (A_p)_{T\lambda}(p - p_0) + (A_\lambda)_{pT}(\lambda - \lambda_0) + \dots \tag{4}$$

$$B = (B_T)_{p\eta}(T - T_0) + (B_p)_{T\eta}(p - p_0) + (B_\eta)_{pT}(\eta - \eta_0) + \dots \tag{5}$$

$$V - V_0 = (v_T)_{p\lambda\eta}(T - T_0) + (v_p)_{T\lambda\eta}(p - p_0) + (v_\lambda)_{pT\eta}(\lambda - \lambda_0) + (v_\eta)_{pT\lambda}(\eta - \eta_0) + \dots \tag{6}$$

where $V_0 = v[p_0 T_0 \lambda(p_0 T_0), \eta(p_0 T_0)] = V(p_0, T_0)$ is the specific volume in the equilibrium state. In what follows we shall neglect higher-order terms in the power series (4)–(6). Taking the derivatives of (6) with respect to time and using equations (4) and (5), it is possible to exclude the internal variables λ, η from the equation of state and to present it in the following form:

$$\ddot{V} - v_p \ddot{p} - v_T \ddot{T} = \left(\frac{1}{\tau^A_{TV}} + \frac{1}{\tau^B_{TV}} \right) v_p \dot{p} + \left(\frac{1}{\tau^A_{pV}} + \frac{1}{\tau^B_{pV}} \right) v_T \dot{T}$$

$$-\left(\frac{1}{\tau^A}+\frac{1}{\tau^B}\right)\dot{V}-\frac{1}{\tau^A\tau^B}(V-V_0)$$
$$+v_T\left[\frac{1}{\tau^B\tau^A_{pV}}+\frac{1}{\tau^A\tau^B_{pV}}-\frac{1}{\tau^A\tau^B}\right](T-T_0)$$
$$+v_p\left[\frac{1}{\tau^B\tau^A_{TV}}+\frac{1}{\tau^A\tau^B_{TV}}-\frac{1}{\tau^A\tau^B}\right](p-p_0), \tag{7}$$

where $\dot{(.)}=d(.)/dt$, $\ddot{(.)}=d^2(.)/dt^2$, $(\tau^A_{x,y})^{-1}=-a\partial A(x,y,\lambda)/\partial\lambda$, $(\tau^B_{x,y})^{-1}=-b\partial B(x,y,\eta)/\partial\eta$, (x,y) is any pair of external variables, different from (p,T) (we omit lower indices in this case). In accordance with De Groot and Mazur (1962) and Danevych, Danylenko et al. (1992), $\tau^A_{x,y}$ and $\tau^B_{x,y}$ are called relaxation times.

A passage from p,T,V to p,S,V variables can be easily performed in (5) if one uses the representation

$$T=T_0(V/V_0)^{-\Gamma_{V\infty}}exp\{(S_e-S_0)/c_{V\infty}\}, \tag{8}$$

which is valid for weakly nonequilibrium processes (Danevych, Danylenko et al., 1992). Here $\Gamma_{V\infty}$ is the isochoric Gruneisen coefficient (Danevych, Danylenko et al., 1992), $c_{V\infty}$ is the thermal capacity for constant volume in the state of frozen equilibrium. In what follows we shall restrict ourselves to consideration of adiabatic processes, where $S_e=const.$ In this case, using the representation (8), one obtains the governing equation containing only p,V variables and their time derivatives. The equation obtained after the substitution of (8) into (7) is rather difficult to analyse. It becomes simpler if $\Gamma_{V\infty}<<1$ or if $\Gamma_{V\infty}>>1$. In these cases the constitutive equation together with the balance equation for mass and momentum, taken in hydrodynamical approximation, gives a closed system of the following form:

$$\bar\rho\frac{d\bar u}{d\bar t}+\frac{\partial\bar p}{\partial\bar x}=\Im \qquad \frac{d\bar\rho}{d\bar t}+\bar\rho\frac{\partial\bar u}{\partial\bar x}=0$$
$$\tau\left(\frac{d\bar p}{d\bar t}-\sigma\frac{d\bar\rho}{d\bar t}\right)=\kappa\bar\rho^n-\bar p-b\left\{\frac{d^2\bar p}{d\bar t^2}+\left[\frac{2}{\bar\rho}\left(\frac{d\bar\rho}{d\bar t}\right)^2-\frac{d^2\bar\rho}{d\bar t^2}\right]\right\}, \tag{9}$$

where $\Im$ is the mass force, $\bar t=t/\tau_0$, τ_0 is the characteristic time of the process, $\bar x=x/(\tau_0C_{T\infty})$ is the dimensionless spatial variable, $\bar\rho=V_0/V$ is the dimensionless density $\bar p=(p-p_0)/(\rho_0C^2_{T\infty})+\kappa$, $\bar u=u/C_{T\infty}$ is the dimensionless velocity $n=1+\Gamma_{V\infty}$, $d/d\bar t=\partial/\partial\bar t+\bar u\partial/\partial\bar x$, C_{T0} and $C_{T\infty}$ are the isothermal sound velocities in the states of complete and frozen equilibrium, respectively, κ, b, σ and τ are dimensionless parameters, depending on sound velocities and relaxation times. In the case where $\Gamma_{V\infty}<<1$ they can be expressed as follows:

$$\kappa=\frac{C^2_{T0}}{C^2_{T\infty}(2-\sigma)},\quad b=\frac{\tau_A\tau_B\sigma}{\tau_0{}^2(2-\sigma)},\quad \sigma=\frac{\tau^A_{TV}}{\tau^A_{Tp}}\equiv\frac{\tau^B_{TV}}{\tau^B_{Tp}},\quad \tau=\frac{\tau_A+\tau_B}{(2-\sigma)\tau_0}$$

So, we have obtained the system describing compressible media with two relaxation processes in hydrodynamical approximation and containing parameters that can be estimated experimentally. It was shown in our previous works (Vladimirov, Danylenko, Korolevich, 1990, Danevych, Danylenko et al., 1992) that system (9) can be employed, under certain conditions, to describe shock wave propagation in multicomponent relaxing media. Note that in the case where $b << 1$, the third (constitutive) equation of system (9) becomes identical with Lyakhov's equation for multicomponent media with one relaxing component (Lyakhov, 1982), while in the limiting case where $\tau << 1, b << 1$ it becomes identical with the so called Tait equation of state, used in the absence of relaxation processes.

In the following section we shall analyse the influence of relaxation effects as well as spatial inhomogeneity on the properties of travelling wave solutions of system (9).

3. Reduced system and pattern formation

It is well known that symmetry properties of a given system of PDE can be used to reduce the number of independent variables (Ovsyannikov, 1982). In the case of one spatial variable this procedure gives rise to an ODE system. By straightforward calculation one can check that system (9) is invariant under the Galilei algebra $AG(1,1)$, spanned by the following operators: $P_0 = \partial/\partial t, \quad P_1 = \partial/\partial x, \quad G = t\partial/\partial x + \partial/\partial u$. If $\Im = \gamma\rho$ and $n = 1$ ($\Gamma_{V\infty} << 1$), the system (9) admits an extra one-parameter group, generated by the operator $\Re = \rho\partial/\partial\rho + p\partial/\partial p$. In this work we shall use the following ansatz

$$u = D + U(w) \quad \rho = exp[\xi t + S(w)] \quad p = \rho Z(w) \quad w = x - Dt, \tag{10}$$

connected in the standard way (Ovsyannikov, 1982) with the symmetry group, generated by the operator $\Lambda = P_0 + DP_1 + \xi\Re$. The expression (10) describes a travelling wave, moving with a constant velocity D. The parameter ξ turns out to be connected with spatial inhomogeneity ahead of the wave front. To demonstrate it, let us formulate the initial value problem for system (9). So, we look for the conditions leading to the existence of self-similar solutions describing shock wave propagation. The initial value problem will be self-similar provided that both states of the medium ahead of and behind the shock front are expressed by the formula (10). Assuming the state ahead of the front to be independent of time we obtain

$$u_1 = 0, \qquad \rho_1 = exp\{\xi x/D + S_0\}, \qquad p_1 = Z_0\rho_1,$$

where S_0, Z_0 are constant parameters. These functions will satisfy the initial system if $Z_0 = \kappa$ and $\gamma = \kappa\xi/D$. Now let us use the ansatz (10). In order to simplify further analysis we shall assume from now on that $\tau_{TV} = \tau_{Tp}$ and $\tau_0 = \tau^A + \tau^B$. Inserting (10) into the equation (9) we obtain an ODE system, cyclic with respect to variable S. If one introduces the new variable $W = dU/dW \equiv \dot{U}$, then the following dynamic system is obtained:

$$U\dot{U} = UW, \qquad U\dot{Z} = \gamma U + \xi Z + W(Z - U^2) \equiv \phi,$$
$$U\dot{W} = [\beta(1 - U^2)]^{-1}\{M\phi + Z - \kappa + W[1 - MZ]\} - W^2, \tag{11}$$

where $\beta = -b < 0, M = 1 - \beta\xi$.

The only critical point of system (11) belonging to the physical parameter range (i.e. lying in the half-space $Z > 0$ beyond the manifold $U\beta(1-U^2) = 0$) is the point having the following coordinates: $U_0 = -\kappa\xi/\gamma \equiv -D, Z_0 = \kappa, W_0 = 0$. Let us introduce new variables $X = U - U_0, Y = Z - Z_0$ and separate the linear part of the system (11) from the nonlinear terms:

$$\frac{d}{d\mu}\begin{pmatrix} X \\ Y \\ W \end{pmatrix} = \begin{pmatrix} 0, & 0, & U_0 \\ \gamma, & \xi, & \Delta \\ L\gamma, & L\xi + K^{-1}, & \sigma \end{pmatrix}\begin{pmatrix} X \\ Y \\ W \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \tag{12}$$

where $d(\cdot)/d\mu = Ud(\cdot)/d\omega, L = M/K, K = \beta(1 - U_0^2), \Delta = \kappa - U_0^2, \sigma = (1 - MD^2)/K$, and

$$H_1 = WX \qquad H_2 = W[Y - X(X + 2U_0)]$$

$$\begin{aligned} H_3 = {} & \beta[2U_0X + (1 + 4\beta U_0{}^2/K)X^2](L\gamma X + L\xi Y + \sigma W)/K \\ & -W[2U_0LX + W + LX^2(1 + 4U_0{}^2\beta/K)] \\ & +O(|X||Y||W|)^3. \end{aligned} \tag{13}$$

We are going to formulate conditions that assure the existence of periodic and quasiperiodic solutions of system (12). This can be done by analytical means if the matrix $\hat{M}$ standing in the RHS of equation (12) has one zero and two pure imaginary eigenvalues. It is so if the following conditions are satisfied:

$$D^2 = 1 + \beta\xi \tag{14}$$

$$\gamma = \kappa\xi/D = 0, \tag{15}$$

$$(L\xi + K^{-1})D^2 - \xi^2 = \Omega^2 > 0 \tag{16}$$

The only way to satisfy (14)–(16) is to take κ equal to zero. Strictly speaking, this is impossible, but there will be no contradiction in assuming that $\kappa << 1$. So we put $\kappa = 0$ for a while and then introduce it into the equation as a small perturbation.

By virtue of (14) and the previous definitions, $K = -\xi\beta^2$, and inequality (16) takes on the form

$$\Omega^2 = -[1 + \xi(\beta + 1)]/(\xi\beta^2) > 0. \tag{17}$$

Our further step will be as follows. If $\kappa = 0$, D^2 is expressed by (16) and inequality (17) is satisfied, then matrix $\hat{M}$ is doubly degenerated, having one zero and two pure imaginary eigenvalues. A general analysis of $(0, \pm i\Omega)$ bifurcation given e.g., by Guckenheimer and Holmes (1984) provides the conditions leading to the existence of periodic and quasiperiodic solutions arising after the removal of degeneracy. Although the classification scheme of Guckenheimer and Holmes (1984) cannot be applied to the system (12), the canonical form technique presented there turns out to be useful. It allows for stating similar results in the case under consideration.

The possibility of passing to a very simple canonical form is directly linked with the degeneracy of the linear part of dynamical system (12). To obtain the canonical form, we first use the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\Omega^2 D} \begin{pmatrix} 0, & \xi^2, & -\xi D^2 \\ 0, & \xi D, & D \\ D\Omega^2, & -(L\xi + K^{-1})D^2, & \xi D^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ W \end{pmatrix} \tag{18}$$

leading to the system with the quasidiagonal linearization matrix $\hat{M}$: $\hat{M}_{ij} = \Omega(\delta_{i2}\delta_{j1} - \delta_{i1}\delta_{j2})$. Then we use the following local asymptotic transformation

$$x^i = \sum_{j \leq k} P^i_{jk} y^j y^k + y^i. \tag{19}$$

The coefficients P^i_{jk} can be chosen in such a way that variables y satisfy, up to $0(|y|^2)$, the system:

$$\begin{aligned} y_1' &= -\Omega y_2 + y_3(M_1 y_1 + S_1 y_2), \\ y_2' &= \Omega y_1 + y_3(S_2 y_1 + M_2 y_2), \\ y_3' &= N_1(y_1 y_1 + y_2 y_2) + N_2 y_3 y_3. \end{aligned} \tag{20}$$

(for details see Christenyuk, Danylenko et al., 1993, Appendix 2). Using the transformation $r = \sqrt{y_1^2 + y_2^2}$, $\theta = \arcsin\{y_2/\sqrt{y_1^2 + y_2^2}\}$, followed by the averaging over the "fast" variable θ we obtain the standard system (Guckenheimer and Holmes, 1984)

$$r' = a_1 r y_3^2, \qquad y_3' = N_1 r^2 + N_2 y_3^2, \tag{21}$$

where

$$a_1 = D/K, \qquad N_1 = 2(\Omega^2 + \xi^2)/[D\xi^2\beta], \qquad N_2 = 0. \tag{22}$$

The singularity of coefficient N_2 makes it impossible to apply the classification scheme of Guckenheimer and Holmes (1984). Yet the system (21) is quite easy to handle. Note that the system

$$r^{-1}r' = a_1 y_3 \qquad r^{-1}y_3' = N_1 r,$$

formally equivalent to (21), has a center if $a_1 N_1 < 0$. This center remains after the unfolding, moreover, its origin may be replaced outside of the line $r = 0$ by a proper choice of the parameters. It follows from (22) that the only way to satisfy inequality $a_1 N_1 < 0$ is to choose $\xi < 0$, since $\beta = -b < 0$. This allows to simplify the relation (17):

$$1 + \xi(\beta + 1) \equiv 1 + |\xi|(|\beta| - 1) > 0 \tag{23}$$

Fulfillment of the above inequality assures the existence of the center which can obviously be destroyed by an arbitrary small perturbation, giving rise to various localized regimes of the original system (limit cycles, toroidal attractors, etc.).

The following family of small perturbations

$$-D = U_0 \to -(D + \varepsilon), \qquad \gamma \to \delta\xi/D + \eta, \qquad \kappa \to \delta + \sigma > 0. \tag{24}$$

is introduced to remove the degeneracy of system (21). If $\eta = \sigma\xi/D$ then the perturbed system takes on the form

$$r' = \mu_1 r + a_1 r y_3, \qquad y_3' = \mu_2 y_3 + N_1 r^2 \tag{25}$$

with

$$\mu_1 = -(\lambda + D\Omega^2\varepsilon)/(K\Omega^2), \quad \mu_2 = 2\lambda/(K\Omega^2),$$

where $\lambda = -(DQ\eta + \xi\delta)/2$, $Q = 1 + 2|\beta|(\Omega^2 + \xi^2)$. Unfortunately, system (25) does not give the complete information about the regimes arising after the unfolding, for it does not contain third-order terms. Without any loss of generality we can write down the canonical form up to $O(|r|^3, |y_3|^3)$ as follows (see Guckenheimer and Holmes, 1984, Ch. VII):

$$r' = \mu_1 + a_1 r y_3, \qquad y_3' = \mu_2 y_3 + N_1 r^2 + f y_3^3. \tag{26}$$

Although the expression for f is available in Danylenko and Vladimirov (1995), it is too complicated for the analytical treatment. Nevertheless, it is still possible, based on the representation (26), to identify the regimes that are certain to arise after the unfolding.

System (26) will have a critical point belonging to the half-plane $r > 0$ provided that the following inequality holds:

$$\mu_1(\mu_2 + z_0^2 f) < 0, \tag{27}$$

where $z_0 = -\mu_1/a_1$. Assuming that condition (27) is satisfied we can rewrite system (26) in coordinates $z_1 = m(y_3 - z_0)$, $z_2 = n(y_3 - z_0) + e^q(r - r_0)$ as follows:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} l/2, & -m\tilde{\Omega} \\ m\tilde{\Omega}, & l/2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} f z_1^3/m^2 \\ a_1 z_1 z_2/m \end{pmatrix}, \tag{28}$$

where $r_0 = \sqrt{\mu_1(\mu_2 + f z_0^2)/(a_1 N_1)}$, $e^{2q} = 2|N_1/a_1|$, $l = \mu_2 + 3z_0 f$, $\tilde{\Omega}^2 = (a_1 r_0 e^q)^2 = -2\mu_1(\mu_2 + f z_0^2) > 0$, $n = -l/(2\tilde{\Omega})$ and $m = \sqrt{1 - n^2}$. Note that we have dropped the higher-order terms in the nonlinear part of the expression (28).

System (28) will possess a limit cycle provided that $|l/2| << |m\tilde{\Omega}|$ and $\mathrm{Re}\ \lambda_{1,2}\ \mathrm{Re}\ C_1 < 0$, where $\lambda_{1,2}$ are complex eigenvalues of the system linearization matrix, while C_1 is the first Floquet index, that can be easily calculated by means of standard procedure (Guckenheimer and Holmes, 1984) :

$$\mathrm{Re}\ C_1 = 6f/16 + O(|\mu_1, \mu_2|).$$

It is obvious that $\mathrm{Re}\ \lambda_{1,2} = l/2$, so the relation (27) together with the condition $\mathrm{Re}\ \lambda\ \mathrm{Re}\ C_1 < 0$ forms the following system:

$$\mu_1(\mu_2 + f z_0^2) < 0, \quad (\mu_2 + 3f z_0^2) f < 0. \tag{29}$$

If $f < 0$ and $|\mu_1, \mu_2| << 1$ this system has the solution

$$\mu_1 < 0, \qquad \mu_2 > 0, \tag{30}$$

while in the case $f > 0$ the solution is as follows

$$\mu_1 > 0, \qquad \mu_2 < 0. \tag{31}$$

The condition $|\mu_1, \mu_2| << 1$ together with the condition $\mu_2^2 << \mu_1$, assuring the validity of the above estimation can be rewritten in the form

$$(\Lambda + D\tilde{\Omega}^2\varepsilon)^2 \ll 2D\tilde{\Omega}^2 \mid \Lambda \mid \ll 8D\tilde{\Omega}^2 \mid \Lambda + D\tilde{\Omega}^2\varepsilon \mid, \tag{32}$$

where $\Lambda = K\Omega^2\mu_2/2 = -\xi[\delta + \sigma + 2 \mid \beta \mid \sigma(\Omega^2 + \xi^2)]/2$. So the obtained result can be summarized as follows:

THEOREM 3.1 *Let $\xi < 0$, D be expressed by the formula (14) and let inequality (23) hold. Then system (12) perturbed in the way described by the formula (24) possesses toroidal attractor if inequalities (32) are satisfied and either $f < 0$ and relations (30) hold, or $f > 0$ and conditions (31) are fulfilled.*

It is easy to see that in both cases a proper choice of the parameters μ_1, μ_2 will lead to appearance of periodic solutions, corresponding to the critical points of the equations (26), lying in the half-plane $r > 0$. Note that only in the case if $f < 0$ these solutions are stable, while $f > 0$ corresponds to unstable periodic and quasiperiodic motions, as can be deduced from the behavior of the canonical system (26) under the time reversal transformation.

Numerical solutions of system (11) have been obtained for the values of the parameters that satisfy the conditions of the above theorem. For $b = -0.8$, $\xi = -1.25$, $\varepsilon = 0.052$, $\delta = 0.004$ and $0.005 < \sigma < 0.1$ a stable quasiperiodic movements were observed (Fig. 1), while for $b = -0.8$, $\xi = -1.25$, $\varepsilon = -0.02$, $\delta = 0.01$ and $-0.2 < \sigma < -0.03$ unstable toroidal attractors were obtained.

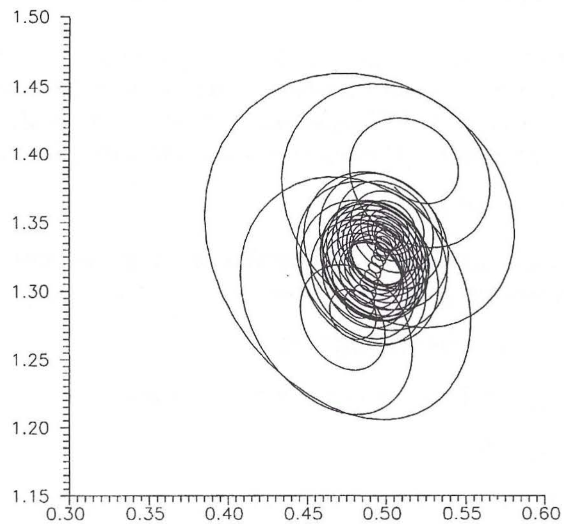

Fig. 1

For $b = -0.95$, $\xi = 0.2$, $\varepsilon = 0.065$, $\delta = 0.2$ and $\sigma = -0.00255$ a homoclinic loop was observed, corresponding to the soliton-like solution of the initial PDE system (Fig. 2). This regime turns out to be extremely sensitive to small changes of the parameters values. These changes result in creation of transient chaotic trajectories near the origin. A Rossler-type strange attractor was also observed for $b = -0.8$, $\xi = -1.25$, $\varepsilon = 0.02$, $\delta = 0.012$ and $-1.2755 < \sigma < -0.6$ (Fig. 3).

4. Concluding remarks

In this paper a system of PDE aimed at describing high-rate processes in multicomponent relaxing media has been proposed. Based on the concept of internal variables we have obtained governing equations, containing physically measurable quantities that characterize multicomponent media with two relaxing processes, in the case where the characteristic wavelength is large in comparison with the characteristic sizes of medium components and deviations from the state of complete equilibrium are not large (Vladimirov, Danylenko, Korolevich, 1990, Danevych, Danylenko et al., 1992).

The modelling system (9), obtained on the basis of the above governing equation and describing relaxing processes in single–velocity hydrodynamical approximation was studied by means of the qualitative theory methods. The

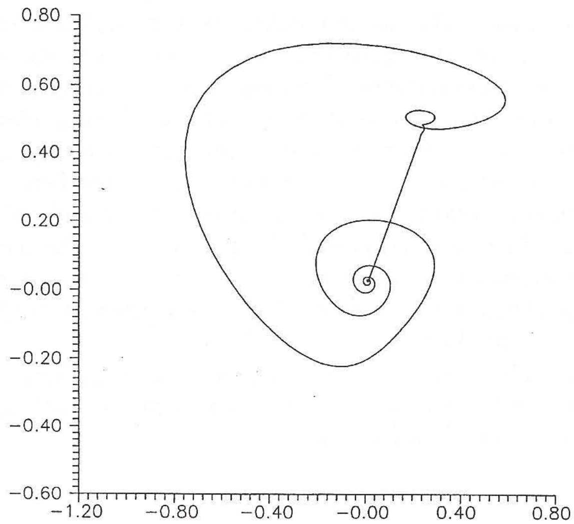
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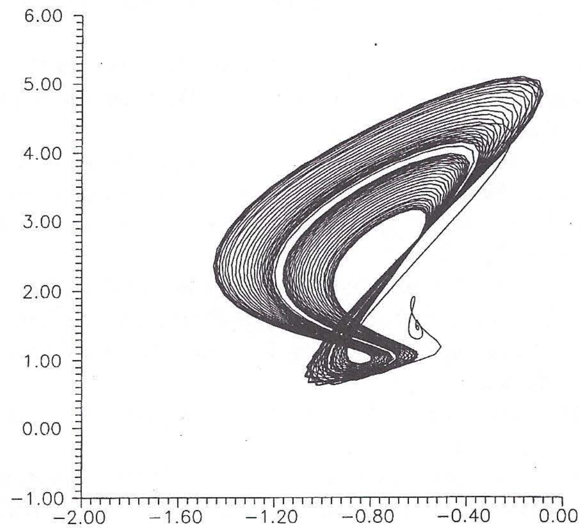
Fig. 2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−1.60
−1.20
−0.80
−0.40
0.00

Fig. 3

results of qualitative analysis as well as direct numerical simulations show that, in contrast to the classical hydrodynamical system, system (9) possesses families of periodic, quasiperiodic, chaotic and soliton-like self-similar solutions, allowing to describe self-organization phenomena widely observed in the nonequilibrium high-rate processes. The existence of the above regimes is possible due to complex interaction of nonlinear terms with the terms describing relaxing and dissipative properties of the medium. Of special interest is the fact that oscillating self–similar solutions of the dynamical system, considered here, correspond to the self-similar initial value problem of the system (9) if and only if stationary state ahead of the wave front is spatially inhomogeneous. For technical reasons the spatial inhomogeneity in this work was connected with the external force, but, generally speaking, it can be attributed to any other source (e.g., the wave moving in the opposite direction).

So the spatial inhomogeneity plays creative role in pattern formation and one is able to control the shape of the shock wave by varying the slope of inhomogeneity ahead of the wave front.

References

CHRISTENYUK, V.O., DANYLENKO, V.A ET AL. (1990) *On the Selforganization Phenomena in the Models of Relaxing and Reacting Media.* Preprint, Subbotin Institute of Geophysics, Kiev.

DANEVYCH, T.B., DANYLENKO, V.A. ET AL. (1992) *The Equations of Hydrodynamics for Active Media.* Preprint, Subbotin Institute of Geophysics, Kiev.

DANYLENKO, V.A., KUDINOV, V.M. AND MAKARENKO, A.S. (1984) Influence of Memory Effects on the Processes of Burning and Explosion. *Fizika Gorenija i Vzryva*, **20**, 3, 52-56.

DANYLENKO, V.A., SOROKINA, V.V., AND VLADIMIROV, V.A. (1993) On the Governing Equations in Relaxing Media Models and Self-Similar Quasiperiodic Solutions. *Journal of Physics, ser. A: Mathematical and General*, **26**, 9, 7125-7135.

DANYLENKO, V.A. AND VLADIMIROV, V.A. (1995) On the Self-Similar Solutions of the Generalized Hydrodynamics Equations. *Proceedings of the International Conference "Symmetry in Nonlinear Mathematical Physics"*, July 4-9, 1995, Kiev.

DE GROOT, S.R. AND MAZUR, P. (1962) *Non-Equilibrium Thermodynamics.* North-Holland, Amsterdam.

GLANSDORFF, P. AND PRIGOGINE, I. (1971) *Thermodynamic Theory of Structure, Stability and Fluctuations.* John Wiley, New York.

GUCKENHEIMER, J. AND HOLMES, P. (1984) *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* Springer-Verlag, New York.

LYAKHOV, G.M. (1982) *Waves in Soils and Multicomponent Porous Media.*

Nauka, Moscow.
MAXWELL, J.C. (1890) On the Dynamical Theory of Gases. *Sci. Papers*, **2**, Cambridge Univ. Press.
OVSYANNIKOV, L.V. (1982) *Group Analysis of the Differential Equations.* Academic Press, New York.
SAMARSKIJ, A.A. ET AL. (1987) *The Blow-up Regimes in the Quasilinear Parabolic Equations Problems.* Nauka, Moscow.
VLADIMIROV, V.A., DANYLENKO, V.A. AND KOROLEVICH, V.YU. (1990) *Nonlinear Models of the Multicomponent Relaxing Media. Qualitative Analysis and Wave Structures Evolutions.* Preprint, Subbotin Institute of Geophysics, Kiev.