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Contact problem for a plate having a crack of minimal opening[1]

by

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Abstract. The paper concerns the equilibrium problem for a plate contacting with a rigid punch and having a vertical crack. Two conditions of inequality type are assumed to be imposed on the solution. These conditions describe a mutual nonpenetration in the plate–punch system and a nonpenetration of crack faces. The first one is of the form $w \geq \varphi$, where w is the vertical displacement of the plate, and φ corresponds to the punch shape. The second one can be written as $\left| \left[\frac{\partial w}{\partial \nu} \right] \right| \leq \frac{1}{\varepsilon}[W]\nu$, where $W = (w^1, w^2)$ is the horizontal displacement, ν is the normal to the crack shape curve, 2ε is the thickness of the plate, and $[\cdot]$ is the jump of a function at crack faces. The aim of the paper is to study the solution properies of the optimal control problem of the punch shape φ. The existence theorem is proved as providing the minimal jump of the displacement $\chi = (W, w)$. The solution regularity is analyzed up to the interior crack points. In particular, the inclusion $\chi \in C^\infty$ is stated to be valid for the crack points having a zeroth jump. The convergence of solution is investigated as $\varepsilon \to 0$.

Keywords: variational inequality, optimal control, solution regularity, solution convergence, nonsmooth boundary, crack.

1. Introduction

The object of this paper is to analyze the solution properties of the variational inequality describing the equilibrium state of the elastic plate. The plate is assumed to have a vertical crack and, simultaneously, to contact with a rigid punch. The vertical section of the plate-punch system is represented in Fig. 1.

Considering the crack, we impose the nonpenetration condition of the inequality type at the crack faces. The nonpenetration condition for the plate-punch system also has the inequality type. It is well known that, in general,

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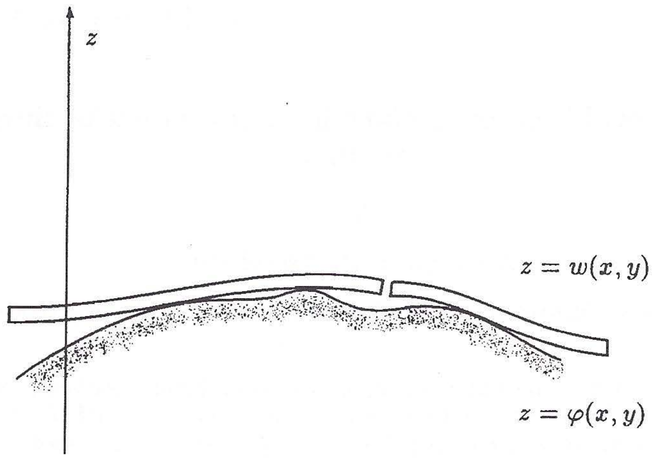

Figure 1.

solutions of problems having restrictions of inequality type are not smooth. In this paper, we establish new existence and regularity results related to the problem considered. Namely, the following questions are under consideration:

1. The existence of punch shape which provides the minimal opening of the crack.
2. The regularity of solutions in the case of minimal opening of the crack.
3. The solutions properties related to the case where the thickness of the plate trends to zero.

We consider the well-known Kirchhoff-Love model of the plate for which both vertical and horizontal displacements of the middle surface points are to be found. The displacements of other points of the plate can be easily found by the corresponding formulae (see further on).

Let us introduce the notations and give the appropriate formulae of the Kirchhoff-Love model which can be found, for instance, in Donnell (1976), Vol'mir (1972). Denote a bounded domain with a smooth boundary Γ by $\Omega \subset R^2$, and $y = \psi(x)$ signifies the function describing a crack face, $x \in [0, 1]$, $(x, y) \in \Omega$. Let Γ_ψ be the graph of the function $y = \psi(x)$ and $\Omega_\psi = \Omega \setminus \Gamma_\psi$.

The domain Ω_ψ is identified with the middle surface of the plate in its nondeformable state. The displacement vector of the middle surface points is denoted by $\chi = (W, w)$, $W = (w^1, w^2)$ is the horizontal displacement , w is the vertical one.

We next assume that the graph $z = \varphi(x, y)$ corresponds to the punch shape, $(x, y) \in \Omega$. Then the nonpenetration condition for the plate-punch system can

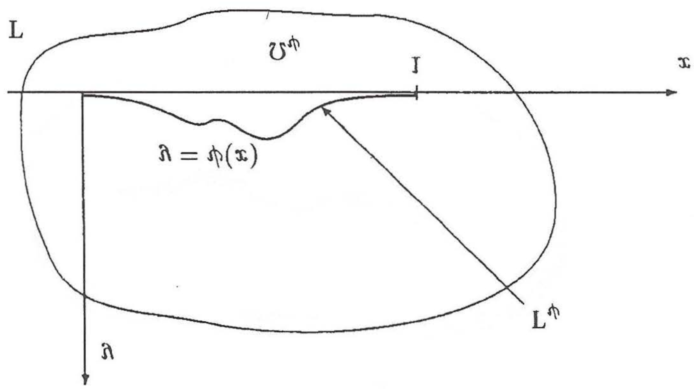

Figure 2.

be written as

$$w \geq \varphi \quad \text{in} \quad \Omega_\psi. \tag{1}$$

The Kirchhoff-Love model of the plate is characterized by the linear dependence of the horizontal displacements on the distance from the middle surface, that is

$$W(z) = W - z\nabla w, \quad -\varepsilon \leq z \leq \varepsilon,$$

where $z = 0$ corresponds to the middle surface, and the axis z is orthogonal to the (x, y)–plane, 2ε is the thickness of the plate. Denote the normal to the graph Γ_ψ by $\nu = \frac{(-\psi_x, 1)}{\sqrt{1+\psi_x^2}}$, $\nu = (\nu_1, \nu_2)$. In this case, the nonpenetration condition of Signorini type imposed at the crack faces is as follows

$$[W - z\nabla w]\nu \geq 0 \quad \text{on} \quad \Gamma_\psi, \quad |z| \leq \varepsilon,$$

where $[V] = V^+ - V^-$ is the jump of V, and $V^\pm$ correspond to the positive and negative directions of ν, respectively. As evident from above, the nonpenetration condition can be rewritten in the equivalent form

$$\varepsilon \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \leq [W]\nu \quad \text{on} \quad \Gamma_\psi. \tag{2}$$

Thus, we see that there is no penetration for all points of the crack faces since condition (2) is independent of $z \in [-\varepsilon, \varepsilon]$. The general view of the plate having the vertical crack is depicted in Fig. 3.

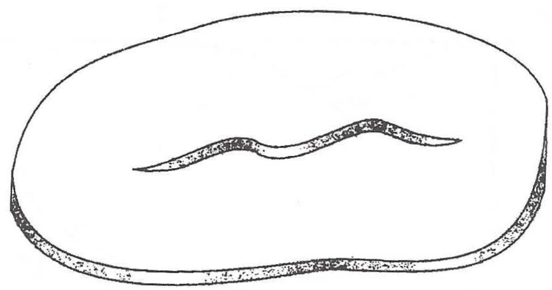

Figure 3.

The strain and integrated stress tensors are denoted by $e_{ij} = e_{ij}(W), \sigma_{ij} = \sigma_{ij}(W)$, respectively:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\right), \quad i,j = 1,2, \; x_1 = x, x_2 = y,$$

$$\sigma_{11} = e_{11} + \kappa e_{22}, \; \sigma_{22} = e_{22} + \kappa e_{11}, \; \sigma_{12} = (1-\kappa)e_{12}.$$

Here $\kappa = const$ is the Poisson's ratio, $0 < \kappa < \frac{1}{2}$.

The following boundary conditions are assumed to be fulfilled at the external boundary

$$w = \frac{\partial w}{\partial n} = W = 0 \quad \text{on} \quad \Gamma.$$

Let the subspace $H^{1,0}(\Omega_\psi)$ of the Sobolev space $H^1(\Omega_\psi)$ consist of functions equal to zero on Γ. Analogously, the functions of $H^{2,0}(\Omega_\psi)$ are equal to zero on Γ together with the first derivatives, $H^{2,0}(\Omega_\psi) \subset H^2(\Omega_\psi)$. Define the space $H(\Omega_\psi) = H^{1,0}(\Omega_\psi) \times H^{1,0}(\Omega_\psi) \times H^{2,0}(\Omega_\psi)$ and consider the energy functional of the plate

$$\Pi(\chi) = \frac{1}{2}B(w,w) + \frac{1}{2}\langle \sigma_{ij}(W), e_{ij}(W)\rangle - \langle f, \chi\rangle.$$

Here $f = (f_1, f_2, f_3) \in L^2(\Omega)$ is the given vector of exterior forces, the brackets $\langle \cdot, \cdot \rangle$ mean the integration over Ω_ψ,

$$B(u,v) = \int_{\Omega_\psi} (u_{xx}v_{xx} + u_{yy}v_{yy} + \kappa u_{xx}v_{yy} + \kappa u_{yy}v_{xx} + 2(1-\kappa)u_{xy}v_{xy})d\Omega_\psi.$$

The above formula for $\Pi(\chi)$ contains three different terms which correspond to the bending energy of the plate, to the deformation energy of the middle surface, and to the work of the exterior force f, respectively. Also, we introduce the set of admissible displacements

$$K_\varepsilon^\varphi = \left\{ (W,w) \in H(\Omega_\psi) \;\middle|\; (W,w) \text{ satisfy } (1),(2) \right\}.$$

The equilibrium problem for the plate can be formulated as variational, namely, it corresponds to the minimum of the functional Π over the set of admissible displacements. To minimize the functional Π over the set K_ε^φ we can consider the variational inequality

$$\begin{aligned} &B(w,\bar{w}-w) + \langle \sigma_{ij}(W), e_{ij}(\bar{W}-W)\rangle \geq \\ &\quad \geq \langle f, \bar{\chi}-\chi\rangle, \quad \chi=(W,w)\in K_\varepsilon^\varphi, \quad \forall\, \bar{\chi}\in K_\varepsilon^\varphi. \end{aligned} \tag{3}$$

Nonemptiness of the set K_ε^φ depends on the values of the function φ on Γ. For further consideration we should note that the following inequality holds in Ω_ψ

$$B(w,w) \geq c\|w\|^2_{2,\Omega_\psi} \quad \forall\, w \in H^{2,0}(\Omega_\psi), \tag{4}$$

and the first Korn inequality takes place

$$\langle \sigma_{ij}(W), e_{ij}(W)\rangle \geq c\|W\|^2_{1,\Omega_\psi} \quad \forall\, W \in H^{1,0}(\Omega_\psi) \tag{5}$$

with the constants independent of w, W, respectively. The relations (4), (5) provide for the coercivity of the functional Π on $H(\Omega_\psi)$. Thus, considering the weak lower semicontinuity of Π, one concludes that there exists a solution of (3). Moreover, the solution is unique.

In the sequel we shall study an optimal control problem. Let $\Phi \subset H^2(\Omega)$ be a convex, bounded and closed set. Assume that $\varphi < 0$ on Γ for each $\varphi \in \Phi$. In particular, this condition secures nonemptiness of K_ε^φ . Denote the solution of (3) by $\chi = \chi(\varphi)$, and introduce the cost functional which characterizes the opening of the crack

$$J_\varepsilon(\varphi) = \int_{\Gamma_\psi} |[\chi]| d\Gamma_\psi.$$

The problem of finding an obstacle providing the minimal opening of the crack can be formulated as follows

$$\inf_{\varphi\in\Phi} J_\varepsilon(\varphi). \tag{6}$$

The crack shape is defined by the function ψ. This function is assumed to be fixed. It is noteworthy that the problems of choice of the so-called extreme crack shapes were considered by Khludnev (1992, 1994). In particular, a nonpenetration condition considered in the first paper exactly corresponds to (2) with $\varepsilon = 0$

and can be considered as approximate one. The solution regularity for biharmonic variational inequalities was analyzed by Frehse (1973), Caffarelli et al. (1979), Schild (1984). The last paper also contains the results on the solution smoothness in the case of thin obstacles. As for general solution properties for the equilibrium problem of the plates having cracks, one may refer to the book by Morozov (1984). Referring to this book, the boundary conditions imposed on crack faces have the equality type. In this case there is no interaction between the crack faces. Asymptotic properties of solutions for problems having a nonsmooth boundary were analyzed by Kondratiev et al. (1982), Oleinik et al. (1981). In these papers the boundary conditions also have the equality type.

In the next two sections the parameter ε is supposed to be fixed. The convergence of solutions of the optimal control problem (6) as $\varepsilon \to 0$ will be analyzed in section 4. For this reason the ε-dependence of the cost functional is indicated.

Also, we have to note at this point that within the Kirchhoff model the unilateral condition (2) cannot properly describe the behaviour of a through-the-thickness fissure in a plate subject to bending when ε is equal to zero. From the physical point of view our considerations apply rather to membrane states, thus precluding bending. Other possibilities require very special types of loadings, for instance compressed plates when the fissure is always closed.

2. Existence of solutions

Let ε be fixed. Before proving the theorem an auxiliary statement is to be established. It is formulated as lemma.

LEMMA 2.1 *Let a sequence $\varphi_m \in \Phi$ possess the properties*

$$\varphi_m \to \varphi \quad \text{weakly in} \quad H^2(\Omega) \quad \text{and uniformly in} \quad \bar{\Omega}. \tag{7}$$

Then for any fixed $\bar{\chi} = (\bar{W}, \bar{w}) \in K_\varepsilon^\varphi$ there exists a sequence $\bar{\chi}_m = (\bar{W}_m, \bar{w}_m) \in K_\varepsilon^{\varphi_m}$ such that

$$\bar{\chi}_m \to \bar{\chi} \quad \text{strongly in} \quad H(\Omega_\psi). \tag{8}$$

Proof. Without loss, the following inequality

$$|\varphi_m - \varphi| < \frac{1}{m} \quad \text{in} \quad \bar{\Omega}$$

is supposed to hold. Set $\tilde{w}_m = \bar{w} + \frac{1}{m}$. In this case $\tilde{w}_m \geq \varphi_m$ in Ω_ψ. There exists a neighbourhood $\mathcal{O}$ of the boundary Γ such that the relation

$$\varphi < -\delta < 0$$

holds in $\mathcal{O} \cap \Omega$ with a constant $\delta > 0$. Let $\Gamma_\psi \cap \mathcal{O} = \emptyset$. In view of the uniform convergence of φ_m the following inequality holds

$$\varphi_m < -\frac{\delta}{2} \quad \text{in} \quad \mathcal{O} \cap \Omega.$$

It is easy to construct a sequence $\xi_m \in C^\infty$ such that the supports of ξ_m belong to $\mathcal{O}$ and

$$\xi_m = \frac{1}{m}, \quad \nabla \xi_m = 0 \quad \text{on} \quad \Gamma,$$
$$|D^\alpha \xi_m| \le \frac{c}{m} \quad \text{in} \quad \mathcal{O}, \quad |\alpha| \le 2,$$

with a constant c independent of m. Now, we can define

$$\bar{w}_m = \tilde{w}_m - \xi_m.$$

It is clear that $\bar{w}_m \ge \varphi_m$ in Ω_ψ and $\left[\frac{\partial \bar{w}_m}{\partial \nu}\right] = \left[\frac{\partial \bar{w}}{\partial \nu}\right]$ on Γ_ψ. Thus the functions $\bar{\chi}_m = (\bar{W}, \bar{w}_m)$ belong to $K_\varepsilon^{\varphi_m}$ for all m. Moreover, convergence (8) takes place. The proof is completed.

We now are in a position to establish solvability of the optimal control problem (6), (3).

THEOREM 2.1 *There exists a solution of the problem (6), (3).*

Proof. Let $\varphi_m \in \Phi$ be a minimizing sequence. It is bounded in $H^2(\Omega)$, hence convergence (7) can be assumed. For every m, the solution of the following variational inequality can be found:

$$B(w_m, \bar{w}_m - w_m) + \langle \sigma_{ij}(W_m), e_{ij}(\bar{W}_m - W_m)\rangle \ge$$
$$\ge \langle f, \bar{\chi}_m - \chi_m \rangle, \quad \chi_m = (W_m, w_m) \in K_\varepsilon^{\varphi_m}, \quad \forall\, \bar{\chi}_m \in K_\varepsilon^{\varphi_m}. \tag{9}$$

By virtue of the uniform convergence of φ_m there exists a function $\bar{\chi}$ such that $\bar{\chi} \in K_\varepsilon^{\varphi_m}$ for all m. Substituting this function in (9) as $\bar{\chi}_m$ implies

$$\|\chi_m\|_{H(\Omega_\Psi)} \le c$$

uniformly in m. In deriving this estimate we make use of the inequalities (4), (5). Hence choosing a subsequence, if necessary, we assume as $m \to \infty$

$$\chi_m \to \chi \quad \text{weakly in} \quad H(\Omega_\psi). \tag{10}$$

Let $\bar{\chi} \in K_\varepsilon^\varphi$ be any fixed element where φ is the function from (7). Lemma ensures the existence of a sequence $\bar{\chi}_m \in K_\varepsilon^{\varphi_m}$ strongly converging to $\bar{\chi}$ in $H(\Omega_\psi)$. Bearing in mind (10), this allows us to carry out the limiting procedure in (9). The resulting relation precisely coincides with (3), i.e. $\chi = \chi(\varphi)$. An additional assumption

$$\chi_m^\pm \to \chi^\pm \quad \text{weakly in} \quad L^1(\Gamma_\psi)$$

easily yields the relations

$$\inf_\Phi J_\varepsilon(\bar{\varphi}) = \underline{\lim} \int_{\Gamma_\psi} |[\chi_m]| d\Gamma_\psi \ge \int_{\Gamma_\psi} |[\chi]| d\Gamma_\psi \ge \inf_\Phi J_\varepsilon(\bar{\varphi}).$$

This means that the function φ is a solution of the problem (6), (3) which completes the proof.

3. Solution regularity

Let $Q \subset R^2$ be a bounded domain with a smooth boundary γ. An external normal to γ is denoted by $n = (n_1, n_2)$. Introduce the following operators defined at the boundary γ

$$m(u) = \kappa \Delta u + (1-\kappa)\frac{\partial^2 u}{\partial n^2},$$
$$t(u) = \frac{\partial}{\partial n}\Delta u + (1-\kappa)\frac{\partial^3 u}{\partial n \partial^2 s}, \quad s = (-n_2, n_1).$$

It is well known (John et al., 1976) that for any fixed $u \in H^2(Q), \Delta^2 u \in L^2(Q)$ the values $m(u), t(u)$ can be considered as elements of $H^{-\frac{1}{2}}(\gamma)$ and $H^{-\frac{3}{2}}(\gamma)$, respectively. Moreover, the Green formula

$$B_Q(u,v) = \langle m(u), \frac{\partial v}{\partial n}\rangle_{\frac{1}{2},\gamma} - \langle t(u), v\rangle_{\frac{3}{2},\gamma} + \langle \Delta^2 u, v\rangle_Q \tag{11}$$

takes place for all $v \in H^2(Q)$. Here, the integration is carried out over Q, and $\langle \cdot, \cdot \rangle_{p,\gamma}$ means a duality pairing between $H^{-p}(\gamma)$ and $H^p(\gamma)$. Besides, one more Green's formula holds good (Temam, 1979). Namely, for any $U \in H^1(Q), \sigma_{ij} = \sigma_{ij}(U), \frac{\partial \sigma_{ij}}{\partial x_j} \in L^2(Q), i = 1,2$, one has $\sigma_{ij} n_j \in H^{-\frac{1}{2}}(\gamma)$ and

$$\langle \frac{\partial \sigma_{ij}}{\partial x_j}, v\rangle_Q = \langle \sigma_{ij} n_j, v\rangle_{\frac{1}{2},\gamma} - \langle \sigma_{ij}, \frac{\partial v}{\partial x_j}\rangle_Q$$
$$\forall\, v \in H^1(Q),\ i = 1,2. \tag{12}$$

Assume next that

$$w > \varphi \quad \text{in} \quad \mathcal{W}, \tag{13}$$

where $\mathcal{W}$ is a neighbourhood of the graph Γ_ψ. In this case the inequality (3) implies that the following equations are satisfied in the sense of distributions in $\mathcal{W} \setminus \Gamma_\psi$

$$\Delta^2 w = f_3, \tag{14}$$
$$-\frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad i = 1,2, \tag{15}$$

where $\sigma_{ij} = \sigma_{ij}(W)$. The relation (13) means, in particular, that the inequality $w > \varphi$ holds at both crack faces. This last fact will be written as $w > \varphi$ on $\Gamma_\psi^{\pm}$.

Consider a connected curve such that it confines the bounded simply connected domain $Q \subset \mathcal{W} \setminus \Gamma_\psi$ and contains Γ_ψ as its part. According to the aforesaid the equations (14), (15) hold in Q, hence

$$m(w) \in H^{-\frac{1}{2}}(\gamma),\ t(w) \in H^{-\frac{3}{2}}(\gamma),\ \sigma_{ij} n_j \in H^{-\frac{1}{2}}(\gamma),\ i = 1,2. \tag{16}$$

Obviously, the domain Q can be constructed in different ways. Nevertheless, in any case one of the inclusions $\Gamma_\psi^+ \subset \gamma$, $\Gamma_\psi^- \subset \gamma$ will be valid, and (16) will take place. The exact form of the boundary conditions on Γ_ψ was obtained by Khludnev (1995). We omit the derivation of these conditions here. The only thing we want to do is to discuss shortly their general form in connection with the subsequent regularity result. These conditions are as follows. Let

$$\{\sigma_{ij}\nu_j\} = \sigma_\nu \nu + \sigma_s s, \quad s = (-\nu_2, \nu_1),$$

be a decomposition of the vector $\{\sigma_{ij}\nu_j\}, i = 1, 2$, into the sum of normal and tangential components on Γ_ψ^-. Then, assuming $\varepsilon = 1$ on account of the reasons shown at the beginning of section 4, we have

$$\begin{aligned}
&\sigma_s = 0, \ \sigma_\nu \le 0 \quad \text{on} \quad \Gamma_\psi, \\
&t(w) = 0, \quad |m(w)| \le -\sigma_\nu \quad \text{on } \Gamma_\psi, \\
&m(w)\left[\frac{\partial w}{\partial \nu}\right] + \sigma_\nu [W]\nu = 0 \quad \text{on } \Gamma_\psi, \\
&[m(w)] = 0, \quad [\sigma_{ij}\nu_j] = 0, \ i = 1, 2.
\end{aligned}$$

The first term of the second line means that for any smooth function θ in Ω with a compact trace on $\Gamma_\psi \setminus \partial\Gamma_\psi$ the relation

$$\langle t(w), \theta \rangle_{\frac{3}{2},\gamma^+} = 0 \tag{17}$$

holds, where a domain Q^+ is chosen in such a way that $\Gamma_\psi^+ \subset \gamma^+$. A similar relation takes place in the case $\Gamma_\psi^- \subset \gamma^-$. The zeroth jumps of $m(w), \sigma_{ij}\nu_j$ on Γ_ψ mean that

$$\begin{aligned}
&\langle m(w), \frac{\partial \theta}{\partial \nu} \rangle_{\frac{1}{2},\gamma^+} = \langle m(w), \frac{\partial \theta}{\partial \nu} \rangle_{\frac{1}{2},\gamma^-}, \\
&\langle \sigma_{ij}\nu_j, \theta \rangle_{\frac{1}{2},\gamma^+} = \langle \sigma_{ij}\nu_j, \theta \rangle_{\frac{1}{2},\gamma^-}, \ i = 1, 2.
\end{aligned}$$

In general, the above boundary conditions hold provided that (13) is fulfilled and the solution is quite regular. In fact, some part of the boundary conditions can be considered as holding in the strong sense without any additional assumptions on regularity. In particular, as proved in the mentioned paper by Khludnev, if $x^0 \in \Gamma_\psi \setminus \partial\Gamma_\psi$ and $\mathcal{O}(x^0)$ is a neighbourhood of x^0 such that $\mathcal{O}(x^0) \subset \mathcal{W}$ and $\Gamma_\psi \cap \mathcal{O}(x^0)$ is a segment, the following inclusions take place

$$W \in H^2(\mathcal{O}(x^0) \cap \Omega_\psi), \quad w \in H^3(\mathcal{O}(x^0) \cap \Omega_\psi).$$

Hence

$$m(w), \ \sigma_{ij}\nu_j \in L^2(\Gamma_\psi^\pm \cap \mathcal{O}(x^0)), \ i = 1, 2.$$

The condition $[\chi] = 0$ on Γ_ψ implies that the cost functional of the problem (6), (3) is equal to zero, i.e.

$$J_\varepsilon(\varphi) = \int_{\Gamma_\psi} |[\chi]| d\Gamma_\psi = 0.$$

In this case the crack is said to have a zeroth opening. The cracks of a zeroth opening prove to possess a remarkable property which is the main result of the present section. Namely, the solution χ is infinitely differentiable in a vicinity of $\Gamma_\psi \backslash \partial\Gamma_\psi$ provided that f is infinitely differentiable. This statement is interpreted as a removable singularity property. In what follows this assertion is proved. Let $x^0 \in \Gamma_\psi \setminus \partial\Gamma_\psi$ and $w > \varphi$ in $\mathcal{O}(x^0)$, where $\mathcal{O}(x^0)$ is a neighbourhood of x^0. For convenience, the boundary of the domain $\mathcal{O}(x^0)$ is assumed to be smooth.

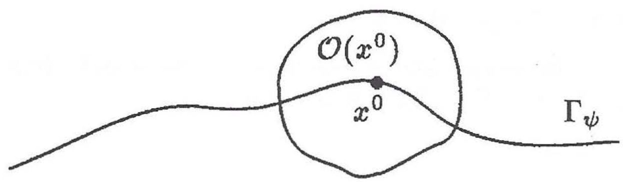

Figure 4.

THEOREM 3.1 *Let* $f \in C^\infty(\mathcal{O}(x^0))$ *and* $[\chi] = 0$ *on* $\mathcal{O}(x^0) \cap \Gamma_\psi$. *Then*

$$\chi = (W, w) \in C^\infty(\mathcal{O}(x^0)).$$

Proof. In view of (2) the hypotheses of the theorem imply $\left[\frac{\partial w}{\partial \nu}\right] = 0$ on $\mathcal{O}(x^0) \cap \Gamma_\psi$. Consequently, (see Mikhailov, 1976)

$$W \in H^1(\mathcal{O}(x^0)), \;\; w \in H^2(\mathcal{O}(x^0)). \tag{18}$$

Equations (14), (15) hold in $\mathcal{O}(x^0) \cap \Omega_\psi$ in the sense of distributions, hence

$$\Delta^2 w \in L^2(\mathcal{O}(x^0) \cap \Omega_\psi), \;\; \frac{\partial \sigma_{ij}}{\partial x_j} \in L^2(\mathcal{O}(x^0) \cap \Omega_\psi), \; i = 1, 2.$$

Let us show that the equation

$$\Delta^2 w = f_3 \tag{19}$$

holds in $\mathcal{O}(x^0)$. The brackets $(\cdot, \theta)$ will mean the value of a distribution evaluated at the point θ. The inclusions (18) are essential in our further reasoning. Let

$\theta \in C_0^\infty(\mathcal{O}(x^0))$ be any fixed function. In view of the Green formula (11) one has

$$\begin{aligned}(\Delta^2 w - f_3, \theta) &= B(w,\theta) - (f_3,\theta) = \\ &= B_+(w,\theta) + B_-(w,\theta) - (f_3,\theta) = \\ &= \langle m(w), \frac{\partial\theta}{\partial\nu}\rangle^-_{\frac{1}{2}} - \langle m(w), \frac{\partial\theta}{\partial\nu}\rangle^+_{\frac{1}{2}} - \\ &-\langle t(w),\theta\rangle^-_{\frac{3}{2}} + \langle t(w),\theta\rangle^+_{\frac{3}{2}} + \langle \Delta^2 w - f_3, \theta\rangle_\pm . \end{aligned} \tag{20}$$

The signs $+, -$ mean that the formulae are concerned with the domains $\mathcal{O}^+(x^0)$, $\mathcal{O}^-(x^0)$, respectively, where

$$\mathcal{O}^+(x^0) = \mathcal{O}(x^0) \cap \{y > \psi(x)\}$$

and $\mathcal{O}^-(x^0)$ is defined in a similar way. The presence of two corner points at the boundaries $\partial\mathcal{O}^\pm(x^0)$ is not essential since θ has a compact support in $\mathcal{O}(x^0)$. In view of (17) the boundary terms of (20) containing $t(w)$ are equal to zero. Besides, the equation (14) holds in $\mathcal{O}^\pm(x^0)$ so that two last terms of (20) are equal to zero. Finally, the condition $[m(w)] = 0$ on $\mathcal{O}(x^0) \cap \Gamma_\psi$ provides for vanishing of two terms of (20) containing $m(w)$. Thus, (20) yields

$$(\Delta^2 w - f_3, \theta) = 0.$$

The proof of (19) is completed.

Analogously, the Green formula (12) and the first relation of (18) imply

$$\begin{aligned}\left(-\frac{\partial\sigma_{ij}}{\partial x_j} - f_i, \theta\right) &= \left(\sigma_{ij}, \frac{\partial\theta}{\partial x_j}\right) - (f_i,\theta) = \\ &= \langle \sigma_{ij}, \frac{\partial\theta}{\partial x_j}\rangle_\pm - (f_i,\theta) = \langle \sigma_{ij}\nu_j, \theta\rangle^-_{\frac{1}{2}} - \\ &-\langle \sigma_{ij}\nu_j, \theta\rangle^+_{\frac{1}{2}} - (f_i,\theta) - \langle \frac{\partial\sigma_{ij}}{\partial x_j}, \theta\rangle_\pm = \\ &= \langle -\frac{\partial\sigma_{ij}}{\partial x_j} - f_i, \theta\rangle_\pm = 0, \quad i = 1,2. \end{aligned}$$

In so doing the equations (15) are used as holding good in $\mathcal{O}^\pm(x^0)$. The equations $[\sigma_{ij}\nu_j] = 0$ on $\mathcal{O}(x^0) \cap \Gamma_\psi$ are also used. Consequently, the following equations

$$-\frac{\partial\sigma_{ij}}{\partial x_j} = f_i, \quad i = 1,2, \tag{21}$$

hold in $\mathcal{O}(x^0)$ in the sense of distributions. The results on the internal solution regularity of (19), (21) (see Fichera, 1972; Lions et al., 1968) provide the validity of the theorem's assertion. The proof is completed. □

4. Convergence of solutions as $\varepsilon \to 0$

Consider the limit case corresponding to $\varepsilon = 0$ in (2). The restriction obtained in such a way describes approximately mutual nonpenetration of the crack faces. Note that in reality a complete account of the thickness implies the dependence of the energy functional on ε. This dependence is as follows (Vol'mir, 1972)

$$\Pi(\chi) = \frac{\varepsilon^3}{2}B(w,w) + \frac{\varepsilon}{2}\langle\sigma_{ij}(W), e_{ij}(W)\rangle - \langle f,\chi\rangle. \tag{22}$$

Moreover, in this case $m(w), t(w), e_{ij}(W)$ should also depend on ε. In spite of this, in section 4 the parameter ε is equal to 1 in the formula (22) just as in the previous sections. Thus, the case $\varepsilon = 0$ in (2), in fact, means both the approximate description of the nonpenetration condition and a fixed thickness. Hence, in the case under consideration a solution should satisfy the following restriction

$$w \geq \varphi \quad \text{in} \quad \Omega_\psi, \tag{23}$$

$$[W]\nu \geq 0 \quad \text{on} \quad \Gamma_\psi. \tag{24}$$

As a result the set of feasible displacements is as follows

$$K_0^\varphi = \left\{(W,w) \in H(\Omega_\psi) \;\middle|\; (W,w) \quad \text{satisfy} \quad (23),(24)\right\}.$$

Herewith the problem of minimizing Π over the set K_0^φ is equivalent to the variational inequality

$$B(w,\bar{w}-w) + \langle\sigma_{ij}(W), e_{ij}(\bar{W}-W)\rangle \geq$$
$$\geq \langle f, \bar{\chi}-\chi\rangle, \quad \chi = (W,w) \in K_0^\varphi, \quad \forall\, \bar{\chi} \in K_0^\varphi. \tag{25}$$

Let the set Φ be the same as in section 2. Consider the optimal control problem

$$\inf_{\varphi\in\Phi} J_\varepsilon(\varphi) = \inf_{\varphi\in\Phi} \int\limits_{\Gamma_\psi} |[\chi]|\,d\Gamma_\psi. \tag{26}$$

There exists a solution of (26), (25). We omit the arguments.

So, instead of precise nonpenetration condition (2) we consider the approximate condition (24) in this section. In practical applications this approach is interesting since it is easier to find the solutions of (25) as compared to (3). In particular, it is possible to find solutions of (25) by using the penalty operator relative to the restriction (24). The displacements W and w are uncoupled in (25), and one can write down two variational inequalities for finding W and w, respectively. Meanwhile, when the optimal control problem (26) is solved, the solution φ depends on the pair (W,w) what, actually, means the coupling of W and w. The problem is to prove the solution proximity of (6), (3) and (26), (25), as $\varepsilon \to 0$.

As for other problems related to the elastic plates having cracks with the approximate nonpenetration condition (24), we refer the reader to Khludnev (1992, 1995).

A relationship between the solutions of (6), (3) and (26), (25) is characterized by the following statement. Introduce the notations

$$j_0 = \inf_{\varphi \in \Phi} J_0(\varphi), \quad j_\varepsilon = \inf_{\varphi \in \Phi} J_\varepsilon(\varphi).$$

Let φ_ε be the solution of (6), (3), and χ_ε correspond to φ_ε.

THEOREM 4.1 *From the sequence $\varphi_\varepsilon, \chi_\varepsilon$ one can choose a subsequence such that as $\varepsilon \to 0$*

$$\begin{aligned} &\varphi_\varepsilon \to \varphi \quad \textit{weakly in} \quad H^2(\Omega), \\ &\chi_\varepsilon \to \chi \quad \textit{weakly in} \quad H(\Omega_\psi), \\ &j_\varepsilon \to j_0, \end{aligned}$$

where φ is a solution of (26), (25) and $\chi = \chi(\varphi)$ is defined from (25).

Proof. Consider first any fixed element $\varphi \in \Phi$ and prove that

$$J_\varepsilon(\varphi) \to J_0(\varphi), \quad \varepsilon \to 0. \tag{27}$$

Let $\chi_\varepsilon(\varphi)$ be the solution of (3). There exists an element $\bar{\chi}$ such that $\bar{\chi} \in K_\varepsilon^\varphi$ for all ε. Substituting $\bar{\chi}$ in (3) as a test function implies

$$\|\chi_\varepsilon(\varphi)\|_{H(\Omega_\psi)} \leq c$$

with a constant c independent of ε. Without loss of generality as $\varepsilon \to 0$ we assume that

$$\begin{aligned} &\chi_\varepsilon(\varphi) \to \tilde{\chi} \quad \text{weakly in} \quad H(\Omega_\psi), \\ &\chi_\varepsilon^\pm(\varphi) \to \tilde{\chi}^\pm \quad \text{strongly in} \quad L^1(\Gamma_\psi). \end{aligned} \tag{28}$$

Moreover, the pair $(\varphi, \tilde{\chi})$ is a solution of the variational inequality

$$\begin{aligned} &B(\tilde{w}, \bar{w} - \tilde{w}) + \langle \sigma_{ij}(\tilde{W}), e_{ij}(\bar{W} - \tilde{W}) \rangle \geq \\ &\qquad \geq \langle f, \bar{\chi} - \tilde{\chi} \rangle, \quad \tilde{\chi} = (\tilde{W}, \tilde{w}) \in K_0^\varphi, \quad \forall\, \bar{\chi} \in K_0^\varphi. \end{aligned}$$

To verify this it suffices to fulfill the limiting transition in (3) as $\varepsilon \to 0$. Thus, $\tilde{\chi} = \chi(\varphi)$. In view of (28) we arrive at the desired convergence (27).

Let φ be a solution of the optimal control problem (26), (25). The above arguments imply

$$j_\varepsilon \leq J_\varepsilon(\varphi) \to J_0(\varphi) = j_0.$$

Whence

$$\limsup j_\varepsilon \leq J_0(\varphi) = j_0. \tag{29}$$

On the other hand, the boundedness of the set Φ provides the estimate

$$\|\varphi_\varepsilon\|_{2,\Omega} \leq c \tag{30}$$

which is uniform in ε. Consequently, the inequality

$$\begin{aligned} &B(w_\varepsilon, \bar{w} - w_\varepsilon) + \langle \sigma_{ij}(W_\varepsilon), e_{ij}(\bar{W} - W_\varepsilon)\rangle \geq \\ &\qquad \geq \langle f, \bar{\chi} - \chi_\varepsilon\rangle, \quad \chi_\varepsilon = (W_\varepsilon, w_\varepsilon) \in K_\varepsilon^{\varphi_\varepsilon}, \quad \forall\, \bar{\chi} \in K_\varepsilon^{\varphi_\varepsilon} \end{aligned} \tag{31}$$

enables us to derive the following estimate

$$\|\chi_\varepsilon\|_{H(\Omega_\psi)} \leq c \tag{32}$$

being uniform in ε. Choosing subsequences, still denoted by $\varphi_\varepsilon, \chi_\varepsilon$, we assume that as $\varepsilon \to 0$

$$\begin{aligned} &\varphi_\varepsilon \to \tilde{\varphi} \quad \text{weakly in } H^2(\Omega), \quad \text{uniformly in } \bar{\Omega}, \\ &\chi_\varepsilon \to \tilde{\chi} \quad \text{weakly in } H(\Omega_\psi). \end{aligned}$$

Moreover, it can be proved that for every fixed $\bar{\chi} \in K_0^{\tilde{\varphi}}$ there exists a sequence $\bar{\chi}_\varepsilon \in K_\varepsilon^{\varphi_\varepsilon}$ such that

$$\bar{\chi}_\varepsilon \to \bar{\chi} \quad \text{strongly in} \quad H(\Omega_\psi).$$

Combining this convergence and Lemma, one can pass on to the limit in (31). Hence the following relation results

$$\begin{aligned} &B(\tilde{w}, \bar{w} - \tilde{w}) + \langle \sigma_{ij}(\tilde{W}), e_{ij}(\bar{W} - \tilde{W})\rangle \geq \\ &\qquad \geq \langle f, \bar{\chi} - \tilde{\chi}\rangle, \quad \tilde{\chi} = (\tilde{W}, \tilde{w}) \in K_0^{\tilde{\varphi}}, \quad \forall\, \bar{\chi} \in K_0^{\tilde{\varphi}}, \end{aligned}$$

that is $\tilde{\chi} = \chi(\tilde{\varphi})$. Furthermore, just as in the proof of (27) the convergence

$$J_\varepsilon(\varphi_\varepsilon) \to J_0(\tilde{\varphi})$$

holds. Hence

$$\liminf j_\varepsilon \geq J_\varepsilon(\tilde{\varphi}). \tag{33}$$

A comparison of (29) and (33) results in the conclusion that $\tilde{\varphi}$ is a solution of (26), (25) and $j_\varepsilon \to j_0$. As noted above, $\tilde{\chi} = \chi(\tilde{\varphi})$. Theorem 3 has been proved. □

5. Conclusion

The condition $[\chi] = 0$ is shown to provide the infinite differentiability of the solution only for $\varepsilon > 0$. For the problem (25), corresponding to $\varepsilon = 0$, one cannot state that $w \in H^2(\mathcal{O}(x^0))$ provided that $[\chi] = 0$ on $\mathcal{O}(x^0) \cap \Gamma_\psi$, since, in general, in this case $\left[\frac{\partial w}{\partial \nu}\right] \neq 0$ on $\mathcal{O}(x^0) \cap \Gamma_\psi$.

The result of Theorem 2 on C^∞-regularity actually shows that the condition $[\chi] = 0$ provides the disappearance of singularity which takes place in view of a presence of the crack. It means that under the condition mentioned, we can "forget" about the crack since the behaviour of the plate is the same as that without the crack. This property of the cracks of minimal opening is interesting from the standpoint of mechanics. It can be used in applications of the crack theory.

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