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Discrete-time approximation of optimal control problems for delayed equations

by

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Abstract. We deal with the infinite horizon optimal control problem for a nonlinear system of differential equations with constant delay focussing our attention on its numerical approximation. Dynamic programming is used to get an approximation scheme based on the time discretization of the dynamics and of the pay–off. We prove that the resulting discrete–time approximation scheme converges to the value function of the continuous problem with rate 1.

Keywords: optimal control, delayed systems, dynamic programming, approximation schemes.

1. Introduction

The control of systems of differential equations with delay is a frequent problem in many applications in engineering and biological models. We consider this problem from the point of view of its numerical approximation and we establish a convergence rate for its discrete–time approximation obtained by applying a discrete version of dynamic programming. The proof of the *a priori* estimate giving the rate of convergence of the scheme is based on direct control arguments.

As it is well known, the dynamic programming approach leads to a characterization of the value function v in terms of a Hamilton–Jacobi equation set in the space of initial conditions. For this characterization the choice of the space of initial functions ϕ for the dynamics is crucial.

A general theory on delayed differential systems with initial conditions in $C(([-\tau,0],\mathbb{R}^N)$ or in the product space $\mathbb{R}^N \times L^2([-\tau,0],\mathbb{R}^N)$ and some applications to control problems can be found in Bensoussan, Da Prato, Delfour and Mitter (1992) (see also Delfour, Karrakchou, 1987, and Delfour, Mitter, 1972). In particular, the linear–quadratic optimal control problem for hereditary differential systems has been studied extensively from the theoretical and from the numerical point of view by applying Nédelec method and semigroups techniques (see e.g. Delfour, 1977, and Delfour, Trochu, 1977).

Here we take the initial conditions in the space $V = C([-\tau, 0]; \mathbb{R}^N)$ essentially because the result on the continuous dependence on the data in that case (see Lemma 2) is more precise and this is particularly useful when applying the dynamic programming principle to the fully nonlinear case.

The drawback of this choice is that $C([-\tau, 0]; \mathbb{R}^N)$ is not an Hilbert space and that it does not satisfy the Radon-Nikodym property which is crucial in the theory of weak solutions in the "viscosity" sense developed in Crandall and Lions (1986) (see also Barbu and Da Prato, 1983, for a different notion of weak solution). We refer to Soner (1988) for the analysis of a control problem for a delayed system in the framework of viscosity solutions.

Although the problem of defining a solution to the Hamilton–Jacobi equation is difficult in the general nonlinear case, we can still use dynamic programming to obtain convergence of an approximation scheme, if the scheme has a control interpretation. In fact, our scheme is constructed via a natural discretization in time of the original control problem which permits the interpretation of the approximate solution v_h as the value function of a discrete–time control problem. In this respect our convergence result is an extension of the results in Capuzzo Dolcetta (1983), Capuzzo Dolcetta and Ishii (1984) and Falcone and Ferretti (1994) related to the approximation of the infinite horizon problem for systems of ordinary differential equations. Adding to the above mentioned approximation a discretization of the space V, e.g. with spline functions, one can finally get a finite dimensional problem which can be solved using the methods described in Falcone (1987). Although a detailed analysis for the space approximation will be developed in a forthcoming paper some directions and a numerical experiment can be found in Falcone and Rosace (1995).

We should also mention that a different approximation of optimal control problems for systems with delay (or hereditary systems) has been previously developed by Banks and his co-authors at the end of the 70's. Their approach is different in two respects. They establish convergence results (without rates) mainly for linear hereditary systems, although some results have been extended to a particular class of nonlinear systems (see Banks, 1979). Moreover, they write the linear functional differential equation as an abstract equation in a Banach space (the space of initial conditions) and use the Trotter–Kato approximation theorem for linear semigroups to establish the convergence of the approximate solutions to the value function. The interested reader can find the detail of this approach in Banks and Kappel (1979), Banks, Burns and Cliff (1979) and in the references therein.

2. The optimal control problem

Let us fix a constant delay $\tau \in \mathbb{R}^+$. We consider the system

$$\begin{cases} \dot{y}(t) = b(t, \alpha(t), y(t), y(t-\tau)) & \text{a.e.} \quad t > 0, \\ y(t) = \phi(t) & \text{in} \quad [-\tau, 0], \end{cases} \tag{1}$$

where the initial function $\phi \in V = C([-\tau, 0], \mathbb{R}^N)$. Our set of admissible controls is

$$\mathcal{A} = \{\alpha : [0, \infty) \longrightarrow A, \quad \text{measurable,}\} \tag{2}$$

where

$$A \text{ is a compact subset of } \mathbb{R}^M. \tag{3}$$

We suppose that the vector field b satisfies the following hypotheses

$$b(\cdot, a, v, w) \text{ is measurable,} \qquad \forall (a, v, w) \in A \times \mathbb{R}^N \times \mathbb{R}^N, \tag{4}$$

$$|\, b(t, a, v, w) \,| \leq M_b, \qquad \forall (t, a, v, w) \in \mathbb{R}^+ \times A \times \mathbb{R}^N \times \mathbb{R}^N, \tag{5}$$

$$|b(t, a, x, y) - b(t, a, z, w)| \leq L_b(|x - z| + |y - w|), \qquad \text{for any } a \in A, \tag{6}$$

where M_b and L_b are positive real costants.

THEOREM 1 *Let (4)-(6) hold true. Then, for any fixed $\phi \in V$ and $\alpha \in \mathcal{A}$ there exists a unique function $y_\phi = y_\phi(t, \alpha(t)) \in AC_{loc}([0, +\infty), \mathbb{R}^N)$ (the space of absolutely continuous functions) which satisfies (1) in the integral sense, i.e.*

$$y_\phi(t) = \phi(0) + \int_0^t b(s, \alpha(s), y_\phi(s), y_\phi(s - \tau))\, ds, \quad \forall t > 0. \tag{7}$$

For the proof see Hale (1971) and Delfour, Mitter (1972).

We choose $V = C([-\tau, 0], \mathbb{R}^N)$ so that we can obtain (by Gronwall's lemma) a precise estimate of the continuos dependence on the data for the system (1). This helps us when applying the dynamic programming principle to our problem. We will denote by $\| \cdot \|_\infty$ the sup norm in V. The proof of the following Lemma, obtained by adapting the arguments for systems without delay, can be found in Rosace (1994).

LEMMA 2 *Let (4)-(6) hold true and $\alpha \in \mathcal{A}$ be fixed. Then for any $\phi, \psi \in V$,*

$$|\, y_\phi(t, \alpha(t)) - y_\psi(t, \alpha(t)) \,| \leq \| \phi - \psi \|_\infty e^{L_b t}, \qquad \text{for all } t > 0. \tag{8}$$

We want to minimize the pay-off

$$J(\phi, \alpha) = \int_0^\infty f(\alpha(s), y_\phi(s, \alpha(s)))\, e^{-\lambda s} ds, \tag{9}$$

where $\lambda \in \mathbb{R}^+$ and $f : A \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$|\, f(a, x) \,| \leq M_f, \qquad \text{for any } (a, x) \in A \times \mathbb{R}^N \tag{10}$$

$$|f(a, x) - f(a, z)| \leq L_f |x - z|, \qquad \text{for any } a \in A, \tag{11}$$

where M_f and L_f are positive real constants. Using standard arguments it is possible to prove that the value function

$$v(\phi) = \inf_{\alpha \in \mathcal{A}} J(\phi, \alpha) \tag{12}$$

satisfies the following dynamic programming principle.

THEOREM 3 *For every $t > 0$, we have*

$$v(\phi) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t f(\alpha(s), y_\phi(s, \alpha(s)) \mathrm{e}^{-\lambda s} \mathrm{ds} + \mathrm{v}(\phi_t)\ \mathrm{e}^{-\lambda t} \right\}, \tag{13}$$

where $\phi_t : [-\tau, 0] \to \mathbb{R}^N$ is defined as $\phi_t(s) = y_\phi(t+s, \alpha(t+s))$, for $s \in [-\tau, 0]$.

Note that by its definition and by Theorem 1, $\phi_t \in V$. The value function has some properties which will be useful to establish the convergence of our scheme. They are stated in the following result.

THEOREM 4 *Let (4)–(6)and (10)–(11) be satisfied. Then, for any $\phi, \psi \in V$ we have*

$$i) \quad | v(\phi) | \leq \frac{M_f}{\lambda} \tag{14}$$

$$ii) \quad | v(\phi) - v(\psi) | \leq C \parallel \phi - \psi \parallel_\infty^\gamma, \tag{15}$$

where C and $\gamma \in \mathbb{R}^+$ are costants depending on L_b, λ, M_f, L_f. More precisely, $\gamma = 1, \frac{1}{2}, \frac{\lambda}{L_b}$ respectively, where $L_b < \lambda$, $L_b = \lambda$ and $L_b > \lambda$.

Proof. For any $\phi \in V$ we have

$$| v(\phi) | \leq \int_0^\infty | f(\alpha(s), y_\phi(s, \alpha(s)) | \ \mathrm{e}^{-\lambda s} ds \qquad \forall \alpha \in \mathcal{A},$$

then by (10) we get immediately *i)*.

Let us prove *ii)*. Take ϕ and ψ in V, by the dynamic programming principle we obtain

$$| v(\phi) - v(\psi) | \leq \inf_{\alpha \in \mathcal{A}} \int_0^t | f(\alpha(s), y_\phi(s, \alpha(s))) - f(\alpha(s), y_\psi(s, \alpha(s))) | \ \mathrm{e}^{-\lambda s} ds$$
$$+ | \ v(\phi_t) \ - \ v(\psi_t) \ | \ \mathrm{e}^{-\lambda t}.$$

By (11) and (8) we have

$$| f(\alpha(s), y_\phi(s, \alpha(s)) \ - \ f(\alpha(s), y_\psi(s, \alpha(s))) | \leq L_f \| \phi - \psi \|_\infty \mathrm{e}^{L_b s} \qquad \forall s > 0.$$

Then we conclude,

$$| v(\phi) \ - \ v(\psi) | \leq L_f \| \phi - \psi \|_\infty \int_0^t \mathrm{e}^{-(\lambda - L_b)s} ds + \frac{2M_f}{\lambda} \mathrm{e}^{-\lambda t}$$

$$= \frac{L_f}{L_b - \lambda} \| \phi - \psi \|_\infty (\mathrm{e}^{-(\lambda - L_b)t} - 1) + \frac{2M_f}{\lambda} \mathrm{e}^{-\lambda t}. \tag{16}$$

Depending on the sign of $(\lambda - L_b)$, we have three cases.

1) Let $\lambda - L_b > 0$, then we pass to the limit for t going to $+\infty$ in (16) and we conclude that
$| v(\phi) \ - \ v(\psi) | \leq C \| \phi - \psi, \|_\infty$
where $C := \dfrac{L_f}{\lambda - L_b}$.

2) Let $\lambda - L_b < 0$, then we have

$$| v(\phi) - v(\psi) | \leq \frac{L_f}{L_b - \lambda} \| \phi - \psi \|_\infty e^{(L_b - \lambda)t} + \frac{2M_f}{\lambda} e^{-\lambda t}. \quad (17)$$

If $\| \phi - \psi \|_\infty \geq 1$, we have that $e^{-\lambda t} \leq \| \phi - \psi \|_\infty{}^{\frac{\lambda}{L_b}}$ for every $t > 0$. Setting $t = \frac{1}{L_b}\ln(\| \phi - \psi \|_\infty)$ and replacing in (17) we get

$$| v(\phi) - v(\psi) | \leq C \| \phi - \psi \|_\infty{}^{\frac{\lambda}{L_b}}, \quad (18)$$

where $C = \frac{L_f}{L_b - \lambda} + \frac{2M_f}{\lambda}$. If $\| \phi - \psi \|_\infty < 1$, we repeat the same argument setting $t = -\frac{1}{L_b}\ln(\| \phi - \psi \|_\infty)$ and replacing t in (17).

3) Let $L_b = \lambda$, then we have

$$| v(\phi) - v(\psi) | \leq L_f \| \phi - \psi \|_\infty t + 2\frac{M_f}{\lambda} e^{-\lambda t}. \quad (19)$$

If $\| \phi - \psi \|_\infty \geq 1$, then $e^{-\lambda t} \leq \| \phi - \psi \|_\infty{}^{\frac{1}{2}}$ for every $t > 0$. We choose $t = \frac{1}{\lambda} \| \phi - \psi \|_\infty{}^{-\frac{1}{2}}$ and we replace it in (19), obtaining

$$| v(\phi) - v(\psi) | \leq C \| \phi - \psi \|_\infty{}^{\frac{1}{2}}, \quad (20)$$

where $C = \frac{L_f}{\lambda} + 2\frac{M_f}{\lambda}$. If $\| \phi - \psi \|_\infty < 1$, then we set

$$t = -\frac{1}{\lambda}\ln(\| \phi - \psi \|_\infty{}^{\frac{1}{2}})$$

and replace t in (19). Since $-\ln(\| \phi - \psi \|_\infty{}^{\frac{1}{2}}) \leq \| \phi - \psi \|_\infty{}^{-\frac{1}{2}}$, we have

$$| v(\phi) - v(\psi) | \leq C \| \phi - \psi \|_\infty{}^{\frac{1}{2}}, \quad (21)$$

where $C = \frac{L_f}{\lambda} + 2\frac{M_f}{\lambda}$. □

Theorem 4 shows that the value function $v \in BUC(V)$, i.e. it belongs to the space of bounded and uniformly continuous functions, so we can define the following norm

$$\| v_1 - v_2 \|_\infty^V = \sup_{\phi \in V} | v_1(\phi) - v_2(\phi) | . \quad (22)$$

3. Time–discretization and convergence

The discrete approximation of the value function can be constructed starting from a discretization in time of the dynamics and of the cost functional with a positive time step h. Let $h = \Delta t$ be positive and let $t_n = nh$. Naturally, to have a meaningful discretization we choose $h < \tau$. Our set of admissible controls for the discrete dynamics is the set of piecewise constant controls

$$\mathcal{A}_h := \{ \alpha \in \mathcal{A} : \alpha(t) = a_n \in A \subset \mathbb{R}^M \text{ with } t \in [t_n, t_{n+1}[\}. \quad (23)$$

For any $n \in A$ we define the discrete dynamical system with delay

$$\begin{cases} \eta_{n+1}(\phi) = \eta_n(\phi) + h\Phi(t_n, a_n, \eta_n(\phi), y(t_n - \tau); h), & n = 0, 1, 2..., \\ \eta_n(\phi) = \phi(nh), \qquad n = 0, -1, -2... - n_\tau, \end{cases} \tag{24}$$

where $\eta_n(\phi)$ is a short notation for $\eta_n(t_n, a_n, \phi)$ and $n_\tau = [\tau/h]$. Note that (24) corresponds to a one-step scheme for (1) and that the function Φ changes according to the numerical method (see e.g. Cryer, 1972, Oppelstrup, 1976, and Oberle and Pesch, 1981, for a detailed analysis of several approximation schemes for systems with delay). For example, one can take the explicit Euler scheme given by

$$\begin{cases} \eta_{n+1}(\phi) = \eta_n(\phi) + h\, b(t_n, a_n, \eta_n(\phi), \eta_{n-m}(\phi)), & n = 0, 1, 2..., \\ \eta_n(\phi) = \phi(nh), & n = 0, -1, -2... - m, \end{cases} \tag{25}$$

where, for simplicity, we assume that $\tau = m\,h$ with $m \in \mathbb{N}$.

In general, we will assume that Φ satisfies the following hypotheses

$$\Phi(t, a, x, y; \cdot) \text{ is continuous }, \quad \forall (t, a, x, y) \in \mathbb{R}^+ \times A \times \mathbb{R}^N \times \mathbb{R}^N, \tag{26}$$

$$\Phi(t, a, x, y; 0) = b(t, a, x, y), \quad \forall (t, a, x, y) \in \mathbb{R}^+ \times A \times \mathbb{R}^N \times \mathbb{R}^N, \tag{27}$$

i.e. the method is consistent. Moreover, we assume that there exists a positive constant L_Φ such that

$$| \Phi(t, a, x, y; h) - \Phi(t, a, s, z; h) | \leq L_\Phi(| x - s | + | y - z |). \tag{28}$$

Let us define the discrete cost functional

$$J_h(\phi, \alpha_h) = h \sum_{i=0}^{\infty} \beta^i f(a_i, \eta_i(\phi)), \tag{29}$$

where $\beta = e^{-\lambda h}$ and $\alpha_h \in \mathcal{A}_h$. The corresponding value function is $v_h : V \longrightarrow \mathbb{R}$,

$$v_h(\phi) = \inf_{\alpha_h \in \mathcal{A}_h} J_h(\phi, \alpha_h). \tag{30}$$

We denote by $\eta(t) = \eta(t, \alpha_h(t), \phi)$ the linear interpolate of the nodes $\eta_n(\phi)$ defined in (24) and for any $p \in \mathbb{N}$ define the function $\phi_p : [-\tau, 0] \longrightarrow \mathbb{R}^N$,

$$\phi_p(s) = \begin{cases} \phi(s + t_p) & \text{for } s + t_p \leq 0, \\ \eta(s + t_p, \alpha_h(s + t_p), \phi) & \text{for } s + t_p > 0. \end{cases} \tag{31}$$

The following result is a key tool for the construction of the approximation. Its proof is standard, so it will be omitted (see e.g. Capuzzo Dolcetta and Falcone, 1988).

THEOREM 5 *Let v_h be defined as in (30), for any integer $p \geq 1$ we have*

$$v_h\,(\phi) = \inf_{\alpha_h \in \mathcal{A}_h} [h \sum_{i=0}^{p-1} \beta^i\, f(a_i, \eta_i(\phi)) \;+\; \beta^p\, v_h(\phi_p)], \tag{32}$$

where $a_i \;=\; \alpha_h(t) \;\in\; A$ *with* $t \in [t_i, t_{i+1})$.

Taking $p = 1$ in (32) we obtain

$$v_h\,(\phi) = \inf_{\alpha_h \in \mathcal{A}_h} [\,\beta\, v_h(\phi_1) + h\, f(a_0, \phi(0))] \;=\; \inf_{a \in A} [\beta\, v_h(\phi_1) + h\, f(a, \phi(0))]. \tag{33}$$

THEOREM 6 *Let the assumptions (3), (6) and (11) be satisfied and $\phi \in V$. Then v_h is the unique bounded solution of*

$$u(\phi) \;=\; \inf_{a \in A} [\beta\, u(\phi_1) \;+\; h\, f(a, \phi(0))]. \tag{34}$$

Proof. We have to prove that v_h is bounded in V. Take $\phi \in V$, by definition (30) and assumption (11) we immediately get

$$v_h(\phi) \;\leq\; h\, M_f \sum_{i=0}^{\infty} \beta^i \;\leq\; \frac{h\, M_f}{1 -\; e^{-\lambda\, h}}. \tag{35}$$

An easy computation shows that $|\, v_h(\phi)\,| \leq \dfrac{M_f}{\lambda}$, for every $\phi \in V$.

To prove uniqueness we proceed by contradiction. Let us assume that there are two bounded solutions of (34) denoted by u_1 and u_2. For any $\epsilon > 0$ there is a control $\tilde{\alpha}_h \;\in\; \mathcal{A}_h$ such that $u_2(\phi) \;+\epsilon \;\geq\; \beta\, u_2(\tilde{\phi_1}) + hf(\tilde{a}, \phi(0))$ where $\tilde{a} = \tilde{\alpha}_h(s)$ with $s \in [0, h)$ and

$$\tilde{\phi_1}(s) \;=\; \begin{cases} \phi(s+h) & \text{for } s+h \;\leq\; 0, \\ \eta(s+h, \tilde{\alpha}_h(s+h), \phi) & \text{for } s+h \;>\; 0. \end{cases} \tag{36}$$

For every $\phi \in V$ and $\epsilon > 0$, we obtain $u_1(\phi) - u_2(\phi) \leq \beta u_1(\tilde{\phi_1}) - \beta u_2(\tilde{\phi_1}) + \epsilon$.

Inverting u_1 and u_2 we can conclude that $\|\, u_1 \;-\; u_2\,\|_\infty \;\leq\; \beta \|\, u_1 \;-\; u_2\,\|_\infty$ that implies $\|\, u_1 \;-\; u_2\,\|_\infty \;=\; 0$. □

The following theorem shows that if the approximation schemes adopted for the dynamics and the cost functional are accurate enough, then the rate of convergence of the approximation scheme defined in (34) is 1. We need the following assumptions

(A1) For every initial function $\phi \in V$ and for every measurable control $\alpha \in \mathcal{A}$, there exist a discrete control $\alpha_h \in \mathcal{A}_h$ and two positive constants K_1 and K_2 such that

$$|\; y_\phi(s, \alpha(s)) \;-\; \eta(s, \alpha_h(s), \phi) \;| \leq\; K_1 h^2, \quad s \;\in\; [0, h], \tag{37}$$

$$|\; \int_0^h f(\alpha(s), y_\phi(s, \alpha(s))\; e^{-\lambda\, s} ds \;-\; hf(a_0, \phi(0)) \;| \leq\; K_2 h^2 \tag{38}$$

where $a_0 = \alpha_h(s)$ with $s \in [0, h]$.

(A2) For every $\phi \in V$ and for every discrete control $\alpha_h \in \mathcal{A}_h$, there exists a measurable control $\alpha \in \mathcal{A}$ such that (37) and (38) are satisfied.

Note that the assumptions $(A1)$ and $(A2)$ introduced in Falcone and Ferretti (1994) for controlled systems without delay can be interpreted as specific requests on the accuracy of the numerical schemes adopted respectively for the dynamics and for the costs.

THEOREM 7 *Let us assume that (5), (6), (10), (11), (26)- (28) and (A1)-(A2) hold true. Moreover, assume that* $L_b < \lambda$ *. Then*

$$\| \, v - v_h \, \|_\infty \leq C\, h.$$

for some positive constant C.

Proof. We know that v_h satisfies (33). For any positive ϵ, there exists $\tilde{\alpha}_h \in \mathcal{A}_h$ such that $v_h(\phi) + \epsilon > \beta v_h(\tilde{\phi_1}) + h\, f(\tilde{a}, \phi(0))$, where $\tilde{a} = \tilde{\alpha}_h(s)$ with $s \in [0, h]$ and

$$\tilde{\phi_1}(s) = \begin{cases} \phi(s+h) & \text{if } s+h \leq 0, \\ \eta(s+h, \tilde{\alpha}_h(s+h), \phi) & \text{if } s+h > 0. \end{cases}$$

The continuous dynamic programming principle implies

$$v(\phi) \leq \int_0^h f(\tilde{\alpha}(s), y_\phi(s, \tilde{\alpha}(s)))\, e^{-\lambda s} ds + e^{-\lambda h} v(\tilde{\phi_h}),$$

where $\tilde{\alpha} \in \mathcal{A}$ is the measurable control corresponding to $\tilde{\alpha}_h \in \mathcal{A}_h$ in *(A2)* and

$$\tilde{\phi_h}(s) = \begin{cases} \phi(s+h) & \text{if } s+h \leq 0, \\ y_\phi(s+h, \tilde{\alpha}(s+h)) & \text{if } s+h > 0. \end{cases}$$

Then, for every $\phi \in V$, we have

$$v(\phi) - v_h(\phi) \leq \left| \int_0^h f(\tilde{\alpha}(s), y_\phi(s, \tilde{\alpha(s)}))\, e^{-\lambda s} ds - hf(\tilde{a}, \phi(0)) \right|$$
$$+ \beta \mid v(\tilde{\phi_h}) - v(\tilde{\phi_1}) \mid + \beta \mid v(\tilde{\phi_1}) - v_h(\tilde{\phi_1}) \mid + \epsilon.$$

Theorem 4 and assumption (37) imply

$$\mid v(\tilde{\phi_h}) - v(\tilde{\phi_1}) \mid \leq L_v \| \, \tilde{\phi_h} - \tilde{\phi_1} \, \|_\infty \leq L_v\, K_1\, h^2\,,$$

where $L_v = \dfrac{L_f}{\lambda - L_b}$. By (38) we get

$$v(\phi) - v_h(\phi) \leq K_2\, h^2 + \beta\, L_v\, K_1\, h^2 + \beta \, \| \, v - v_h \, \|_\infty^V + \epsilon. \tag{39}$$

Following the same argument, we can prove that

$$v_h(\phi) - v(\phi) \leq K_2 h^2 + \beta L_v K_1 h^2 + \beta \| v - v_h \|_\infty^V + \epsilon. \quad (40)$$

By (39), (40) for any fixed ϵ we have

$$(1-\beta) \| v - v_h \|_\infty^V \leq K_2 h^2 + \beta L_v K_1 h^2 + \epsilon.$$

Passing to the limit for ϵ going to 0^+ and recalling that $1-\beta = \lambda h - O(h^2)$, we get

$$\| v - v_h \|_\infty^V \leq C h,$$

where $C = \dfrac{K_2}{\lambda} + \dfrac{L_v K_1}{\lambda}$. □

REMARK 1 *Similar results for higher order approximations can be obtained as in Falcone and Ferretti (1994). The reader can also find there some sufficient conditions assuring (A1) and (A2) for systems without delay, i.e. for $\tau = 0$.*

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