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Finite element analysis with mesh refinement for shape optimization[1]

by

Nickolay V. Banichuk*, F.-J. Barthold, A. Falk** and E. Stein****

*Laboratory of Modelling and Design of Mechanical Systems and Structures, Institute for Problems in Mechanics, Russian Academy of Sciences, 101, pr. Vernadskogo, 117526, Moscow, Russia, e-mail: `banichuk@ipm.msk.su`

**Universität Hannover, Institut für Baumechanik und Numerische Mechanik, Hannover, Germany, e-mail: `fjb@bach.ifbnm.uni-hannover.de`

Abstract. In this paper a new integrated approach to mesh refinement for shape optimal design is considered. The local measure of residuals in necessary optimality conditions is proposed as an additional error estimator for controlling the quality of mesh. Comparison of the developed double refining procedures with the conventional Z^2–refining have been made. Computation performed confirmed that the proposed approach can be effectively applied to a wide class of shape optimal design problems.

Keywords: finite element analysis, mesh refinement, shape optimization

1. Optimal shape design problems. Continuous and discrete formulations.

Consider a deformed elastic body occupying a region Ω whose boundary consists of a surface $\Gamma = \Gamma_u \cup \Gamma_\sigma$. Displacements are assigned to the part Γ_u of this surface, while tractions are given on the part Γ_σ . The corresponding boundary conditions and the basic equations of the theory of elasticity are

$$u_i = U_i \ on \ \Gamma_u, \qquad \sigma_{ij} n_j = T_i \ on \ \Gamma_\sigma \tag{1}$$

$$\sigma_{ij,j} + q_i = 0, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

where σ_{ij}, ε_{ij}, u_i and n_j denote, respectively, the components of the stress tensor, the strain tensor, the displacement vector, and the unit vector pointing

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in the direction of an outward normal to the surface of the body, while U_i and T_i are given functions of the coordinates x_j. Constants C_{ijkl} are the components of the elastic moduli's tensor. We use a Cartesian reference frame $\{0, x_1, x_2, x_3\}$ and the indices i, j assume the values $1, 2, 3$. The summation is performed over repeated indices.

The optimization problem consists in finding surface Γ_v, $\Gamma_v \subset \Gamma$, such that the optimized functional attains a minimum

$$J = \int_\Omega f(u_k,\ \sigma_{ij})\, d\Omega \to \min_{\Gamma_v} \tag{3}$$

while satisfying prescribed bounds on some properties of the stressed and deformed state

$$g_\nu(u_k,\ \sigma_{ij}) \le C_\nu, \qquad \nu = 1, 2, \ldots \tag{4}$$

where g_ν are given functions and C_ν are given constants.

Applying the approach of finite element modelling we discretize the domain Ω and boundary Γ and approximate the state and design variables. The optimal design problem now assumes the form

$$I(u, h) \Rightarrow \min_h \tag{5}$$

$$\Psi_i(u, h) = 0, \qquad i = 1, 2, \ldots, m \tag{6}$$

$$\Psi_i(u, h) \le 0, \qquad i = m + 1, \ldots, k \tag{7}$$

Here u and h are vectors of the state and design variables and $I(u, h)$, $\Psi_i(u, h)$ are given functions.

2. Sensitivity analysis

Consider the problem of optimal design formulated in (5)-(7) and evaluate the effect of variation of design variables δh . We shall observe both the state and the design variables u and h and their perturbed values $u + \delta u$, $h + \delta h$. We linearize the relations between the variations of the quality function and the constraints as well as the vectors δu and δh

$$\delta I = (\frac{\partial I}{\partial u}, \delta u) + (\frac{\partial I}{\partial h}, \delta h) \tag{8}$$

$$\delta \Psi_i = (\frac{\partial \Psi_i}{\partial u}, \delta u) + (\frac{\partial \Psi_i}{\partial h}, \delta h) \quad i = m + 1, \ldots, k \tag{9}$$

and linearize (6)

$$A\delta u + B\delta h = 0 \qquad A = \begin{bmatrix} \frac{\partial \psi_1}{\partial u_1} & \cdots & \frac{\partial \psi_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_m}{\partial u_1} & \cdots & \frac{\partial \psi_m}{\partial u_m} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial \psi_1}{\partial h_1} & \cdots & \frac{\partial \psi_1}{\partial h_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_m}{\partial h_1} & \cdots & \frac{\partial \psi_m}{\partial h_n} \end{bmatrix} \tag{10}$$

The quantities $\partial I/\partial u$, $\partial I/\partial h$, $\partial \Psi_i/\partial u$ and $\partial \Psi_i/\partial h$, which occur in (8), (9) as well as the matrices A and B, must be computed for unperturbed values of the arguments. Matrix A is assumed to be nonsingular. The following approach for eliminating the dependence of functions on δu has found very wide acceptance. To eliminate the expressions $(\partial I/\partial u)\delta u$ and $(\partial \Psi_i/\partial u)\delta u$ let us introduce $k - m + 1$ vectors λ_i

$$A^T\,\lambda^0 = \frac{\partial I}{\partial u}, \qquad A^T\,\lambda^i = \frac{\partial \Psi_i}{\partial u} \qquad i = m+1,\dots,k \tag{11}$$

where the superscript T denotes the transpose of a vector or matrix. The functions on the right hand side of (11) are computed for the current values of the vectors u and h. Using (11) and (10) we derive

$$-(\lambda^0, B\delta h) = (\frac{\partial I}{\partial u}, \delta u), \qquad -(\lambda^i, B\delta h) = (\frac{\partial \Psi_i}{\partial u}, \delta u) \tag{12}$$

Substituting from (12) the expressions for $\partial I/\partial u$ and $\partial \Psi_i/\partial u$ in (8), (9) we obtain

$$\delta I = \left(\frac{\partial I}{\partial h} - B^T\lambda^0, \delta h\right) \tag{13}$$

$$\delta \Psi_i = \left(\frac{\partial \Psi_i}{\partial h} - B^T\lambda^i, \delta h\right) \qquad i = m+1,\dots,k \tag{14}$$

The components of the vectors

$$\nabla_h I = \frac{\partial I}{\partial h} - B^T\lambda^0 \tag{15}$$

$$\nabla_h \Psi_i = \frac{\partial \Psi_i}{\partial h} - B^T\lambda^i \tag{16}$$

are called the sensitivity coefficients of the constraints and the quality functional with respect to the corresponding design variables.

The sensitivity coefficients in (15), (16) are very useful in a shape design process, since they contain information how a change in design affects the quality functional and the constraints. In particular the computation of the values of these coefficients permits us to identify those design variables whose changes have the most significant influence for the objective function and other characteristics.

3. Successive optimization and motivation for design refinement

An effective solution of the problem of designing an optimal structure can be obtained only after a decomposition of the original problem into a sequence of simpler problems of constructing improved approximations. So we pay particular attention to the iterative optimization algorithms, which can be conveniently applied to shape optimization problems.

In solving shape optimal design problems we apply FEM for computing displacement and stress fields and gradient projection technique for improving the unknown shape. For effective computations of the state fields we can apply mesh refinement using the Zienkiewicz-Zhu criterion (Zienkiewicz and Zhu, 1987, 1991) as an a posteriori error estimator

$$\eta_j^2 = \int_{\Omega_j} (\sigma - \hat{\sigma})^T C^{-1} (\sigma - \hat{\sigma}) \, d\Omega \tag{17}$$

Here σ is the computed stress distribution and $\hat{\sigma}$ is a continuous stress distribution which is obtained from postprocessing. This criterion gives us a possibility to refine the mesh by minimizing the stress jumps at the element edges. As it follows from the direct computations of the stress fields by FEM and from theoretical analysis these jumps grow and concentrate nearby the stress concentrators (boundary angles, cracks, holes) and for the regions with high stress gradients. The refined mesh will be concentrated at special regions of the domain Ω to satisfy the requirement $\eta_j^2 \leq \varepsilon_\sigma$, where ε_σ is a small positive number. It is very reasonable from the theoretical and engineering point of view and improves meaningfully the quality of the solution of the analysed problem. It means also that our additional degrees of freedom were exhausted by the special regions.

For shape optimization problems the error of the solution is a sum of errors from the finite element analysis and the process of improving of the shape. The latter can be evaluated by a norm of the residuals in necessary optimality conditions. To minimize this error it is necessary to have an accurate (detailed) information concerning the state functions for the regions close to the varied boundary. So we also need additional degrees of freedom (additional refinement) for the optimized boundary. One of the simplest design error estimators which can be taken as an a posteriori refinement criterion is the norm of the design gradient of the Lagrange function

$$\mu_j^2 = \int_{\Gamma_j} (\nabla_h I^L)^2 d\Gamma \tag{18}$$

In accordance with the requirement $\mu_j^2 \leq \varepsilon_\Gamma$ (ε_Γ is a given small positive number) the refined mesh will be concentrated nearby optimized boundary. Note that the exact expressions for the design gradients of the Lagrange functions are presented in the books (Banichuk, 1990; Haug and Aurora, 1979).

The algorithm of iterative optimization consists of obtaining successive approximations of the optimal solution and comes from the idea of "small" perturbations of the design vector h (shape of the boundary) and multiple solution of the "direct" problem of finding state variables (displacements, stresses and strains) with the values of h regarded as fixed. For a chosen small positive values ε_Γ and ϵ_σ the algorithm consists of specified steps:

Step 1. As a result of computations completed in the previous cycle with number k, we regard the design and state variables as known and solve the boundary value problem of the theory of elasticity. Then go to Step 2.

Step 2. Estimate an a posteriori error, using Zienkiewicz - Zhu criterion. If $\max_j(\eta_j^2) > \epsilon_\sigma$ for new state variables $(u)^{k+1}$, $(\sigma)^{k+1}$, then perform mesh refinement and return to Step 1. Otherwise, go to Step 3.

Step 3. Compute the new values of Lagrange multipliers $(\lambda)^{k+1}$.

Step 4. Compute the new values h^{k+1}, I^{k+1} of design variables and objective function.

Step 5. Estimate the incompatibility of the state and design variables in satisfying the necessary optimality conditions. If $\max_j(\mu_j^2) \leq \varepsilon_\Gamma$, terminate the program. Otherwise go to Step 6.

Step 6. Perform boundary mesh refinement and return to Step 1.

To derive the new values of design variables we use design sensitivity analysis and the derivatives of the cost function I and the augmented Lagrange function I^L with respect to design parameters. The values $\partial I^L/\partial h_i$ are used also to identify the boundary elements for which the refinement is necessary. So, the element of the boundary must be refined if the local gradient of I^L with respect to coordinates of the element is greater than a given positive constant ε_Γ; $\mu_j^2 > \varepsilon_\Gamma$.

4. Numerical example

The developed technique of shape optimization with mesh refinement is illustrated in the following example. The example concerns weight optimization of an in-plane loaded plate under displacement constraints. The initial design of the plate is rectangular ($\Omega : 0 \leq x_1 \leq \alpha, \quad 0 \leq x_2 \leq 1$). The plate is loaded by a uniformly distributed load $q = 1$ at the edge $0 \leq x_1 \leq \alpha$, $x_2 = 0$ and is clamped ($u_1 = u_2 = 0$) along the edges $x_1 = 0$, $0 \leq x_2 \leq 1$ and $x_1 = \alpha$, $0 \leq x_2 \leq 1$. The load free boundary is taken as unknown and has to be found to minimize the plate area under the displacement constraint $u_2 \leq 1$. For the initial design $S = \text{meas}\Omega = \alpha$ the mesh consists of $N = 80$ elements. Bilinear quadrilaterals (4–node isoparametric elements) are used in the example and the material constants (Young's module, Poisson ratio) are taken as $E = 2{,}1 \cdot 10^8 kN/m^2$ and $\nu = 0{,}3$.

The shape is modeled as a COONS patch with a Bezier curve of degree 8 for the load free boundary. The x_2 - coordinate of 8 *Bézier* control points are the discrete design variables marked with stars in the Figure 2.

The optimal shape was computed using two different mesh refinement techniques within the optimization procedure. In a first approach the conventional Zienkiewicz and Zhu error indicator was applied twice within the optimization

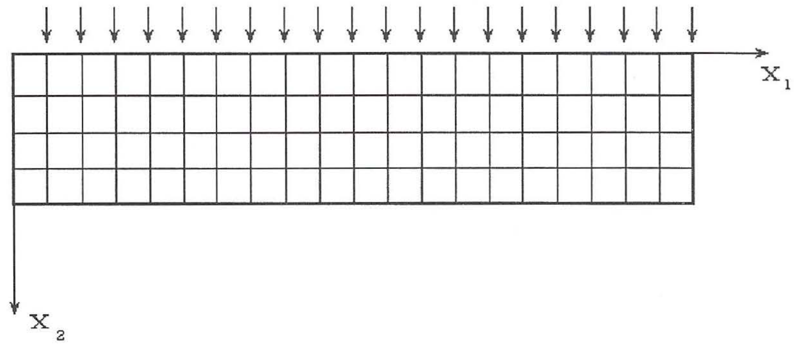

Figure 1. Initial design and mesh

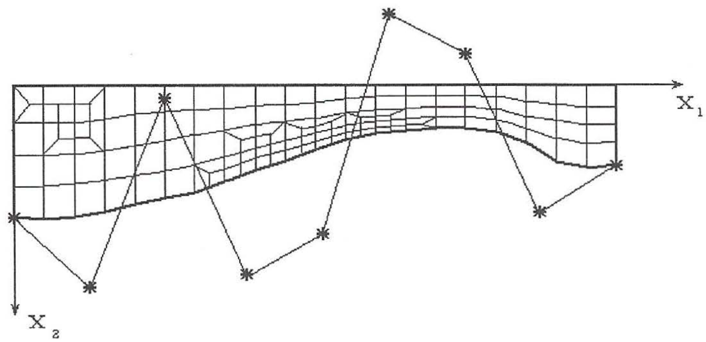

Figure 2. Optimal design and the final mesh.

process to improve the accuracy of the finite element analysis of the considered intermediate design. The minimal value of the objective function is $J = 0{,}68\alpha$. In the second approach we applied our newly derived criterion together with the above mentioned error indicator to improve the overall optimization procedure. The computed optimal shape and the final mesh are shown in Figure 2. We end up with the same objective function value. The refined regions are not only the regions with stress concentration but also the part of the boundary with nonhomogeneous energy distribution. Thus this mesh is more adapted to our optimization problem and leads to a faster convergence of the algorithm, i.e. the optimum was obtained with only 85% of the computational time compared to the first approach.

5. Some notes and conclusions

The computations showed fast convergence of the algorithm. Comparison of the developed double refining procedures with the conventional FE refining have been made. Extensive analysis performed confirmed that the proposed approach can be effectively applied to a wide class of shape optimal design problems, but it appears to be especially effective in the case of nonhomogeneous state field distributions. Another important feature, which is revealed in applying the proposed approach to shape optimal design problems, consists in the following. Even though the "new" boundary nodes obtained in optimization refinement are not considered as an additional degrees of freedom for the shape, the approximate solution so found remains to be good. It makes possible to bound the dimensionality of the design vector and to use effectively the parametrization of the boundaries.

The proposed approach can be directly applied to the problems of internal structure optimization and finding the optimal thickness for thin-walled structures, where the design variables enter the coefficients of the governing equations.

This paper concentrated on a new iterative approach to the solution of shape optimization problems. By coupling the optimal design error estimator with the FEM a posteriori estimator we developed the successive optimization algorithm and performed the computational experiment. It is worthwhile to note that theoretical and practical interests for future investigation are the following:

(i) How to develop the procedure with a flexible mesh refinement scheme taking into account that the errors arise both from the approximation of the displacement fields and their derivatives (strains and stresses)?
(ii) What is the optimum strategy for domain and boundary mesh refinement?
(iii) What a posteriori refinement criteria can be applied for effective reduction of error in double iteration procedures?

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