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Mathematical modeling of memory effects' influence on fast hydrodynamic and heat conduction processes[1]

by

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Abstract. This paper describes the results of mathematical modelling of hydrodynamic processes on the basis of generalized equations. The hyperbolic heat conduction equation, hyperbolic modification of Burgers equation and finite dimensional systems of Lorentz's type are considered. We discuss blow-up solutions, solutions with decreasing length and some unusual types of chaotic behavior. Applications to combustion, turbulence and heat conduction are described. The problems of correct numerical simulation are also considered.

Keywords: generalized hydrodynamics, memory effects, blow-up solutions, combustion, dissipative structures, turbulence, Lorentz type systems, chaos, singular perturbations, fast processes

1. Introduction

In the vast fields of investigations devoted to heat and mass transfer processes in extended systems, the models for such phenomena are mainly based on parabolic equations of heat and mass transport. These equations are derived from the local equilibrium assumptions and phenomenological laws, which reflect the local correspondence in space and time between the thermodynamic flux and forces (such as Fourier's law, etc.). In such a case the local state of a medium is described by the governing equations, which do not depend upon gradients. In many models the kinetic transport coefficients are assumed constant. Parabolic equations with constant coefficients admit physically irrelevant solutions with infinitely large speed of propagation and infinite streams in the initial moment. In spite of such peculiarities these equations describe well enough many experimental results in nature. But these equations lose their applicability in extended

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media description for fast processes when gradients are changing during correlation time and inside the correlation length. In this case essential dispersive effects take place. The hypothesis of local equilibria fails and one needs to take the nonlocality and memory effects into account.

There are a number of investigations on the derivation of fluid motion equations from first principles in the situation when the hydrodynamic level of description is valid. Certain basic variables such as stress, velocity and temperature are on such a level in general. Investigations show that far from equilibrium in such circumstances the equations of motion have universal form, and constitutive equations have rather general form – Picirelli (1968), Zubarev and Tishcenko (1972), Resibois and De Lener (1980), Jou, Perez-Garcia, Garcia-Colin at al (1985). For example the equation which connects the stress and strain in the unidimensional case takes the form

$$\sigma(x,t) = -\int_{-\infty}^{t} dt \int dx K(x,x';t't')\partial V/\partial x$$

where σ is stress and V denotes strain. For the temperature there is a similar state equation

$$q(x,t) = -\int_{-\infty}^{t} dt \int dx K1(x,x';t,t')\partial T/\partial x$$

where T is temperature, q is heat flux. Such expressions take into account the influence of prehistory of processes. This is called memory effects (or relaxation or, in some cases, the dispersion). Nonlinearity and space nonlocality are also taken into account.

The above mentioned expressions are postulated in another well defined theory, namely in the rational thermodynamics of media with memory – Gurtin and Pipkin (1968), Coleman and Noll (1960), Nunziato (1971), Day (1974), Joseph and Preziozi (1989). Note that an expression of the above form often takes place in experiments in many fields of science and technology - for example in neutron scattering in liquids, in turbulence, in motion of visco-elastic fluid, in heat conduction and in many other cases.

In addition, the equations for strongly nonequilibrium processes are integro-differential in the general case. Particular types of equations can be obtained under the particular choice of kernels in such equations (cf. Makarenko, 1987; Danilenko, Korolevich, Makarenko and Christenuk, 1992; Makarenko, 1994). In many cases the integro-differential equation kernels are close to the equilibrium ones. This implies that the characteristic relaxation time and nonlocality scales are relatively small. Then, after the reduction of integro-differential equations to the differential one, they take the form of a parabolic equation, which is completed by the high derivatives both in time and space, but with small parameters (singularly perturbed equations). The simplest examples are obtained with the exponential kernels $exp(-t/\tau)$. The parameter τ in this kernel is the so called relaxation time. Relaxation time is the characteristic time of fading processes in systems, Kaliski (1965), Luikov (1965), Joseph and Preziozi (1989). The values of relaxation times can vary from 10 to 10^{-13} sec. in solids, 10 to 10^{-8} sec. in liquids to 10^3 in turbulent liquids and polymers. The simplest examples obtained with exponential kernels are the hyperbolic equation of heat conduction and the generalized hydrodynamics with memory effects. The generalized hydrodynamics also gives the generalization of the Lorentz type systems by the usual Galerkin-type procedure.

The present report is devoted to a brief review of our results for the cases when a strong difference emerges between the results of classical and generalized theories. Such phenomena take place in problems with collapsing (blow-up) solutions, travelling wave stability, chaos, etc., under strong nonequilibrium conditions.

2. Hyperbolic heat conduction equation

As we mentioned above, there can be a large difference from classical description in regions with fast variation. A typical example is that of blow-up solutions (see for example Samarsky, Galaktionov, Kurdumov and Michailov, 1987, and references in it). This solution has the interpretation of nonstationary dissipative structures. In such structures solutions grow with increasing rate. Such nonstationary structures were investigated mainly by parabolic equations. It was found that solutions blow-up when the amplitude is increased to infinity in finite time because of nonlinear source, and boundary regimes collapse when the value on the boundary is increased to infinity in finite time. In this part there is a very brief description of some results for blow-up solutions for hyperbolic heat conduction equation.

2.1. Blow-up solutions

The typical mathematical problem is to find a solution to the problem

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + f(T), \tag{1}$$

$$T(x,0) = \phi(x), \quad \partial T/\partial t(x,0) = \psi(x), \quad x \in \Omega \tag{2}$$

for initial-value problem (if $\Omega = (-\infty, \quad +\infty)$), plus

$$T(\partial\Omega) = \omega(t,x) \tag{3}$$

for initial-boundary problems ($\Omega = [-a,+a]$).

For the sources of increasing type ($f(T) = T^m$, $m > 1$, ($f(T) = exp(T)$ and so on) there are some purely mathematical theorems in which the existence of blow-up solutions is connected with the nonexistence of global in time classical or generalized solutions (Danilenko, Kudinov and Makarenko, 1983; Makarenko, 1990). The formulations of these theorems are long and include some integrals of initial values and nonlinear sources. Some interesting properties were also found. Let us indicate by $t(\tau)$ the upper estimate of blow-up time. Then there are the following cases in problems (1) - (3): a) if $\psi(x) = 0$ and τ in (1) tends to ∞, then $t(\tau_1)/t(\tau_2) = (\tau_1/\tau_2)^{1/2}$; b) if $\phi, \psi > 0$ and $\tau \to \infty$ then $t(\tau) \to \infty$; c) if $\tau = \varepsilon \to 0$ and there are blow-up solutions as in problems with $\tau = 0$ as with $\tau \neq 0$, then $t(\varepsilon) \to t(0)$, $\varepsilon \to 0$.

The numerical results confirm such properties. It is easy to see three stages of reorganization of blow-up solution from the graphics of numerical solutions: 1) slow reconstruction of profile, 2) slow growth of profile, 3) fast blow-up regime. There is also a dropping of structure which is impossible in the parabolic case. In our case, a one-humped initial distribution evolves on some stage to two-humped solution. For smaller initial data two travelling waves arose without blow-up solutions (see the graphics in Danilenko, Kudinov and Makarenko, 1983a, 1984b, 1984a; Danilenko, Korolevich, Makarenko and Christenuk, 1992).

2.2. Boundary blow-up regimes

Such regimes are described by problems with infinite growth rate on the boundary in finite time. The simplest case is described by the problem (1) - (3) with $\Omega = [0, +\infty)$, $f(T) = 0$ and auxiliary condition

$$T(0,t) = \mu(t), \quad t \geq 0, \quad \mu(t) \to +\infty, \quad t \to T_1 < +\infty, \quad \mu(0) = 0. \qquad (4)$$

with monotonically increased $\mu(t)$. A dependence on growth rate of μ was found before that for the parabolic case ($\tau = 0$), amplitude profiles which evolve in regimes with diminishing, constant, or growing effective width, denoted respectively by LS, H, HS solution (Samarsky, Galaktionov, Kurdumov and Mikhailov, 1987).

The hyperbolic and parabolic cases are essentially different. First of all, we can see from theorems that in the first case there is only LS boundary blow-up regime. The case $0 < \tau = \varepsilon << 1$ is especially interesting. A bounded classical solution was constructed in Danilenko, Kudinov and Makarenko (1983 b) by Vishik - Lusternik asymptotic method. There is a hyperbolic boundary layer near the moment $t = 0$. Divergence from the solution of the parabolic equation is large in this layer. Then after $t > \varepsilon$, the solution tends to the solution of the parabolic equation. Blow-up cases are principally different from bounded cases. At first there exists a hyperbolic boundary layer, then the solution tends to a solution of the parabolic equation. Then, with the growth of solution, the difference became large. In the case with $\tau << 1$ this difference concentrates on times $|t - T_1| < \tau$. Therefore in this strip there is a new mathematical object, a second boundary layer in time (first layer has been near 0). There are also some connections of blow-up regimes with the theory of generalized functions and operator theory, Kudinov and Makarenko (1985), Makarenko (1994). Since for $t \to T_1$ the solution becomes infinitely large and is not classical, it is natural to consider the limit profile as a generalized function. The problem can also be reformulated with the selfadjoint extension of operators with exit from the space.

3. Model equations for generalized hydrodynamics with the memory effects

The standard object of many mathematical investigations in hydrodynamics are Navier-Stokes systems. However, this system has a drawback which was mentioned in the introduction. It consists of the motion equations and constitutive equation connecting the stress and strain in Newtonian fluid. There are a great number of investigations on the derivation the equations from the first principles in the situation when the hydrodynamic level of description still is valid. The general constitutive equations have an integral form. The application of exponential kernels leads to the hydrodynamic equations of visco-elastic fluids (see references to part 1). In other methods one investigates Navier-Stokes and generalized hydrodynamic systems, for example, and applies model equations.

As it is known in many cases a good model equation for the Navier- Stokes systems is the Burgers equation. The more general case with exponent-type kernels leads to the so called hyperbolic modification of Burgers equation, Pyatkov, Rudyak and Smagulov (1982), Makarenko and Moskalkov (1992), Makarenko and Levkov (1993):

$$\tau\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{5}$$

If $\tau = 0$, then we have Burgers equation. In this section we briefly describe the properties of solutions to (5). Some can be derived from the linearized version with the term $a\partial u/\partial x$, $a \equiv const$ instead of $u\partial u/\partial x$. Let us call the quantity $M = |a|/c$, $c = \sqrt{\nu/\tau}$ the Mach number. The analysis with the assistance of harmonics shows that when $M < 1$ the linearized equation does not have the growth solutions. When $M > 1$ there is a branch of growing harmonics with exponential growth. The linearized equation also has exact solutions constructed by use of Bessel functions. There is much information about the equation in such a solution. When $M > 1$ there is growth with the rate proportional to $\tau^{-1/2}$ and oscillations of solutions of wave packet type. This form of solutions follows from Bessel function properties. Characteristic lengths of oscillations is determined by zeros of Bessel functions and is equal approximately

$$\lambda \simeq 2.5/\sqrt{c}, \quad c = \frac{1}{\tau}\left(\frac{1}{4\tau} - \frac{a^2}{4\nu}\right). \tag{6}$$

We can see from (6) that when $c \to -\infty$ the number of oscillations per unit length increases (the dropping or nucleation of length scale). The nonlinear equation (5) has a set of travelling wave solutions of the form $u(x,t) = u(p)$, $p = x - Dt$. For this solution we have (for the particular case $\tau = \nu = 1$)

$$\theta_{pp}(1 - D^2) + (D + \theta) = 0. \tag{7}$$

The equation (7) has solution with $|D| > M$ and $1/(x - x_0)$ type singularity. There are some types of comparison theorems for equation (5) (Makarenko and Moskalkov, 1992) where the solutions of (7) with singularities can serve as "lower solutions" in comparison theorems. This can lead to the existence of blow-up (or collapse) solutions for (5) in some cases. This conclusion was confirmed by numerical computations with initial data of hump type. There are also some oscillations in solutions of wave packet type, Makarenko and Moskalkov (1992), Danilenko, Korolevich, Makarenko and Christenuk (1992), Makarenko and Levkov (1993). In these papers also model systems for two- dimensional flows are described. For the case with a vortex as initial condition and $M > 1$ the blow- up of vorticity with generation of new vortex with smaller characteristic lengths was described.

4. Finite–dimensional systems of O.D.E. of Lorentz type

One of the approaches to the investigation of hydrodynamic equations (for example Navier-Stokes equations) is the Galerkin method. By this method it is easy to construct a low-dimensional dynamical system. For example, the well known Lorentz system arises from the Navier-Stokes equations.

In this section of the paper we consider some results on the construction of low-dimensional analogs for generalized hydrodynamics with memory effects and compare them with the local equations under the same initial and boundary conditions.

We note that in Boldrighini and Franceschini (1979) a five–dimensional system on a torus was described (we refer to it below as BF). We derived analogous systems for the generalized hydrodynamics under the same conditions. For example, the ten–dimensional system has the form

$$\begin{aligned}
&\tau dx_1/dt = (-x_1 - 2x_6 + 4x_7x_8 + 4x_9x_{10} \\
&\qquad +4\tau(x_4x_{10} + x_9x_5), \\
&\tau dx_2/dt = (-x_2 - 9x_7 + 3x_6x_8) + 3(x_1x_8 + x_6x_3)\tau, \\
&\tau dx_3/dt = (-x_3 - 5x_8 - 7x_6x_7) - 7(x_1x_7 + x_6x_2)\tau + R, \\
&\tau dx_4/dt = (-x_4 - 5x_9 - x_6x_{10}) - (x_1x_{10} + x_6x_5)\tau, \\
&\tau dx_5/dt = (-x_5 - x_{10} - 3x_6x_9) - 3(x_1x_9 + x_6x_4)\tau, \\
&dx_6/dt = x_1, \quad dx_7/dt = x_2, \quad dx_8/dt = x_3, \quad dx_9/dt = x_4, \\
&dx_{10}/dt = x_5.
\end{aligned} \tag{8}$$

When $\tau = 0$ (no memory effects) the system (8) coincides with BF. In the case $0 < \tau << 1$ it is a singular perturbation of it. We mention that in Makarenko

(1994) the systems of o.d.e. for generalized hydrodynamics with memory effects for three- dimensional flows with slip boundary conditions are considered.

In Danilenko, Korolevich, Makarenko and Christenuk (1992), Makarenko (1994 a, b) some properties of such systems and results of computer simulation of them are presented. The first distinction with respect to the case without memory is in the appearance of the neutrally stable oscillations. Another one concerns the types of chaotic behavior. In the case $\tau = 0$ the attractor of "butterfly's" type is typical as in Lorentz's system. With $\tau \neq 0$, a complex behavior of new type arises. The trajectory fills densely some bounded volume ("container") and has a broken form in many points (see Makarenko, 1994). Visually that behavior is similar to the one in two–dimensional mappings with homoclinic tangency and quasi-attractors described in Gonchenko, Shilnikov and Turaev (1993). The first results of our continued investigations on bifurcation points of a system (creation of the pairs of conjugate roots of Jacobian) and on existence of neutrally stable oscillations support the possibility of such a mechanism.

5. Conclusions

In the above we considered some results concerning a more correct description of heat and mass transfer for nonequilibrium processes taking into account memory effects. Three types of model equations and some results on blow-up solutions, oscillations and chaos have been described. These results also have a real range of applications. For example, in Danilenko, Kudinov and Makarenko (1983 c, 1984 b), there are some problems from combustion theory. In Makarenko and Levkov (1992), Makarenko, Moskalkov and Levkov (1995) some ideas of memory effects in turbulence are considered. Analogs of boundary blow-up regimes take place in fast energetic influence on the surfaces of specimen. Moreover, there are many other phenomena which need a more correct description with memory and nonlocality effects (Davidenko, Kudinov and Makarenko, 1985; Joseph and Preziosi, 1989; Peszyńska, 1995 and so on). Note that the complex behavior described in section 4 can serve as a prototype of new possible type of chaos in the media with memory or in the media with finite speed of disturbances.

The unusual form of heat and mass transfer equations and their solutions posed some problems with the computation and foundation of computation. Some results on the foundations for blow-up computations are presented in Makarenko (1983b, 1991). The problems of dispersion of numerical methods, fighting with nonphysical oscillations, and interpretation of numerical solutions in hyperbolic equations, are described in Makarenko and Moskalkov (1984), Makarenko (1982). The problem of complexity measure of solutions is discussed in Makarenko (1992), with the new definition and the properties of information and complexity of nonprobability objects included there.

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