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Representation of bargaining games as simple distribution problems

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Abstract: In this paper we show that the set of all bargaining problems is isomorphic to the set of all simple distribution problems.

1. Introduction

Beginning with the seminal work of Nash (1950), a bargaining problem has been conceived as a compact, convex, comprehensive subset of a finite dimensional Euclidean space, each such subset containing a strictly positive vector. The intuitive motivation behind the bargaining problem has been a rather common place economic problem: dividing a dollar between a group of claimants. Following Nash (1950), there has been a plethora of research in this area, with almost all of them dealing with the same or similar problem. A recent survey of such research is the book by Peters (1992) and more importantly the special issue of this journal dedicated to bargaining and negotiations (see Stefański, 1992). But a technical issue, concerning the class of all bargaining problems as conceived by Nash (1950) and all money division problems as is usually the intuitive motivation behind bargaining game theory, remains yet unanswered: the problem of mathematical isomorphism.

Let $N = \{1, 2, \ldots, n\}$ be a set of agents. A simple distribution problem for N is a pair $[(u_i)_{i \in N}, W]$ satisfying the following properties:

(a) W is a positive real number.

(b) For each $i \in N$, $u_i : \mathbf{R}_+ \to \mathbf{R}_+$ is a function which is continuous, concave, non-constant and non-decreasing.

Let E denote the set of all simple distribution problems. Generic elements of E will be denoted e.

Given $e = [(U_i)_{i \in N}, W] \in E$, it is well known that the set $S(e) = \{x \in \mathbf{R}^n_+ / x_i \leq u_i\,(y_i),\ \sum_{i \in N} y_i = W,\ y \in \mathbf{R}^n_+\}$ satisfies the following property:

(1) $S(e)$ is non empty, compact, convex, comprehensive and contains a strictly positive vector.

This result may be found in Peters (1993) for instance. Let C be the set of all sets S in $\mathbf{R}_+^n$ satisfying property (1). Thus corresponding to each e in E, there is a unique $S \in C$ such that $S = S(e)$.

The converse result i.e. corresponding to each $S \in C$, there is an $e \in E$ such that $S = S(e)$, is what we propose to establish in this paper. Our technique of proof mimics that of Billera and Bixby (1973;1974) whose result established an equivalence relation between the set of non-transferable utility games and distribution problems with the number of commodities varying with the number of agents.

2. Polyhedral problems

A set $S \in C$ is said to be a polyhedral problem if it is the convex comprehensive hull of a finite number of points in $\mathbf{R}_+^n$.

LEMMA 2.1 *If S is a polyhedral problem then there exists an $e \in E$ such that $S = S(e)$.*

Proof: Without loss of generality assume that $D^{n-1} = \{y \in \mathbf{R}_+^n / \sum_{i \in N} y_i = 1\}$ is a subset of S where S is a polyhedral problem. This is possible since suitable multiplication of the resulting functions $(u_i)_{i \in N}$ in e, by positive scalars will have the desired property.

Let $A = \{a^1, \ldots, a^k\} \subseteq \mathbf{R}_+^n$ be such that $S =$ convex hull of $AU\{0\}$, and that there does not exist B (a strict subset of A) such that $S =$ convex hull of $BU\{0\}$. Let l_j be the half line connecting the origin to a^j. Let l_j intersect D^{n-1} at b^j. For $i \in N$, assume without loss of generality that $b_i^1 < b_i^2 < \ldots < b_i^k$. Define $u_i : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ as follows:

$$u_i(b_i^j) = a_i^j \text{ for } j \in \{l, \ldots, k\}$$

$$u_i(tb_i^j + (1-t)b_i^{j+1}) = ta_i^j + (1-t)a_i^{j+1} \text{ for } j \in \{1, \ldots, k-1\}$$

$$u_i(k) = u_i(b_i^k) \text{ for } k \geq b_i^k$$

Observe $b_i^1 = 0$, $b_i^k = 1$. Also u_i is continuous, concave, non-constant and non-decreasing. It is easy to see that $S = S(e)$ where $e = [(u_i)_{i \in N}, 1]$ as each polygon in D^{n-1} defined by the intersecting half lines is linearly mapped into a polygon on the efficient frontier of S. This proves the Lemma.

3. The main result

THEOREM 3.1 *If S is a problem in C, then there exist an $e \in E$, such that $S = S(e)$.*

Proof: Let $\{S^k\}_{k=1}^{\alpha}$ be a sequence of polyhedral problems in C, satisfying the following properties:

i) $S^k \subseteq S^{k+1}$, $k \in N$ (the set of natural number)

ii) $\lim_{k\to\infty} S^k = S$, where the limit is taken in the Hausdorff topology on C.

iii) $\forall\, k \in N$, $a_i(S)\, e_i \in S^k$, $i \in \mathbf{N}$, where $a_i(S) = \max\{x_i / x \in S\}$, $i \in N$, and $e_i \in \mathbf{R}^n_+$ is the i-th unit coordinate vector.

By Lemma 2.1, for each $k \in N$, there exists $e^k = \left[(u_i^k)_{i\in N}, 1\right] \in E$, such that $S^k = S(e^k)$.

Let u_i be the uniform limit of $\{u_i^k\}_{k\in N}$ for $i \in N$. This limit exists on $[0,1]$ for the following reasons:

a) $u_i^{k+1}(x) \geq u_i^k(x)\, \forall\, x \in \mathbf{R}_+$, $\forall\, k \in \mathbf{N}$.

b) Each u_i^k is continuous on the compact set $[0,1]$. Hence u_i is continuous on $[0,1]$.

Since beyond 1, each $u_i^k(x) = a_i(S)\ \forall\, i \in \mathbf{N}$ and $k \in \mathbf{N}$, the uniform limit of $\{u_i^k\}_{k\in N}$ exists beyond 1, and is constant. Let $e = [(u_i)_{i\in N}, 1]$. Since $\lim_{k\to\infty} S^k = S$, we have $S = S(e)$.

REMARK 3.1 *It is easy to see in the above that the uniform limit of a sequence of concave functions is concave.*

REMARK 3.2 *Our result sharpens a corresponding result of Arrow and Hahn (1971) who merely establish that the set of efficient points of a problem in C is homomorphic to the unit simplex.*

REMARK 3.3 *Problems in C are often referred to as choice problems or bargaining games.*

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