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On the vibration of an elastic plate embedded with fibers of shape memory alloys: modeling and numerical simulations

by

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Abstract. A new mathematical model is proposed to describe the vibration of an elastic plate embedded with fibers of shape memory alloys. Based on the micromechanical approach of composite we consider the matrix and fibers as individual mechanical systems with interactions on their interfaces. The matrix is treated as vibrating plate and the fibers as beams. Numerical simulations are done for some simplified situations which show that both the frequency and amplitude of the vibration of the composite plate can be effectively controlled by controlling the temperatures of the fibers.

1. Introduction

Shape memory alloys are alloys exhibiting shape memory effect: after heating, such an alloy will recover its original shape however it has been deformed at the lower temperature. Typical alloys are NiTi and Cu based alloys. Close observation shows that the shape memory effect results from a phase transition, called the martensite transformation, occurring during the heating and deforming process. The high temperature phase is called austenite, it has usually a cubic crystal structure. The other phase is called martensite, which is stable at lower temperature or at bigger load and has a less symmetric crystal structure.

When the alloy is cooled from its austenitic phase, it will transform to the martensitic phase, which forms usually twins and therefore the alloy will have macroscopically the same shape as before. Upon loading one variant of the twins will increase and the other will decrease. This produces a relatively big strain, called the transformation strain. After unloading the transformation strain remains unrecovered. However, if the temperature increases, the alloy will transform back to the austenitic phase. Then the alloy recovers its original shape as if it has memorized this shape. This phenomenon is referred to as the shape memory effect and such alloys are called the shape memory alloys.

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Since their invention in 1965, shape memory alloys have attracted a great deal of attentions because of their striking property and their possible applications. In Funakubo (1987), the authors summarized the up-to-date results of the metallurgical and mechanical studies of the alloys as well as many successful applications.

Rogers and Robertshaw (1988) suggested a possible application of shape memory alloy reinforced material in acoustic and vibration control. Their idea is the following: consider an elastic plate in which several long fibers of NiTi alloy are embedded, upon heating and cooling the phase of the fibers will change, this makes in turn a change of the property of the composite plate. Specially, since the elastic modulus of NiTi alloy has a much bigger value at the austenitic phase than at the martensitic phase, the vibration frequency of the composite plate will be bigger at a higher temperature than at a lower temperature. Moreover, if the embedded fibers have some initial transformation strain, they will contract after heating so to produce a stress on the plate. When the fibers are embedded out of the mid-plane of the plate, this stress will produce a moment on it and this moment will then affect the amplitude of its vibration. Therefore, by controlling the temperatures of the fibers one may control both the frequency and the amplitude of the vibration of the composite plate.

Experimental and theoretical studies of such composites have been done by Rogers and associates. In Rogers (1990), the natural frequency, sound radiation, deflection and amplitude control of NiTi hybrid graphite-expoxy plates were investigated experimentally. The results show that the above properties can be strongly influenced by heating or cooling the NiTi fibers. A mathematical model is proposed for this problem, see Rogers et al (1991). In the model they considered the composite plate as homogenized anisotropic elastic material with its elastic modulus changing with the temperature. So classical equations for the vibration of an anisotropic elastic plate can be applied. Using Raleigh-Ritz method and special numerical techniques, they solved the equations for two different groups of elastic modulus, one is supposed to correspond to the low temperature case and the other to the high temperature. The result shows that the natural frequencies, directivity patterns and transmission loss of the two cases are significantly different and the differences agree, at least qualitatively, with the experiments.

To consider material composite of more than one elastic materials as a homogenized elastic material is now a standard approach, see e.g. Zweben et al (1989). Its advantage is certainly simplicity and the possibility to use known results. However, in the case of NiTi fiber reinforced composite, because the NiTi alloy, a shape memory alloy, is not an elastic material, the homogenized material can not be elastic any more. The phase transition of the NiTi fibers will influence the behavior of the composite. This influence, however, is very difficult to be considered in a homogenized model.

At very low or very high temperature, the NiTi fiber behaves indeed elastically for small vibrations of the plate. So the homogenized model may be used.

At a moderate temperature, the fibers may change their phases during vibration. In other words the phases of the fibers depend not only on the change of the temperatures but also on the vibration itself. Moreover, to control the amplitude of the plate it is necessary to control the temperature of each individual fiber. So it seems to us more appropriate to use the so called micromechanical approach of composites (Whitney and McCullough, 1990). Namely, we shall consider the fibers and the matrix as two thermomechanical systems with interactions on their interfaces. And we shall have separated vibration equations for the fibers and for the coat with interaction terms. In this way, we believe, the amplitude control can be easily modeled and the relatively complicated phase transition behavior of the shape memory alloy fibers and its influence on the vibration behavior of the whole system can be better investigated.

In this paper we first present a new mathematical model in which the matrix is treated as a thin plate "composed" of the matrix and the holes in which the fibers are embedded, and the fibers are treated as thin beams. So we should have classical thin plate equations for the matrix and beam equations for the fibers, both with interaction terms. In order to obtain correct formulas for the interaction terms we shall follow one of the possible ways to obtain the plate and beam equations from the balance laws, namely integration. The final equations are in Section 6. After that we give some numerical results for a simplified set of equations to demonstrate that our model is capable to produce the right results of the frequency and amplitude controls as observed in experiments.

2. Statement of the problem

Consider a rectangular plate, with length L (x-axis), width W (y-axis) and thickness $2h$ (z-axis). Inside it, several fibers are embedded regularly along the x-axis on two planes, which are parallel with each other and have distances $\pm\delta_z$ to the mid-plane of the plate. We shall call the matrix with the holes as the coat. Figure 1 shows such a plate with the distance of the fibers on each plane as $2\delta_y$ and the radius of the fibers as $r_\circ$. Typical example as given in Rogers (1990) has the following values: $L = 819.2\,mm$, $W = 21.8\,mm$, $2h = 0.86\,mm$, $\delta_y = 0.79\,mm$ and $r_\circ = 0.19\,mm$. So we shall assume in the following that $h \ll L, W$ and $\delta_y \ll W$.

The relevant balance equations are the balances of momentum and internal energy,

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial t_{ij}}{\partial x_j}\,, \tag{1}$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v_j \frac{\partial \varepsilon}{\partial x_j} = t_{ij}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + p\,, \tag{2}$$

where v_i is the velocity, t_{ij} the stress tensor, ε the internal energy, q_i the heat flux and p the power of electric current.

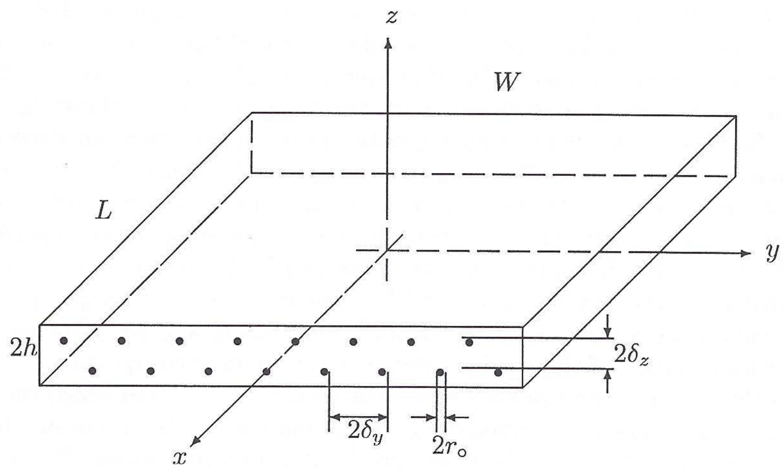


Figure 1. The composite plate.

The boundary conditions can be divided into three categories, one for the upper and lower boundaries of the plate, $\mathcal{S}_h$, another for the four edges $\mathcal{S}_e$ and the other for the interfaces between the fibers and the coat, $\mathcal{S}_i$. For $\mathcal{S}_h$, we assume that the plate is free of traction and in contact with a thermal bath. So we have

$$t_{ij}n_j = 0, \qquad T = T_a, \quad \text{at} \quad z = \pm h, \tag{3}$$

with n_i the outer normal and T the temperature.

Across the interfaces $\mathcal{S}_i$, the traction, displacement u_i and heat flux are continuous. Denote the jump by $[\![\cdot]\!]$, we have

$$[\![t_{ij}]\!]n_j = 0, \quad [\![u_i]\!] = 0, \quad [\![q_i]\!]n_i = 0 \quad \text{on} \quad \mathcal{S}_i. \tag{4}$$

We shall not specify the conditions on the edges $\mathcal{S}_e$ now, just because we do not need them to proceed.

It is usually too complicated to solve the full equations (1, 2). On the other hand we are not interested in knowing the exact solutions inside the plate but rather want to model only the vibration behavior of the plate. Therefore we shall follow the classical idea of solid mechanics to treat this problem. Namely, we shall integrate the equations for the matrix along the z-direction so to get equations for the vertical vibration and the horizontal extensions. Similarly we integrate the equation for the fibers on its cross section and obtain equations for its vertical vibration and longitudinal extension.

Before that, let us assume small velocity, small displacement gradient and small internal energy gradient, so we can neglect the quadratic terms $\rho v_j \frac{\partial v_i}{\partial x_j}$ in

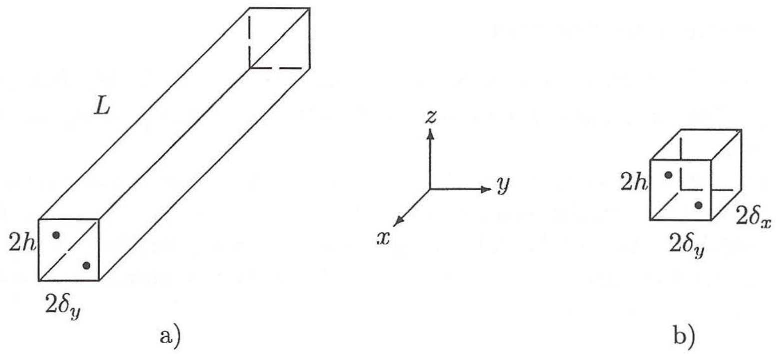


Figure 2. A beam with two fibers and an infinitesimal portion.

(1) and $\rho v_j \frac{\partial \varepsilon}{\partial x_j}$ in (2). Take the density ρ as constants $\rho_{\pm,m}$ for the upper, lower fibers and the matrix, we get the simplified balance equations as

$$\rho_{\pm,m} \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial t_{ij}}{\partial x_j}, \tag{5}$$

$$\rho_{\pm,m} \frac{\partial \varepsilon}{\partial t} = t_{ij} \frac{\partial^2 u_i}{\partial t \partial x_j} - \frac{\partial q_j}{\partial x_j} + p. \tag{6}$$

3. Integrated equations

Observe that the fibers are regularly embedded, we can cut the plate into rectangular beams all (possibly but two on the edge) with length L, width $2\delta_y$ and thickness $2h$, such that each such beam has only two fibers embedded in it. Figure 2a shows such a beam. A cross-section of the beam, $\Omega(x, y_p)$, can be divided into three regions, $\Omega_c(x, y_p)$ for the coat and $\Omega_\pm(x, y_p)$ for the upper and lower fibers, where y_p is the y-coordinate of the center line of the beam.

We consider an infinitesimal portion of such a beam, $\delta V : [x - \delta_x, x + \delta_x]$, which has three parts: the coat δV_c , the upper fiber δV_+ and the lower fiber δV_-, as shown by Figure 2b.

We define in the following the middle value of a field $f(x, y, z, t)$ on $\Omega_{\pm,c}(x, y_p)$ as

$$\overline{f}^{\pm,c}(x, y_p, t) := \frac{1}{A_{\pm,c}} \int\!\!\int_{\Omega_{\pm,c}(x,y_p)} f(x, y, z, t)\, dydz, \tag{7}$$

with $A_{\pm,c}$ the area of $\Omega_{\pm,c}(x, y_p)$, $A_\pm := \pi r_\circ^2$ and $A_c = 4h\delta_y - 2\pi r_\circ^2$. And we denote the volume fractions of the fibers and the matrix as $\nu_{f,m}$. From the geometry of the plate we have $\nu_f = (A_+ + A_-)/A_\Omega$ and $\nu_m = A_c/A_\Omega$ with $A_\Omega = 4h\delta_y$ the area of the cross section Ω.

3.1. Equations for the coat

We integrate the balance of momentum (5) on δV_c and divide the both sides by $2\delta_x A_\Omega$. The left hand side equals to $\rho_c \frac{\partial^2}{\partial t^2}\overline{u_i}^c(x, y_p, t)$ and $\rho_c = \nu_m \rho_m$, as δ_x goes to zero.

Using the divergence theorem, the right hand side can be written as a sum of integrals of $t_{ij}n_j$ on the surfaces $\{z = \pm h\}$, $\Omega_c(x \pm \delta_x, y_p)$, $\{y = y_p \pm \delta_y\}$ and the interfaces $\delta V_c \cap \delta V_\pm$. The integral on $\{z = \pm h\}$ vanishes by (3). On $\Omega_c(x \pm \delta x, y_p)$ the outer normal are $(\pm 1, 0, 0)$, so the corresponding surface integral can be written as

$$\frac{\nu_m}{2\delta_x}[\overline{t_{ix}}^c(x + \delta_x, y_p, t) - \overline{t_{ix}}^c(x - \delta_x, y_p, t)].$$

As δ_x goes to zero, this goes to $\frac{\partial}{\partial x}\nu_m \overline{t_{ix}}^c(x, y_p, t)$.

The normal on $\{y = y_p \pm \delta_y\}$ is $(0, \pm 1, 0)$, so the integral is

$$\frac{1}{2\delta_x A_\Omega}\int_{x-\delta_x}^{x+\delta_x} d\xi \int_{-h}^{h} [t_{iy}(\xi, y_p + \delta_y, z) - t_{iy}(\xi, y_p - \delta_y, z)]\, dz.$$

As δ_x goes to zero, this goes to

$$\frac{1}{A_\Omega}\int_{-h}^{h} [t_{iy}(x, y_p + \delta_y, z) - t_{iy}(x, y_p - \delta_y, z)]\, dz.$$

This is in turn $\frac{\partial}{\partial y}\nu_m \overline{t_{iy}}^c(x, y_p)$. Here, the derivative with respect to the discrete variable y_p should be understood as the difference. However, when $\delta_y \ll W$, the two forms differ little from each other.

Hence we get the integrated equation for the coat as

$$\rho_c \frac{\partial^2 \overline{u_i}^c}{\partial t^2} - \frac{\partial \nu_m \overline{t_{ix}}^c}{\partial x} - \frac{\partial \nu_m \overline{t_{iy}}^c}{\partial y} = \varphi_i^c, \tag{8}$$

where φ_i^c is the limit of the surface integral of the stress on the interfaces. It represents the force of the two fibers acted on the coat and has the form

$$\varphi_i^c := \frac{1}{A_\Omega}[\oint_{\mathcal{C}+} + \oint_{\mathcal{C}-}] t_{ij} n_j \, d\ell, \tag{9}$$

with $\mathcal{C}^\pm := \Omega_c \cap \Omega_\pm$, the intersection of the cross sections.

To describe the vibration of the coat we need also the momentum equations, namely times (5) with z and integrate it over δV_c. Observe that $z t_{ij,j} = (z t_{ij})_{,j} - t_{iz}$ and follow the same procedure as before to treat the divergence term $(z t_{ij})_{,j}$, we obtain the integrated momentum equation as

$$\rho_c \frac{\partial^2 \overline{z u_i}^c}{\partial t^2} - \frac{\partial \nu_m \overline{z t_{ix}}^c}{\partial x} - \frac{\partial \nu_m \overline{z t_{iy}}^c}{\partial y} = \psi_i^c - \nu_m \overline{t_{iz}}^c, \tag{10}$$

with ψ_i^c the momentum of the force of the fibers acted on the coat, which has the form,

$$\psi_i^c := \frac{1}{A_\Omega}[\oint_{\mathcal{C}+} + \oint_{\mathcal{C}-}] z t_{ij} n_j \, d\ell. \tag{11}$$

The balance of the internal energy (6) can be treated in the same way, we have

$$\rho_c \frac{\partial \overline{\varepsilon}^c}{\partial t} - \nu_m \overline{t_{ij} \frac{\partial^2 u_i}{\partial t \partial x_j}}^c + \frac{\partial \nu_m \overline{q_x}^c}{\partial x} + \frac{\partial \nu_m \overline{q_y}^c}{\partial y} = -\tau^c - \tau^a, \tag{12}$$

where τ^c and τ^a are the heat flux through the interfaces $\mathcal{S}_i$ and the outer boundary $z = \pm h$, respectively,

$$\begin{aligned} \tau^c &:= \tfrac{1}{A_\Omega}[\oint_{\mathcal{C}+} + \oint_{\mathcal{C}-}] q_j n_j \, d\ell, \\ \tau^a &:= \tfrac{1}{A_\Omega} \int_{z=\pm h} q_j n_j \, d\ell. \end{aligned} \tag{13}$$

The power of electric current p for the coat is assumed to be zero.

3.2. Equations for the fibers

Integrate (5) over $\delta V_\pm$ and divide both sides by $2\delta A_\pm$, we get that the first term approaches $\rho_\pm \frac{\partial^2}{\partial t^2}\overline{u_i}^\pm$, as δ_x goes to zero. Again using the divergence theorem the other term reduced to surface integrals over the cross sections $\Omega_\pm(x \pm \delta_x, y_p)$ and over the interfaces. So as δ_x goes to zero, we get the equations as

$$\rho_\pm \frac{\partial^2 \overline{u_i}^\pm}{\partial t^2} - \frac{\partial \overline{t_{ix}}^\pm}{\partial x} = \varphi_i^\pm, \tag{14}$$

where $\varphi_i^\pm$ is the force of the coat on the upper (lower) fiber

$$\varphi_i^\pm := \frac{1}{A_\pm} \oint_{\mathcal{C}^\pm} t_{ij} n_j \, d\ell. \tag{15}$$

Since the fibers are on the planes with $z = \pm\delta_z$, the momentum equations are obtained by integrating $(z \mp \delta_z)$ times (5). Similar as the equation for the coat, we have the integrated moment equation for the fibers as

$$\rho_\pm \frac{\partial^2}{\partial t^2} \overline{(z \mp \delta_z) u_i}^\pm - \frac{\partial}{\partial x} \overline{(z \mp \delta_z) t_{ix}}^\pm = \psi_i^\pm - \overline{t_{iz}}^\pm, \tag{16}$$

with $\psi_i^\pm$ the moment of the force of the coat applied on the fibers,

$$\psi_i^\pm := \frac{1}{A_\pm} \oint_{\mathcal{C}^\pm} (z \mp \delta_z) t_{ij} n_j \, d\ell. \tag{17}$$

The equation for the internal energy reduced to

$$\rho_\pm \frac{\partial \overline{\varepsilon}^\pm}{\partial t} - \overline{t_{ij} \frac{\partial^2 u_i}{\partial t \partial x_j}}^\pm + \frac{\partial \overline{q_x}^\pm}{\partial x} = -\tau^\pm + \overline{p}^\pm, \tag{18}$$

with

$$\tau^{\pm} := \frac{1}{A_{\pm}} \oint_{C^{\pm}} q_j n_j \, d\ell. \tag{19}$$

3.3. Relations between the interaction terms

Because the stress and the heat flux are continuous across the interfaces, see (4), the interaction terms, $\varphi_i^{\pm,c}$, $\psi_i^{\pm,c}$ and $\tau^{\pm,c}$ defined by (9, 11, 15, 17, 13, 19) are not independent from each other. Remember that n_i is the outer normal of the interfaces respect to each integration volume, we have

$$\begin{aligned} \varphi_i^c &= -\nu_f(\varphi_i^+ + \varphi_i^-), \\ \psi_i^c &= -\nu_f[\psi_i^+ - \psi_i^- + \delta_z(\varphi_i^+ - \varphi_i^-)], \\ \tau^c &= -\nu_f(\tau^+ + \tau^-). \end{aligned} \tag{20}$$

The last term in the equation for the momentum ψ_i^c shows that the interaction forces $\varphi_i^{\pm}$ will induce a momentum to the coat, if the fibers are not embedded on the middle plane.

To proceed further, we need to know how the stress, the internal energy and the heat flux are related to the displacement and the temperature, namely the constitutive equations.

4. Constitutive assumptions

By the example in Rogers (1990), the matrix is made of graphite-epoxy composites and the fibers are made of NiTi alloy which exhibits shape memory effect.

During the vibration of the plate, we expect that there should be no permanent plastic deformation. This means that the strain in the matrix has to be within its elastic range. From the property of the graphite-epoxy composite, see e.g. Lee (1991), we can conclude that the admissible strain has to be very small. Because the fibers are very thin and are assumed fixedly embedded inside the coat, therefore the admissible strain in the fibers has to be very small as well.

4.1. Constitutive equations for the linear elastic matrix

For small strain, we can assume that the matrix is linear elastic, and so it obeys Hooke's law

$$t_{ij} = \mathbf{C}_{ijkl} e_{kl} \tag{21}$$

with $e_{kl} := (u_{k,l} + u_{l,k})/2$ the strain tensor and $\mathbf{C}_{ijkl}$ the elastic constants.

However, if the matrix has no symmetry, all the 36 components of $\mathbf{C}_{ijkl}$ are independent, it is difficult to proceed further. Fortunately, in many relevant cases, the matrix does have some symmetry. In Rogers (1990), the matrix is made of graphite-epoxy composites, a composite material with orthotropic symmetry, namely symmetry under reflections about three perpendicular planes.

For orthotropic material, the elastic constants have only 9 independent components. In terms of the engineering constants, Young's modules E_i, Poisson's ratios ν_{ij} and the shear modules G_{ij}, we have the strain-stress relation as

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{13} \\ e_{23} \\ e_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & \frac{-\nu_{13}}{E_1} & 0 & 0 & 0 \\ \frac{-\nu_{21}}{E_2} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{31}}{E_3} & \frac{-\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{22} \\ t_{33} \\ t_{13} \\ t_{23} \\ t_{12} \end{pmatrix} \tag{22}$$

where $\nu_{12}/E_1 = \nu_{21}/E_2$, $\nu_{13}/E_1 = \nu_{31}/E_3$ and $\nu_{23}/E_2 = \nu_{32}/E_3$.

The internal energy ε is a function of the temperature T and the strain,

$$\rho_m \frac{\partial \varepsilon}{\partial t} = \rho_m \frac{\partial \varepsilon}{\partial T} \frac{\partial T}{\partial t} + \rho_m \frac{\partial \varepsilon}{\partial e_{ij}} \frac{\partial e_{ij}}{\partial t}.$$

Neglect the thermal expansion, we should have

$$\rho_m \frac{\partial \varepsilon}{\partial e_{ij}} = \rho_m \frac{\partial f}{\partial e_{ij}} = t_{ij}, \tag{23}$$

where f is the free energy. Denote the specific heat per unit volume $\rho_m \frac{\partial \varepsilon}{\partial T}$ by c_v^m, we get that

$$\rho_m \frac{\partial \varepsilon}{\partial t} = c_v^m \frac{\partial T}{\partial t} + t_{ij} \frac{\partial e_{ij}}{\partial t}. \tag{24}$$

We assume finally that the heat flux obeys the Fourier law,

$$q_i = -k_{ij} \frac{\partial T}{\partial x_j}, \tag{25}$$

with

$$k_{ij} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}. \tag{26}$$

4.2. Constitutive assumptions for the fibers of shape memory alloys

As we have pointed out in Introduction, the shape memory alloy has a rather complicated temperature – dependent hysteretic behavior. As far as the authors' knowledge is concerned, there are no general accepted constitutive equations for such material. The difficulty is certainly because of the phase transitions involved. Therefore approximations are unavoidable.

For small strain as in our case, we expect that the strain can be separated into two parts, one is responsible for the stress, and may be called the elastic strain, and the other is due to the phase transition. And the stress may be approximately taken as a linear function of the small elastic strain. Moreover, since the NiTi alloys used in practice are usually isotropic polycrystals, the stress-strain relation would have the following form,

$$t_{ij} = \lambda(e_{kk} - e^{ph}_{kk})\delta_{ij} + 2\mu(e_{ij} - e^{ph}_{ij}), \tag{27}$$

with λ and μ the Lamè constants and e^{ph}_{ij} the strain due to the phase transition.

We know that, see Funakubo (1987), for a single crystal specimen the martensitic phase has many possible variants, each of which contributes a strain at some specific direction. So for the polycrystalline NiTi fiber, the strain due to the phase transition has generally all components. However, since the fibers are very thin and long, we expect that the most relevant strain is the one in the length direction. Therefore, we may assume that all non-diagonal elements of e^{ph}_{ij} are zero and the strains in the y- and z-axis come just from the transverse contraction, or extension, resulting from the extension, or contraction, in the x-axis. Namely, we assume that

$$e^{ph}_{ij} = \begin{pmatrix} e_{ph} & 0 & 0 \\ 0 & -\nu e_{ph} & 0 \\ 0 & 0 & -\nu e_{ph} \end{pmatrix} \tag{28}$$

Now the stress-strain relation (27) reduces to

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - E e_{ph}. \tag{29}$$

Where ν is the Poisson's ratio and E the Young's modulus defined by

$$\nu := \frac{\lambda}{2(\lambda+\mu)} \qquad \text{and} \qquad E := \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \tag{30}$$

The reference configuration is the stress-free one at which the fiber is in the martensitic phase with possible initial transformation strain. e_{ph} is then the strain of phase transition form this reference configuration.

To simplify the problem, we may assume that there are only two kinds of martensite: M_1 and M_2, of which M_1 produces a positive strain in the x-direction and M_2 a negative strain. And e_{ph} is a function of the mass fractions of M_α, denoted in the following by X_α and called as the phase fractions.

Under the above assumptions, the internal energy of the fibers should be a function of the temperature, the strain and the phase fractions. So we have

$$\rho_f \frac{\partial \varepsilon}{\partial t} = \rho_f \frac{\partial \varepsilon}{\partial T}\frac{\partial T}{\partial t} + \rho_f \frac{\partial \varepsilon}{\partial X_\alpha}\frac{\partial X_\alpha}{\partial t} + \rho_f \frac{\partial \varepsilon}{\partial e_{ij}}\frac{\partial e_{ij}}{\partial t}. \tag{31}$$

Since $\varepsilon = f + Ts$ with f the free energy and s the entropy, we can write $\rho_f \frac{\partial \varepsilon}{\partial X_\alpha}$ into a sum of the derivatives of f and s. We denote in the following the latent heat of phase transition $T\rho_f \frac{\partial s}{\partial X_\alpha}$ by $T\Delta_s^\alpha$, the dissipation coefficients $-\rho_f \frac{\partial f}{\partial X_\alpha}$ by D_α and the specific heat $\rho_f \frac{\partial \varepsilon}{\partial T}$ by c_v^f. Neglect again the thermal expansion, we have

$$\rho_f \frac{\partial \varepsilon}{\partial t} = c_v^f \frac{\partial T}{\partial t} + T\Delta_s^\alpha \frac{\partial X_\alpha}{\partial t} - D_\alpha \frac{\partial X_\alpha}{\partial t} + t_{ij}\frac{\partial e_{ij}}{\partial t}. \tag{32}$$

Finally we assume that the heat flux obeys the Fourier law,

$$q_i = -k_f \frac{\partial T}{\partial x_j} \tag{33}$$

and the power of electric current is given by

$$p = j^2 \rho_r^f, \tag{34}$$

with j the electric current density and ρ_r^f the specific electric resistance.

The phase fractions X_α depend generally on the changes of the temperature and stresses with a hysteretic character. Several one-dimensional models exist already, however it seems to us that they are all not very suitable in this or that respect. So we prefer not to specify the equations for X_α here. More about this point will be discussed in the last section.

5. Simplifications

The above constitutive equations can not be directly used in the integrated equations. We need to make further simplifications.

5.1. Thin plate assumptions for the coat

Since the fibers are usually embedded along one of the principal axes of the matrix, we shall assume in the following that the coat is an orthotropic thin plate, "composed" of the matrix and the holes, with its three symmetric planes parallel to the three coordinate planes. In classical bending theory of thin rectangular plates, one assumes that the material exhibits an infinite rigidity to transverse normal and shearing strains, see e.g. Dawson (1976). Consider also tensions in the plate, we assume that the displacements as

$$\overline{u_i}^c = (u_c(x, y_p, t),\ v_c(x, y_p, t),\ w_c(x, y_p, t)),$$

$$u_i = (u_c - z\frac{\partial w_c}{\partial x}, v_c - z\frac{\partial w_c}{\partial y}, w_c), \tag{35}$$
$$\overline{zu_i}^c = -I_c(\frac{\partial w_c}{\partial x}, \frac{\partial w_c}{\partial y}, 0),$$

with $I_c := \frac{1}{A_c}\int\int_{\Omega_c} z^2\, dydz$. The relevant average stresses and moments are assumed accordingly as

$$\begin{aligned}
\nu_m\overline{t_{xx}}^c &= \frac{\tilde{E}_1}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}(\frac{\partial u_c}{\partial x} + \tilde{\nu}_{21}\frac{\partial v_c}{\partial y}),\\
\nu_m\overline{t_{yy}}^c &= \frac{\tilde{E}_2}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}(\frac{\partial v_c}{\partial y} + \tilde{\nu}_{12}\frac{\partial u_c}{\partial x}),\\
\nu_m\overline{t_{xy}}^c &= \tilde{G}_{12}(\frac{\partial u_c}{\partial y} + \frac{\partial v_c}{\partial x}),\\
\nu_m\overline{zt_{xx}}^c &= -\frac{I_c\tilde{E}_1}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}(\frac{\partial^2 w_c}{\partial x^2} + \tilde{\nu}_{21}\frac{\partial^2 w_c}{\partial y^2}),\\
\nu_m\overline{zt_{yy}}^c &= -\frac{I_c\tilde{E}_2}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}(\frac{\partial^2 w_c}{\partial y^2} + \tilde{\nu}_{12}\frac{\partial^2 w_c}{\partial x^2}),\\
\nu_m\overline{zt_{xy}}^c &= -2I_c\tilde{G}_{12}\frac{\partial^2 w_c}{\partial x\partial y}.
\end{aligned} \tag{36}$$

For the internal energy, heat flux and temperature, we assume, by (24) and (25), that

$$\begin{aligned}
\rho_c\frac{\partial\overline{\varepsilon}^c}{\partial t} &= c_v^c\frac{\partial\overline{T}^c}{\partial t} + \overline{\nu_m t_{ij}\frac{\partial e_{ij}}{\partial t}}^c,\\
\nu_m\overline{q_x}^c &= -\tilde{k}_1\frac{\partial\overline{T}^c}{\partial x}, \quad \text{and} \quad \nu_m\overline{q_y}^c = -\tilde{k}_2\frac{\partial\overline{T}^c}{\partial y},\\
\overline{T}^c &= T_c(x, y_p, t)
\end{aligned} \tag{37}$$

where $\rho_c := \nu_m\rho_m$, $c_v^c := \nu_m c_v^m$, and $\tilde{E}_{1,2}$, $\tilde{\nu}_{12,21}$, $\tilde{G}_{12}$, $\tilde{k}_{1,2}$ are the corresponding quantities for the "composed" plate, which are related to the quantities of the matrix and the geometry of the holes. We suggest to use the self-consistent field method of composite mechanics to calculate them, see Section 7 for more details.

5.2. Beam assumptions for the fibers

The fibers will be assumed to be in simple bending in the x-z plane, so we take the average values on $\Omega_\pm(x, y_p)$ as

$$\overline{u_i}^\pm = (u_\pm(x, y_p, t),\ v_\pm(x, y_p, t),\ w_\pm(x, y_p, t)),$$
$$\overline{t_{xx}}^\pm = E_f^\pm(\frac{\partial u_\pm}{\partial x} - e_{ph}^\pm), \tag{38}$$

$$
\begin{aligned}
\overline{(z \mp \delta_z)u_i}^{\pm} &= (-I_{\pm}\frac{\partial w_{\pm}}{\partial x}, 0, 0), \\
\overline{(z \mp \delta_z)t_{xx}}^{\pm} &= -I_{\pm}E_f^{\pm}\frac{\partial^2 w_{\pm}}{\partial x^2},
\end{aligned}
$$

with $I_{\pm} := \frac{1}{A_{\pm}} \int\int_{\Omega_{\pm}} (z \mp \delta_z)^2 \, dydz$.

In the y-direction, we assume simply that the fibers move together with the coat. By (35.2), we have

$$
v_{\pm}(x, y_p, t) = u_y(x, y_p, \pm\delta_z, t) = v_c(x, y_p, t) \mp \delta_z \frac{\partial w_c}{\partial y}. \tag{39}
$$

So $v_{\pm}$ is not an independent variable.

Correspondingly for the internal energy, heat flux, temperature, phase fractions and power of electric current, we assume, by (32), (33) and (34), that

$$
\begin{aligned}
&\rho_f \frac{\partial \overline{\varepsilon}^{\pm}}{\partial t} = c_v^{\pm}\frac{\partial \overline{T}^{\pm}}{\partial t} + \overline{T}^{\pm}\Delta_s^{\alpha}\frac{\partial \overline{X_\alpha}^{\pm}}{\partial t} - D_\alpha^{\pm}\frac{\partial \overline{X_\alpha}^{\pm}}{\partial t} + \overline{t_{ij}\frac{\partial e_{ij}}{\partial t}}^{\pm}, \qquad (40)\\
&\overline{q_x}^{\pm} = -k_f \frac{\partial \overline{T}^{\pm}}{\partial x}, \quad \overline{T}^{\pm} = T_{\pm}(x, y_p, t), \quad \overline{X_\alpha}^{\pm} = X_\alpha^{\pm}(x, y_p, t), \quad \overline{p}^{\pm} = j_{\pm}^2\rho_r^{\pm}.
\end{aligned}
$$

5.3. Simple forms for the interaction terms

The interaction terms depend generally on the quantities of both the coat and the fibers. And the dependencies can be and usually are very complicated. Approximations are therefore necessary.

For the heat of contacting $\tau's$, however, it is classical to assume that they are proportional to the difference of the temperatures of the contacting bodies. Namely,

$$
\begin{aligned}
\tau^a &= C_a(T_c - T_a), \\
\tau^{\pm} &= C_f(T_{\pm} - T_c),
\end{aligned} \tag{41}
$$

with T_a the temperature on the upper and the lower boundary of the plate and $C's$ the coefficients of heat transfers.

For other terms, we suggest in the following the simplest possibility which comes to our mind in the spirit of the above forms for $\tau's$. These forms should only be taken as the first approximations.

For the interaction forces $\varphi's$, we assume simply that they are proportional to the differences of the displacements,

$$
\begin{aligned}
\varphi_x^{\pm} &= -K_x(u_{\pm} - u_c \pm \delta_z \frac{\partial w_c}{\partial x}), \\
\varphi_z^{\pm} &= -K_z(w_{\pm} - w_c), \\
\varphi_y^{\pm} &= 0.
\end{aligned} \tag{42}
$$

The moments of the interaction forces $\psi's$ are also taken as proportional to the differences of the derivatives of the deflections,

$$\psi_x^{\pm} = M_x(\frac{\partial w_{\pm}}{\partial x} - \frac{\partial w_c}{\partial x}),$$
$$\psi_y^{\pm} = M_y(\frac{\partial w_{\pm}}{\partial y} - \frac{\partial w_c}{\partial y}). \tag{43}$$

REMARK 5.1 *Just like the thermal conducting coefficients $C_{a,f}$, all the other coefficients $K_{x,z}$ and $M_{x,y}$ must be measured by experiments.*

6. The final equations

The vibration equations for the coat are obtained by putting (35, 36, 42, 43) into (8, 10, 20). The relevant equations are

$$\rho_c \frac{\partial^2 u_c}{\partial t^2} - \frac{\tilde{E}_1}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} \frac{\partial^2 u_c}{\partial x^2} - \tilde{G}_{12}\frac{\partial^2 u_c}{\partial y^2} - (\frac{\tilde{E}_1\tilde{\nu}_{21}}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} + \tilde{G}_{12})\frac{\partial^2 v_c}{\partial x \partial y}$$
$$= \nu_f K_x(u_+ + u_- - 2u_c), \tag{44}$$

$$\rho_c \frac{\partial^2 v_c}{\partial t^2} - \frac{\tilde{E}_2}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} \frac{\partial^2 v_c}{\partial y^2} - \tilde{G}_{12}\frac{\partial^2 v_c}{\partial x^2} - (\frac{\tilde{E}_2\tilde{\nu}_{12}}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} + \tilde{G}_{12})\frac{\partial^2 u_c}{\partial x \partial y}$$
$$= 0, \tag{45}$$

$$\left\{\begin{array}{l}
\rho_c \dfrac{\partial^2 w_c}{\partial t^2} - \dfrac{\partial \nu_m \overline{t_{zx}}^c}{\partial x} - \dfrac{\partial \nu_m \overline{t_{zy}}^c}{\partial y} = \nu_f K_z(w_+ + w_- - 2w_c), \\
-I_c\rho_c \dfrac{\partial^3 w_c}{\partial t^2 \partial x} + \dfrac{I_c\tilde{E}_1}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}\dfrac{\partial^3 w_c}{\partial x^3} + I_c(\dfrac{\tilde{E}_1\tilde{\nu}_{21}}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} + 2\tilde{G}_{12})\dfrac{\partial^3 w_c}{\partial x \partial y^2} \\
= -\nu_f M_x(\dfrac{\partial(w_+ + w_-)}{\partial x} - 2\dfrac{\partial w_c}{\partial x}) + \nu_f K_x\delta_z(u_+ - u_- + 2\delta_z\dfrac{\partial w_c}{\partial x}) \\
-\nu_m \overline{t_{xz}}^c, \\
-I_c\rho_c \dfrac{\partial^3 w_c}{\partial t^2 \partial y} + \dfrac{I_c\tilde{E}_2}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}\dfrac{\partial^3 w_c}{\partial y^3} + I_c(\dfrac{\tilde{E}_2\tilde{\nu}_{12}}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} + 2\tilde{G}_{12})\dfrac{\partial^3 w_c}{\partial y \partial x^2} \\
= -\nu_f M_y(\dfrac{\partial(w_+ + w_-)}{\partial y} - 2\dfrac{\partial w_c}{\partial y}) - \nu_m \overline{t_{yz}}^c
\end{array}\right. \tag{46}$$

The first two equations govern the extension of the coat in the x-y plane and the last three are for the deflection in the z-direction. Since $\overline{t_{yz}}^c = \overline{t_{zy}}^c$ and $\overline{t_{xz}}^c = \overline{t_{zx}}^c$, we can eliminate them from the above equations and obtain a single equation for w_c as

$$\rho_c\frac{\partial^2 w_c}{\partial t^2} + D_{11}\frac{\partial^4 w_c}{\partial x^4} + 2D_{12}\frac{\partial^4 w_c}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_c}{\partial y^4} = I_c\rho_c(\frac{\partial^4 w_c}{\partial t^2 \partial x^2} + \frac{\partial^4 w_c}{\partial t^2 \partial y^2})$$
$$-\nu_f M_x(\frac{\partial^2(w_+ + w_-)}{\partial x^2} - 2\frac{\partial^2 w_c}{\partial x^2}) - \nu_f M_y(\frac{\partial^2(w_+ + w_-)}{\partial y^2} - 2\frac{\partial^2 w_c}{\partial y^2})$$
$$+\nu_f K_x\delta_z(\frac{\partial(u_+ - u_-)}{\partial x} + 2\delta_z\frac{\partial^2 w_c}{\partial x^2}) + \nu_f K_z(w_+ + w_- - 2w_c). \tag{47}$$

where $D_{\partial\beta}$ are the flexure rigidities defined as

$$\begin{aligned} D_{11} &:= \frac{I_c\tilde{E}_1}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}, \qquad D_{22} := \frac{I_c\tilde{E}_2}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}, \\ D_{12} &:= I_c(\frac{\tilde{E}_2\tilde{\nu}_{12}}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}} + 2\tilde{G}_{12}). \end{aligned} \tag{48}$$

$I_c\rho_c(\frac{\partial^4 w_c}{\partial t^2\partial x^2} + \frac{\partial^4 w_c}{\partial t^2\partial y^2})$ is the inertia term and is usually much smaller than the first term in the above equation, so can be neglected.

The energy balance reduced to the following equation for the temperature by using (37, 41, 20) in (12),

$$c_v^c\frac{\partial T_c}{\partial t} - \tilde{k}_x\frac{\partial^2 T_c}{\partial x^2} - \tilde{k}_y\frac{\partial^2 T_c}{\partial y^2} = \nu_f C_f(T_+ + T_- - 2T_c) - C_a(T_c - T_a). \tag{49}$$

Similarly put (38, 42, 43) into (14, 16), we get the following relevant equations for the fibers,

$$\rho_\pm\frac{\partial^2 u_\pm}{\partial t^2} - \frac{\partial}{\partial x}(E_f^\pm(\frac{\partial u_\pm}{\partial x} - e_{ph}^\pm)) = -K_x(u_\pm - u_c \pm \delta_z\frac{\partial w_c}{\partial x}), \tag{50}$$

$$\left\{\begin{aligned} &\rho_\pm\frac{\partial^2 w_\pm}{\partial t^2} - \frac{\partial\overline{t_{zx}}^\pm}{\partial x} = -K_z(w_\pm - w_c), \\ &-I_\pm\rho_\pm\frac{\partial^3 w_\pm}{\partial t^2\partial x} + I_\pm\frac{\partial}{\partial x}(E_f^\pm\frac{\partial^2 w_\pm}{\partial x^2}) = M_x(\frac{\partial w_\pm}{\partial x} - \frac{\partial w_c}{\partial x}) - \overline{t_{xz}}^\pm \end{aligned}\right. \tag{51}$$

The first equation is for the longitudinal motion and last two for the deflection. Again by eliminating $\overline{t_{xz}}^\pm = \overline{t_{zx}}^\pm$ we get the following equation for $w_\pm$

$$\begin{aligned} &\rho_\pm\frac{\partial^2 w_\pm}{\partial t^2} + I_\pm\frac{\partial^2}{\partial x^2}(E_f^\pm\frac{\partial^2 w_\pm}{\partial x^2}) = I_\pm\rho_\pm\frac{\partial^4 w_\pm}{\partial t^2\partial x^2} + M_x(\frac{\partial^2 w_\pm}{\partial x^2} - \frac{\partial^2 w_c}{\partial x^2}) \\ &\quad -K_z(w_\pm - w_c). \end{aligned} \tag{52}$$

The internal energy equation reduced, by using (40, 41) in (18), to

$$c_v^\pm\frac{\partial T_\pm}{\partial t} + T_\pm\Delta_s^\alpha\frac{\partial X_\alpha^\pm}{\partial t} - D_\alpha^\pm\frac{\partial X_\alpha^\pm}{\partial t} - k_f\frac{\partial^2 T_\pm}{\partial x^2} = -C_f(T_\pm - T_c) + j_\pm^2\rho_r^\pm. \tag{53}$$

To complete we need the equations for the phase fractions $X_\alpha^\pm$. We do not want to specify them here and will discuss this aspect more in the last section.

REMARK 6.1 *The coefficients, $\rho_{c,\pm}$, $I_{c,\pm}$, $\tilde{E}_{1,2}$, $\tilde{\nu}_{12,21}$, $\tilde{G}_{12}$, ν_f, $K_{x,z}$, $M_{x,y}$, c_v^c, $\tilde{k}_{x,y,f}$, $C_{a,f}$ and δ_z are all positive constants. The other coefficients $E_f^\pm$, $e_{ph}^\pm$, $c_v^\pm$, Δ_s^α, $D_\alpha^\pm$ and $\rho_r^\pm$ are generally functions of the phase fractions, the temperature and the strains. And the exact formulas depend on what model we choose for the phase transitions.*

REMARK 6.2 *If the in-plane stresses $\overline{t_{xx}}^c$, $\overline{t_{xy}}^c$ and $\overline{t_{yy}}^c$ of (36) become rather big, they will affect the vertical vibration. So instead of (47) we would have*

$$\rho_c \frac{\partial^2 w_c}{\partial t^2} + D_{11}\frac{\partial^4 w_c}{\partial x^4} + 2D_{12}\frac{\partial^4 w_c}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_c}{\partial y^4}$$
$$-\frac{\partial}{\partial x}(\nu_m \overline{t_{xx}}^c \frac{\partial w_c}{\partial x} + \nu_m \overline{t_{xy}}^c \frac{\partial w_c}{\partial y}) - \frac{\partial}{\partial x}(\nu_m \overline{t_{xy}}^c \frac{\partial w_c}{\partial x} + \nu_m \overline{t_{yy}}^c \frac{\partial w_c}{\partial y})$$
$$= I_c \rho_c (\frac{\partial^4 w_c}{\partial t^2 \partial x^2} + \frac{\partial^4 w_c}{\partial t^2 \partial y^2})$$
$$-\nu_f M_x(\frac{\partial^2 (w_+ + w_-)}{\partial x^2} - 2\frac{\partial^2 w_c}{\partial x^2}) - \nu_f M_y(\frac{\partial^2 (w_+ + w_-)}{\partial y^2} - 2\frac{\partial^2 w_c}{\partial y^2})$$
$$+\nu_f K_x \delta_z(\frac{\partial (u_+ - u_-)}{\partial x} + 2\delta_z \frac{\partial^2 w_c}{\partial x^2}) + \nu_f K_z(w_+ + w_- - 2w_c). \quad (54)$$

Similarly the equation for the fiber (52) changes to

$$\rho_\pm \frac{\partial^2 w_\pm}{\partial t^2} + I_\pm \frac{\partial^2}{\partial x^2}(E_f^\pm \frac{\partial^2 w_\pm}{\partial x^2}) - \frac{\partial}{\partial x}(\overline{t_{xx}}^\pm \frac{\partial w_\pm}{\partial x})$$
$$= I_\pm \rho_\pm \frac{\partial^4 w_\pm}{\partial t^2 \partial x^2} + M_x(\frac{\partial^2 w_\pm}{\partial x^2} - \frac{\partial^2 w_c}{\partial x^2}) - K_z(w_\pm - w_c). \quad (55)$$

REMARK 6.3 *If the plate is vibrating in a viscous fluid, e.g. air, water, oil and so on, we should add a damping term $C_D \frac{\partial w_c}{\partial t}$ to the left hand side of (47). And if there is a distributed force $g(x, y, t)$ applied on the upper or lower faces of the plate, we should add it to the right hand side of (47).*

7. Some numerical results

In order to show that the above set of equations does predict the right experimental results of the frequency and amplitude control of the vibration, we give here some numerical calculations by using finite element methods. We consider only simplified situations here and leave the mathematical analysis and the numerical calculation of the full set of equations to future works.

7.1. Equations for temperature controlled experiments

We assume that the temperatures of the fibers are controlled directly and are either very high such that the fibers are in austenitic phase, or very low such that they are in martensitic phase. Under this assumption we do not need to solve the equations for the temperatures and the phase fractions.

We assume further that the vertical displacement of the fibers and the coat are always the same, so $w_\pm = w_c$. Neglect the inertia terms and the in-plane displacement u_c, we have the following simplified equations for w_c and $u_\pm$,

$$\rho_c \frac{\partial^2 w_c}{\partial t^2} + D_{11}\frac{\partial^4 w_c}{\partial x^4} + 2D_{12}\frac{\partial^4 w_c}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_c}{\partial y^4}$$

$$= \nu_f K_x \delta_z (\frac{\partial(u_+ - u_-)}{\partial x} + 2\delta_z \frac{\partial^2 w_c}{\partial x^2}),$$

$$\rho_\pm \frac{\partial^2 u_\pm}{\partial t^2} - \frac{\partial}{\partial x}(E_f^\pm(\frac{\partial u_\pm}{\partial x} - e_{ph}^\pm)) = -K_x(u_\pm \pm \delta_z \frac{\partial w_c}{\partial x}). \tag{56}$$

Since the above system of partial differential equations is linear, there is no problem in mathematical analysis, e.g. existence and uniqueness of solutions. Convergence of finite element numerical scheme is also guaranteed.

7.2. Relevant parameters

The geometry of the plate is assumed as

$$L = 1\,m, \quad \frac{L}{W} = 4, \quad \frac{2h}{L} = 10^{-2}, \quad \frac{r_\circ}{2h} = 0.2, \quad \frac{\delta_z}{h} = 0.5, \quad \frac{\delta_y}{h} = 0.4\pi,$$
$$\nu_f = 0.2. \tag{57}$$

According to Rogers (1990), the matrix is assumed to be graphite-epoxy fiber-reinforced composite. From Lee (1991, page 116, table 2) we know that for T300, a transverse isotropic composite,

$$\rho_m = 1.6\,g/cm^3, \quad \nu_{12} = 0.28, \; E_1 = 181\,GPa, \; E_2 = 10.3\,GPa,$$
$$G_{12} = 7.17\,GPa. \tag{58}$$

To obtain the corresponding parameters for the coat as a "composite" of the matrix and the holes, we use the so called self-consistent field relationship, see Whitney and McCullough (1990). Assume that the parameters of the holes are all zero and the holes are along the axis of the transverse symmetry of the matrix, we have

$$\rho_c = \nu_m \rho_m, \quad \tilde{\nu}_{12} = \nu_{12}, \; \tilde{E}_1 = \nu_m E_1, \; \tilde{E}_2 = \frac{\nu_m E_1}{1 + 2\nu_f(1 - \nu_{12}\nu_{21})},$$
$$\tilde{G}_{12} = \frac{\nu_m G_{12}}{1 + \nu_f}. \tag{59}$$

The fibers are made of NiTi alloys, by Cross et al (1969) we have

$$\rho_\pm = 6.48 g/cm^3, \qquad E_f = \begin{cases} 8.44\,GPa & \text{for martensite,} \\ 61.9\,GPa & \text{for austenite.} \end{cases} \tag{60}$$

Assume that the fibers have an initial transformation strain $\varepsilon_\circ = 1\%$, so we have

$$e_{ph}^\pm = \begin{cases} 0 & \text{for martensite,} \\ -\varepsilon_\circ = -1\% & \text{for austenite.} \end{cases} \tag{61}$$

We know little about the interaction coefficient K_x, so we choose arbitrarily

$$K_x = 6.11 \times 10^3\,GPa/m^2. \tag{62}$$

7.3. Boundary and initial conditions

We assume that the plate is fixed at $x = 0$ and free on other three edges. So

$$u_\pm = w_c = \frac{\partial w_c}{\partial x} = 0, \quad \text{for} \quad x = 0; \tag{63}$$

$$\begin{cases} t_{xx}^\pm := E_f^\pm(\frac{\partial u_\pm}{\partial x} - e_{ph}^\pm) = 0, \quad M_{xx} := -D_{11}(\frac{\partial^2 w_c}{\partial x^2} + \tilde{\nu}_{21}\frac{\partial^2 w_c}{\partial y^2}) = 0, \\ Q_{xx} := -D_{11}\frac{\partial^3 w_c}{\partial x^3} - D_{12}\frac{\partial^3 w_c}{\partial x \partial y^2} + \nu_f K_x \delta_z (u^+ - u^- + 2\delta_z \frac{\partial w_c}{\partial x}) = 0, \\ \text{for} \quad x = L; \end{cases}$$

$$M_{yy} := -D_{22}(\frac{\partial^2 w_c}{\partial y^2} + \tilde{\nu}_{12}\frac{\partial^2 w_c}{\partial x^2}) = Q_{yy} := -D_{22}\frac{\partial^3 w_c}{\partial y^3} - D_{12}\frac{\partial^3 w_c}{\partial x^2 \partial y} = 0$$
$$\text{for } y = W.$$

$$M_{xy} := -2I_c\tilde{G}_{12}\frac{\partial^2 w_c}{\partial x \partial y} = 0, \text{ for } x = L, y = 0, \text{ or } W.$$

The initial values are chosen as

$$w_c = w_c^\circ(x), \quad u_\pm = u_\pm^\circ(x), \qquad \frac{\partial w_c}{\partial t} = \frac{\partial u_\pm}{\partial t} = 0, \qquad \text{for } t = 0. \tag{64}$$

Where $w_c^\circ(x)$ and $u_\pm^\circ(x)$ are the stationary solutions of the equation (56) under the boundary condition (63.1), (63.3), $t_{xx}^\pm = M_{xx} = 0$ and $w_c = w_m = 0.5\,m$ at $x = L$. When all the fibers have the same temperature so are all either in martensitic or in austenitic phase, we can solve the equations analytically and we have that

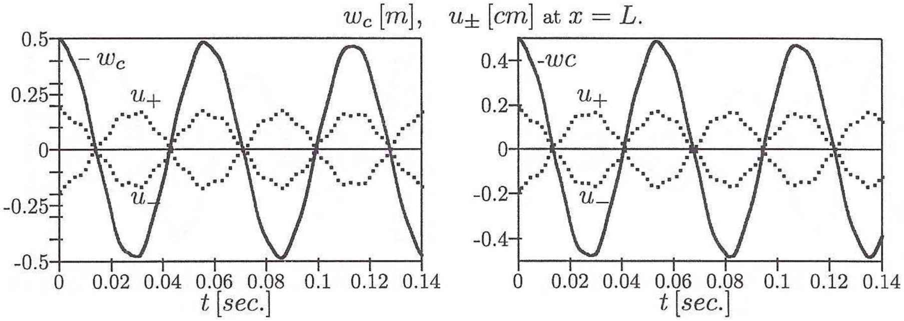
$$\begin{aligned} \frac{w_c^\circ}{w_m} &= \frac{1}{2}A(\frac{x}{L})^2(3 - \frac{x}{L}) + 3\frac{AB}{q^2}\left(\frac{x}{L} + \frac{\sinh(q(1-\frac{x}{L}))}{q\cosh(q)} - \frac{\tanh(q)}{q}\right), \\ \frac{u_\pm^\circ - e_{ph}^\pm x}{\pm\delta_z w_m} &= -\frac{3}{2}A\frac{x}{L}(2 - \frac{x}{L}) + 3\frac{A}{q^2}\left(1 - \frac{\cosh(q(1-\frac{x}{L}))}{\cosh(q)}\right), \end{aligned} \tag{65}$$

with $B = 2\nu_f\delta_z^2 E_f/D_{11}$, $q^2 = \frac{K_x L^2}{E_f}(1+B)$ and $A = 1/(1 + \frac{3B}{q^2}(1 - \frac{\tanh(q)}{q}))$.

7.4. Numerical calculations

We use linear elements for fiber equations and Zienk elements for the coat equation. We use the central difference method for the acceleration terms and decouple the equations by taking the values of the coupling terms from the previous time step. In the following we choose $N_x = 20$, $N_y = 5$ and $dt = t_s \times 10^{-4}$ with $t_s = 3.257 \times 10^{-2}\,sec.$

Figure 3 shows the value of w_c, $u_\pm$ at $x = L$ as functions of the time for two different temperatures. In Figure 3a, the fibers have a lower temperature at which they are in martensitic phases and in Figure 3b they have a higher



a) $T = T_M < T_A$, b) $T = T_A$.

Figure 3. Vibration of the plate at different temperatures. (In b) $u_\pm$ is oscillating around their equilibrium value $u_\pm^e := e_{ph}^\pm L = -1\,cm$.)

temperature such that they are all in austenitic phase. From the figure we conclude that the vibration frequency is bigger at the higher temperature as observed in experiment (Rogers, 1990).

Thus we can control the amplitude by controlling the temperatures of the upper and lower fibers. For example, if we control the temperatures in such a way that the effective moment is always opposite to the direction of the vibration, the amplitude will decrease very rapidly as shown by our calculation in Figure 5a. This coincides qualitatively to the experiment in Rogers (1990).

Since the fibers have initial transformation strains, they will contract so as to produce a force to the coat when they are heated to the austenite. If all fibers are heated simultaneously like in Figure 3b, they produce an in-plane stress which

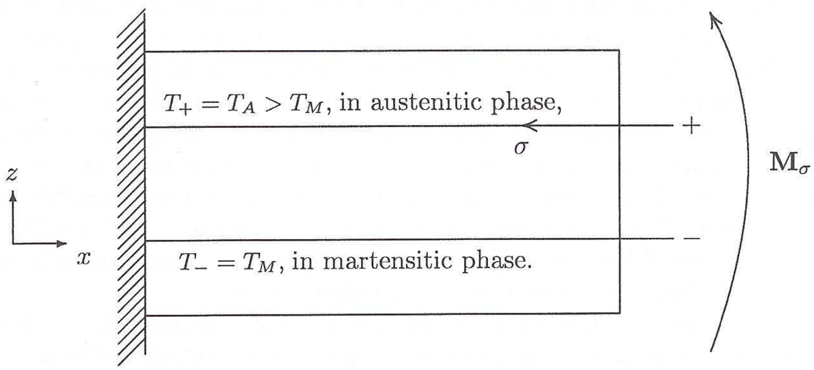
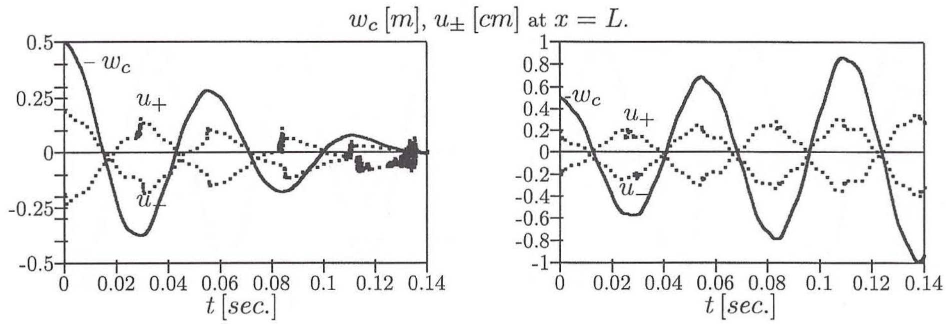


Figure 4. Effective moment produced by the fibers.



a) Decreasing, b) Increasing.

Figure 5. Amplitude control through controlling temperatures.

affects the vibration very little, so we have neglect this effect here. However, if, for example, only the upper fibers are heated, as schematically shown in figure 4, an effective moment will be produced by the contraction forces. This moment will affect the vibration amplitude very strongly.

In figure 5b, the temperatures of the fibers are so controlled to have the effective moment always in the direction of the vibration, thus the amplitude increases strongly.

8. Discussions

The numerical examples in the last section show that the composite plate which we considered is indeed a good candidate to be used in acoustic and vibration control, since its frequency and amplitude can be controlled effectively by controlling the temperatures of the embedded NiTi fibers. However, in practice it is difficult to control the temperature directly, because the fibers are embedded inside the matrix. Rather, one may heat the fibers by electric currents going through them and cool them by keeping the plate in a low temperature thermal bath. And we have to manage the heating–cooling process to make the phase transition of the fibers in such a way that the resulting effective moment fits to our needs. Therefore, the present model with the equations (49) and (53) for the temperatures of the coat and of each fibers gives us the possibility to treat this control problem, in contrast to homogenized models.

Since the austenite–martensite transition is of the first-order, latent heat is released or absorbed during the transition. And this will influence the heating–cooling control process strongly. Therefore it seems very important to have a good mathematical model to describe as accurately as possible the phase transition behavior of the NiTi fibers, if we want our numerical calculations to be somehow close to the reality in order to give information on how to control the frequency and the amplitude of the vibration of the composite plate.

Two properties of the NiTi alloy are essentially used in the above simulations. Namely, its elastic modulus has a much bigger value in the austenitic than in the martensitic phase and an initial transformation strain can be imposed to it. The first property leads to the frequency control of the plate. The second provides us the possibility to control the amplitude. Therefore, any useful mathematical model for the austenite–martensite transition must at least have these two features.

Achenbach and Müller (1985) proposed a one-dimensional model for shape memory alloy based on statistical considerations. However, in this model the elastic modulus is assumed to be smaller in the austenitic than in the martensitic phase and this assumption is crucial and so it cannot be replaced. Similarly, the model based on Landau-Ginzburg free energy, used by Falk (1984) for shape memory alloy, assumes also that the low temperature phase has a bigger elastic modulus than the high temperature phase. In the model proposed by Frémond (1990), the elastic modulus can be chosen to satisfy our criterion. However, the states of phase mixtures are assumed there to be unstable, so according to this model (see Wörsching, 1994), a fiber with some initial transformation strain, which is in a state of mixture of many martensitic variants, cannot remain stable and will transform to have only one variant. Thus none of the above one-dimensional models assumes the two important properties of the NiTi alloys.

It seems to us that in rate-dependent hysteretic models, like all the three mentioned previously, the state of phase mixtures is always assumed to be unstable. So in order to treat the transition between the martensitic phase mixtures and the austenite, one needs rate-independent models. For shape memory alloys, as far as the authors' knowledge is concerned, there are still no satisfactory mathematical models of such a type .

Even when we have a good hysteretic model for the fibers, the mathematical and numerical analyses involved in such a system of nonlinear partial differential equations are still not trivial. Specially in view of the applications, one should look at the problem of optimal control of the vibrations. And it is also very useful to consider forced vibrations of the plate. The force acting on the plate can be even of stochastic nature, e.g. noise. In this case the control of the amplitude becomes certainly much more difficult. Hence, more work has to be done on the above proposed mathematical model.

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