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Stochastic version of the economic lot size model

by

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Abstract: This paper deals with a stochastic version of the classical economic lot size model where demands are random variables. Assuming step function type distribution demands, the problem is shown to be approximated by a linear programming problem having the same structure as the deterministic one.

Keywords: lot sizing, deterministic, stochastic, linear programming.

1. Introduction

This paper presents a stochastic version of the well-known economic lot size model which determines production quantities when the discrete demand pattern over the planning horizon is stochastic with known density functions. Although the deterministic version has been extensively discussed in the literature, the stochastic economic lot size model has not received the same attention.

It is the work of Myung and Tcha (1987) with a stochastic uncapacitated facility location problem which has given the main inspiration for the ideas presented in this paper.

It is well-known that the deterministic economic lot size problem can be formulated as an uncapacitated facility location problem with a special structure (Ferreira and Vidal, 1984). This formulation gives origin to several efficient algorithms. The results along these lines are reviewed in Section 2.

In Section 3 the stochastic economic lot size problem will be formulated and it will be approximated by a mixed $0-1$ linear programming problem.

The main result of this paper, presented in Section 4, is to show that the mixed $0-1$ linear programming problem of Section 3 can be transformated to a problem having exactly the same structure as the deterministic model of Section 2. Finally, the last Section presents the concluding remarks.

2. Model formulation: Deterministic case

Let us first present the deterministic economic lot size problem in which time is divided into N periods and the demand is assumed known for each period. A production schedule, $\{x_1, x_2, \ldots, x_N\}$, has to be determined to satisfy a forecasted demand, $\{d_1, d_2, \ldots, d_N\}$. The objective is to minimize the total cost of production, setup and holding inventory.

The basic model, a generalized Wagner-Whitin model, Wagner and Whitin (1958), has the following form:

$$\min \left(\sum_{k=1}^{N} c_k x_k + S_k \delta(x_k) + h_k i_k \right)$$

subject to:

$$\begin{aligned} i_k &= i_{k-1} + x_k - d_k, \quad k = 1, 2, \ldots, N \\ i_0 &= i_N = 0 \\ i_k, x_k &\geq 0, \quad k = 1, 2, \ldots, N \end{aligned}$$

where

$$\delta(x_k) = \begin{cases} 1, & \text{if } x_k > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The following parameters (all nonnegative) are defined for each period k:
c_k the variable production cost, in \$/unit/period,
S_k the setup cost, in \$
h_k the inventory holding cost, in \$/unit/period, and
d_k the demand, in units/period.

The decision variables are:
x_k the amount to be produced (assumed available at the beginning of the period k), in units/period, and
i_k the inventory level at the end of period k, in units.

Bilde (1970) and Vidal (1970) were the first ones to show that the basic economic lot size model can also be formulated as an uncapacitated facility location model using the following transformation. Define first the new parameters

$$\begin{aligned} C_{ii} &= c_i, \quad i = 1, 2, \ldots, N \\ C_{ij} &= C_{ii} + \sum_{k=i}^{j-1} h_k, \quad i < j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \\ c_{ij} &= C_{ij} d_j, i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \end{aligned}$$

Let the new decision variables be: z_{ij}, $i \leq j$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$, representing the fraction of units to be used in the j-th period which are produced in the i-th period, i.e. $z_{ij} = x_{ij}/d_j$ if x_{ij} is the amount produced in the

i-th period to be used in the j-th period, and $y_i = \delta\left(\sum_{j=i}^{N} z_{ij}\right)$, $i = 1, 2, \ldots, N$, a binary variable that will be equal to 1 if there is any production in the i-th period, otherwise it will be equal to 0.

The basic model can now be presented as a mixed 0-1 linear programming problem as follows

$$\min\left(\sum_{j=1}^{N}\sum_{i=1}^{j} c_{ij}z_{ij} + \sum_{i=1}^{N} S_i y_i\right) \tag{1}$$

subject to

$$\sum_{i=1}^{j} z_{ij} = 1, \quad j = 1, 2, \ldots, N \tag{2}$$

$$y_i - z_{ij} \geq 0, i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \tag{3}$$

$$z_{ij} \geq 0, \quad i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \tag{4}$$

$$y_i = 0, 1, \quad i = 1, 2, \ldots, N \tag{5}$$

For the straightforward details, refer to Ferreira and Vidal (1984). Relaxing the requirements that y_i should be either 0 or 1 the folowing linear programming problem can be obtained

$$\min\left(\sum_{j=1}^{N}\sum_{i=1}^{j} c_{ij}z_{ij} + \sum_{i=1}^{N} S_i y_i\right)$$

subject to (2) and (3)

$$z_{ij}, y_i \geq 0, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N$$

Bilde (1970) and Vidal (1970) have also proved that an optimal solution to this last problem will always give y_i equal either 0 or 1 and therefore, it will also be an optimal solution to the mixed 0-1 linear programming model, that is an optimal solution to the basic model. More formal proofs have later been published in Bilde and Krarup (1975), Vidal (1986), and Ferreira and Vidal (1984). These references also present a very effective dual algorithm to solve the above mentioned linear programming problem. Moreover, Ferreira and Vidal (1987) present a quasi-optimal heuristic approach which seems very promising.

3. Model formulation: the stochastic case

Let us now assume that the demand d_j, $j = 1, 2, \ldots, N$, are stochastic independent random variables with known density function $f_j(d_j)$. Two kinds of penalty costs are introduced according to the amount undersupplied or oversupplied. The objective of the stochastic economic lot size problem is to minimize the total cost which includes setup costs, variable production costs, inventory costs, and expected penalty costs for shortage or overage. We restrict ourselves to search of an open loop solution of the problem which minimized the total expected cost.

The stochastic economic lot size problem may be defined as follows:

$$\min \left(\sum_{j=1}^{N} \sum_{i=1}^{j} c_{ij} z_{ij} + \sum_{i=1}^{N} S_i y_i \right.$$
$$+ \sum_{j=1}^{N} E_{d_j \geq x_j} [l_j^1 (d_j - x_j)]$$
$$\left. + \sum_{j=1}^{N} E_{d_j \leq x_j} [l_j^2 (x_j - d_j)] \right) .$$

subject to

$$\begin{aligned} \bar{b}_j y_i - x_{ij} &\geq 0, \ i \leq j, \ i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, N \\ x_{ij} &\geq 0, \ i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, N \\ y_i &= 0, 1, \ i = 1, 2, \ldots, N \end{aligned}$$

where,
l_j^1 is the penalty cost for each unit in shortage at period j,
l_j^2 is the penalty cost for each unit in overage at period j,
$x_j = \sum_{i=1}^{j} x_{ij}$ is the variable denoting the total production to satisfy demand in period j (backlogging is not allowed), and
$\bar{b}_j$ some upper bound on the values taken on by d_j.

Szwarc (1964) has shown that a transportation problem with stochastic demand can be approximated by a linear program, when the density functions of the random variables representing the demand is decomposed or approximated by a step function. To incorportate his approximation technique for our case, the same assumptions are imposed on the density functions $f_j(d_j)$. We are assuming that our densities all have finite ranges which means that for each j there is an interval $[\underline{b}_j, \bar{b}_j]$ outside of which the step function $f_j(d_j) = 0$. For any such function we can, as in Figure 1, divide this interval into a set of k_j subintervals with size Δ_{sj}, $s = 1, 2, \ldots, k_j$, over each of which $f_j(d_j)$ is constant. Note that the indices of Δ_{sj} are numbered according to $s = 1, 2, \ldots, k_j$ from right to left for each j.

Let p_{sj} denote the probability that d_j takes a value from the subinterval with size Δ_{sj}, $s = 1, 2, \ldots, k_j$. Define new variables y_{sj}, each satisfying $0 \leq y_{sj} \leq \Delta_{sj}$, and for each of them penalty parameters per unit associated with

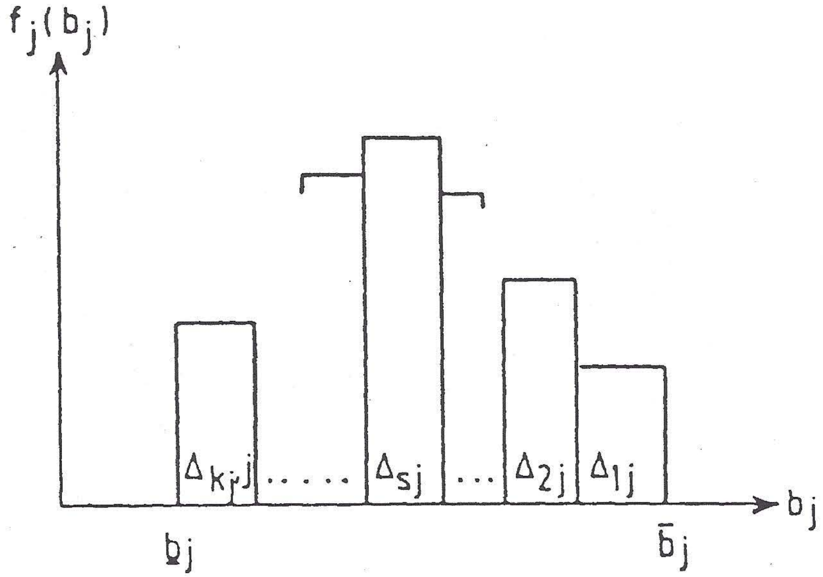

Figure 1. The density function

the amount in shortage or overage r_{sj} defined as

$$\begin{aligned} r_{sj} &= l_j^1 \left(\frac{1}{2} p_{sj} + p_{s-1,j} + \ldots + p_{lj} \right) \\ &\quad - l_j^2 \left(\frac{1}{2} p_{sj} + p_{s+1,j} + \ldots + p_{k_j,j} \right) \end{aligned}$$

for $s = 1, 2, \ldots, k_j$, $j = 1, 2, \ldots, N$

The above formulated stochastic economic lot size problem can then be approximated by the following mixed 0-1 linear programming problem. The desired degree of approximation can be regulated by choosing suitable values for Δ_{sj}.

$$\min \left(\sum_{j=1}^{N} \sum_{i=1}^{j} C_{ij} x_{ij} + \sum_{j=1}^{N} \sum_{i=1}^{k_j} r_{sj} y_{sj} + \sum_{i=1}^{N} S_i y_i \right) \tag{6}$$

subject to

$$\sum_{i=1}^{j} x_{ij} + \sum_{s=1}^{k_j} y_{sj} = \bar{b}_j, \quad j = 1, 2, \ldots, N \tag{7}$$

$$y_{sj} \leq \Delta_{sj}, \quad s = 1, 2, \ldots, k_j, \quad j = 1, 2, \ldots, N \tag{8}$$

$$x_{ij} \leq \bar{b}_j y_i, \quad i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \tag{9}$$

$$x_{ij}, y_{sj} \geq 0, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N, \quad s = 1, 2, \ldots, k_j \tag{10}$$

$$y_i = 0, 1, \quad i = 1, 2, \ldots, N \tag{11}$$

This is a deterministic economic lot size problem, in which demand at period j can be satisfied by "j" uncapacitated production periods and k_j capacitated "pseudo" production facilities in "j" production periods.

To be exact, the objective function (6) should contain some constant terms, but its listing is avoided for the sake of brevity for further details refer to Szwarc (1964).

4. The main result

Now we present an alternative (uncapacitated) compact formulation which is equivalent to (6)-(11) in the sense that their optimal solutions coincide. Let us first define the following additional sets,

$$S_j(i) = \{s : r_{sj} < C_{ij}, \quad i \leq j, \quad s = 1, 2, \ldots, k_j\}$$

The new formulation is

$$\min \left(\sum_{j=1}^{N} \sum_{i=1}^{j} \hat{c}_{ij} z_{ij} + \sum_{i=1}^{N} S_i y_i \right) \tag{12}$$

subject to

$$\sum_{i=1}^{j} z_{ij} = 1, \quad j = 1, 2, \ldots, N \tag{13}$$

$$y_i - z_{ij} \geq 0, \quad i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \tag{14}$$

$$z_{ij} \geq 0, \quad i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \tag{15}$$

$$y_i = 0, 1, \quad i = 1, 2, \ldots, N \tag{16}$$

where

$$\hat{c}_{ij} = C_{ij} \left(\bar{b}_j - \sum_{s \in S_j(i)} \Delta_{sj} \right) + \sum_{s \in S_j(i)} r_{sj} \Delta_{sj} \tag{17}$$

THEOREM 4.1 *Given any binary vector $y \neq 0$, problems (6)-(11) and (12)-(16) yield the same optimal objective value, and the values of the corresponding optimal variables have the following relationships:*

$$x_{ij} = z_{ij}\left(\bar{b}_j - \sum_{s \in S_j(i)} \Delta_{sj}\right), \quad i \leq j, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N \qquad (18)$$

$$y_{sj} = \sum_{i=1}^{j} \sum_{s \in S_j(i)} z_{ij}\Delta_{sj}, \quad s = 1, 2, \ldots, k_j, \quad j = 1, 2, \ldots, N \qquad (19)$$

Proof

For any given binary vector $y \neq 0$, define

$$I^+ = \{i : y_i = 1\}$$

and $i^+(j) \in I^+$ with $C_{i^+(j),j} = \min_{i \in I^+} C_{ij}$, $i \in \{1, 2, \ldots, N\}$, $j = \{1, 2, \ldots, N\}$, $i \leq j$.

Then the optimal sets of $\{x_{ij}, y_{sj}\}$ in (6)-(11) and $\{z_{ij}\}$ in (12)-(16) are easily determined. Thus, since $\bar{b} - \sum_{s \in S_j(i), i \leq j} \Delta_{sj} > 0$, due to the definition of Δ_{sj}, for any y vector, the corresponding optimal variables in (6)-(11) should be such that

$$y_{sj} = \begin{cases} \Delta_{sj}, & \text{if } s \in S_j(i^+(j)), \quad j = \{1, 2, \ldots, N\}, \quad i \leq j, \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} \bar{b}_j - \sum_{s \in S_j(i)} \Delta_{sj}, & \text{if } i = i^+(j), \quad j = \{1, 2, \ldots, N\}, \quad i \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

and the corresponding objective value is

$$\sum_j \left[C_{i^+(j),j} \left\{ \bar{b} - \sum_{s \in S_j(i^+(j))} \Delta_{sj}(j) \right\} + \sum_{s \in S_j(i^+(j))} r_{sj}\Delta_{Sj} \right] + \sum_i S_i \qquad (20)$$

On the other hand, the optimal z_{ij}'s in (12)-(16) are given by

$$z_{ij} = \begin{cases} 1, & \text{if } i = i^+(j), \quad j = \{1, 2, \ldots, N\}, \quad i \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

with the same objective value as (20). Moreover, (19) clearly holds from the above expressions for optimal solutions to (6)-(11) and (12)-(16).

This means that we have been able to construct a suitable approximation model for the stochastic economic lot size problem which has the same structure as the deterministic one. Moreover, the mixed $0-1$ linear programming model (12)-(16) can be relaxed to a linear programming in a similar way as shown in Section 3, therefore, the above mentioned algorithmic and heuristic approaches can also be used to solve the stochastic version of the economic lot size problem.

5. Concluding remarks

The stochatic version of the economic lot size problem can be formulated and solved by stochastic dynamic programming. However, such an approach is too cumbersome and complicated for such a simple problem. In addition, many numerical problems arise because both the state and control variables have to be discretized. The same problem applies to the demand parameter. The approach we present in this paper retains the structure of the problem and can besolved by the matrix algorithm and related heuristic. Moreover, the computational burden to solve the stochastic problem is exactly the same as for the deterministic problem, because the two models have the same dimensions. Some information about the degree of aproximation required and the quality of the optimal solution can be derived by adapting the results obtained in Szwarc (1964). Moreover, if the information about the stochastic demand is available as frequency functions then our approach is not any longer an approximation.

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