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Uncertain dynamical systems with fuzzy constraints

by

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Abstract: The paper demonstrates how to apply belief functions in modelling transition function of a dynamical system. It is explained how to compute a next state of the system and how to find a sequence of fuzzy controls satisfying fuzzy constraints to meet a final state of the system.

1. Introduction

When modelling behaviour of complex systems we must often deal with uncertainty, imprecision and lack of full information. The standard tool for coping with uncertainty is probability theory. Hence, stochastic systems form a well developed branch of control theory. Emergence of fuzzy sets profited in a new discipline - *fuzzy* dynamical systems. At the same time Schweppe (1968) developed another approach, termed uncertain systems. It seems that all these ideas can be put into one formalism - that of belief functions.

In this paper we postulate such a common formalism and we show that depending on the amount of information we have at our disposal it reduces to the approaches already mentioned.

The paper is divided into three main parts. Section 2 provides short introduction to the theory of belief functions. In Section 3 we focus on various representations of the transition function of a dynamical system and in Section 4 we make an attempt towards computing a sequence of optimal controls satisfying certain set of constraints.

2. Belief functions

This section provides a brief introduction to the theory of belief functions, termed also *theory of evidence or Dempster-Shafer Theory (of evidence).* An unacquainted reader is referred to the original works of Shafer (1976;1982) or to a paper by Wierzchoń (1996) where all the notions discussed later are explained.

2.1. Representing a single body of information

Consider a variable X taking the values from a discrete set Θ, referred to as a domain of X or a frame of discernment (this last name is used in Shafer's papers). When assessing a current value of X we may be faced with five generic situations:

- *certain information*: we are sure that X takes on a concrete value $x \in \Theta$,
- *probabilistic information*: we know a distribution function allowing to choose a value $x \in \Theta$ with a probability $p(x) = P(X = x)$,
- *uncertain information*: because of lack of full information we only know that the current value of X should lie somewhere in the subset A of Θ,
- *commonsense information*: due to our experience the value of X *typically* belongs to a subset A of Θ (that is with probability α it lies somewhere in A and in $100 \times (1 - \alpha)$ percent of remaining cases the location of X is absolutely undetermined),
- *lack of information* (total ignorance): we only know that current value of X belongs to the frame Θ.

All the cases mentioned above can be nicely handled within one mathematical formalism - that of belief functions. Our knowledge about possible location of current value of X is expressed now in terms of a so called *belief function* $\mathrm{Bel}_X : 2^\Theta \to [0, 1]$. Actual form of this set function hardly depends on current state of knowledge about the problem under consideration. It was observed by Kohlas and Monney (1994), that Bel stands for a concise representation of hints concerning possible location of the value of X.

From the formal standpoint, belief functions are *monotone of order infinity* set functions defined on the power set of a domain, Θ, and taking the values from the unit interval. A *belief function*, Bel, can be computed effectively from so-called *mass function* (described later), m, according to the equation

$$\mathrm{Bel}(A) = \sum\{m(B) | B \subseteq A, B \neq \emptyset\},\ A \subseteq \Theta \tag{1}$$

Here we have used simplified notation, that is (1) is equivalent to $\sum_{B \subseteq A, B \neq \emptyset} m(B)$.

Equation (1) is valid when the domain Θ is discrete. If Θ is a continuus domain, Bel is defined in the next equation

$$\mathrm{Bel}(A) = \int\int_{[x,y] \subseteq A} m(x, y) dx dy,\ x \leq y \tag{2}$$

where the mass density function, $m(x, y)$, is defined for all the closed intervals $[x, y]$ such that x and y are in Θ, and $x \leq y$ - see Dempster (1968). In other words, m defines a random interval $[X, Y]$ whose endpoints are random variables X and Y such that $X \leq Y$. This interval is an example of a so-called random set. Denoting by $\mathbf{S}$ this random set, the belief function can be interpreted as the probability that $\mathbf{S}$ belongs to a prespecified set A, i.e., $\mathrm{Bel}(A) = \Pr(\mathbf{S} \subseteq A | \mathbf{S} \neq \emptyset)$. Similarly, $m(A) = \Pr(\mathbf{S} = A) / \Pr(\mathbf{S} \neq \emptyset)$. In the sequel we will consider the discrete case only.

In this restricted case the values of mass function are related to a given belief function through the equation

$$m(A) = \sum\{(-1)^{|A-B|}\mathrm{Bel}(B)|B \subseteq A\},\ A \subseteq \Theta \tag{3}$$

A set function $m : 2^\Theta \to [0,1]$ qualifies as a mass function if it satisfies the three postulates: (i) $m(\emptyset) = 0$, (ii) $m(A) \geq 0$ for any subset A of Θ, and (iii) the sum of m-values over all the subsets of Θ (including Θ itself) equals 1. The requirement (i) corresponds to the situation when Θ is a complete list of possible states; otherwise it is justified to assign positive mass to the empty set. In other words, condition (i) corresponds to the so-called *closed world assumption* while it relaxation - to the *open world assumption* - see Smets (1994) or Wierzchoń (1996) for details.

The subsets of Θ with positive m-values are referred to as *focal elements* of a given mass function. Let us denote by $\mathbf{F}(m)$ the set of focal elements of a mass function m. If A does not belong to $\mathbf{F}(m)$ then $m(A) = 0$.

We note that when all focal elements of m are singletons, then such a mass function reduces to a classical probability mass function. Hence, in this case we obtain the so called *Bayesian* belief function, that is the classical probability function.

When $\mathbf{F}(m) = \{\Theta\}$, then m represents a *vacuous* belief function. It describes an agent's total ignorance about the occurrence of a specific elements of Θ.

When $\mathbf{F}(m) = \{\mathrm{A}\}$, where A is a (strict) subset of Θ, we have *categorical* belief function. It represents the agent's belief that the value of X belongs exactly to A. Such functions model uncertain information.

From a practical standpoint, the most important are *simple* support functions characterized by mass functions with two focal elements, $\mathbf{F}(m) = \{A, \Theta\}$, $A \neq \Theta$. They can be viewed as a mixture of categorical and vacuous belief functions and they can be interpreted in the following way: an agent is partially sure (to the extent α) that the truth lies in A; thus the remaining portion of belief (i.e. $1 - \alpha$) must be assigned to the whole domain Θ. Such functions represent default, or commonsense information.

A conjungate to Bel is the *plausibility function*, $\mathrm{Pl}(A) = 1 - \mathrm{Bel}(A^c)$, where A^c is the complement of A in Θ. With our random-set interpretation, Pl, the plausibility function can be interpreted as the probability that the random set $\mathbf{S}$ hits a subset A, i.e. $\mathrm{Pl}(A) = \Pr(\mathbf{S} \cap A \neq \emptyset)$. In other words, the number $\mathrm{Pl}(A)$ determines the extent to which it is *possible* that a current value of X lies somewhere in the set A.

2.2. Combining independent bodies of information

Assume there are two mass functions m_1 and m_2 representing unrelated pieces of information. The resulting (combined) mass function m, being the *orthogonal* *sum* of m_1 and m_2, and denoted $m_1 \oplus m_2$, is defined in the next equation:

$$m(A) = m1 \oplus m2(A) = \kappa^{-1} \sum \{m_1(A_1) \cdot m_2(A_2) | A_1, A_2 \subseteq \Theta, A_1 \cap A_2 = A\} \tag{4}$$

where κ is a normalizing constant, $\kappa = \sum\{m_1(A_1) \cdot m_2(A_2) | A_1, A_2 \subseteq \Theta, A_1 \cap A_2 \neq \emptyset\}$. Hence the rule (4), known as *Dempster's rule of combination*, is applicable only if $\kappa > 0$. Intuitively, the value of κ expresses the degree of agreement between two pieces of information. If the pieces are perfectly consistent then $\kappa = 1$, and if they are absolutely inconsistent then $\kappa = 0$. Obviously, the rule (4) is applicable if the two pieces are at least partially consistent. It is always applicable under the open world assumption mentioned earlier.

It is interesting to note that when two categorical mass functions, with focal elements A and B respectively, are combined then the resulting mass function has exactly one focal element $C = A \cap B$. Hence, Dempster's rule of combination acts like a generalized set intersection. It can be verified that the rule corresponds to the intersection of two independent random sets. Another interesting property of the combination rule was observed by Kohlas and Monney (1990). The authors state on p. 51 that "*Dempster's rule of combination of hints through arithmetic constraints really corresponds to the arithmetic of uncertain intervals (much as the arithmetic of random variables)*".

2.3. Transporting beliefs expressed on different frames

When studying dynamical systems, the set Θ is usually understood as a Cartesian product of few spaces. More concretely, let $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ be a set of variables and let $\Theta(\{X_i\})$ stands for the domain of i-th variable. Then $\Theta(\mathbf{X})$ denotes the Cartesian product of the domains of all variables specified in the set $\mathbf{X}$. Similarly if $\mathbf{A}$ is a subset of $\mathbf{X}$ then $\Theta(\mathbf{A}) = \times\{\Theta(\{X_i\}) | X_i \in \mathbf{A}\}$. Belief functions defined on the frame $\Theta(\mathbf{X})$ are referred to as the multivariate belief functions, and they are studied in Kong (1986).

To combine two multivariate mass function m_1 and m_2 defined on different spaces, say $\Theta(\mathbf{A})$ and $\Theta(\mathbf{B})$, respectively, we must redefine them to the common space $\Theta(\mathbf{A} \cup \mathbf{B})$ - consult Shafer (1976) for a general case. Such an operation is said to be a minimal extension. It is defined in the next equation:

$$m_1^{\uparrow(\mathbf{A}\cup\mathbf{B})}(B) = \begin{cases} m_1(A) & \text{if } B = A \times \Theta((\mathbf{A} \cup \mathbf{B}) - \mathbf{A}) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Particularly it can be verified that if m_1 and m_2 are defined on the spaces $\Theta(\mathbf{A})$ and $\Theta(\mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are disjoint subsets of $\mathbf{X}$, then the combined mass function m has focal elements of the form $A \times B$, $A \in \Theta(\mathbf{A})$, $B \in \Theta(\mathbf{B})$ and

$$m(A \times B) = m_1(A) \cdot m_2(B) \tag{6}$$

This shows that frames being products of disjoint subsets of $\mathbf{X}$ qualify as *independent* frames - see Shafer (1976), for a deeper discussion of this notion.

Knowing a multivariate mass function, m, defined on the space $\Theta(\mathbf{A})$, we may be interested in the information this mass function carries about $\Theta(\mathbf{B})$, where $\mathbf{B} \subset \mathbf{A}$. This task is gained through a marginalization operation defined in the equation (7):

$$m^{\downarrow\mathbf{B}}(B) = \sum\{m(A)|A \subseteq \Theta(\mathbf{A}),\ \mathrm{Proj}(A, \Theta(\mathbf{B})) = B\} \tag{7}$$

where $\mathrm{Proj}(A, \Theta(\mathbf{B}))$ stands for the projection of the set A onto the space $\Theta(\mathbf{B})$. Marginalization is an extension of a similar operation from probability theory.

2.4. Conditional belief functions

Conditional belief functions are belief functions obtained by combining multivariate belief functions with categorical belief functions. To be more precise, consider a mass function defined on the product space $\Theta(\mathbf{X})$. Let $\mathbf{A}$ be a (strict) subset of $\mathbf{X}$ and let $\mathbf{B} = \mathbf{X} - \mathbf{A}$. Denote $m_{\mathbf{b}}$ to be the categorical belief function with focal element $\mathbf{b} \in \Theta(\mathbf{B})$. The conditional mass function $m(\cdot|\mathbf{B} = \mathbf{b})$ defined on $\Theta(\mathbf{A})$ is computed according to the equation[1]

$$m(\cdot|\mathbf{B} = \mathbf{b}) = (m \oplus m_{\mathbf{b}})^{\downarrow\mathbf{A}} \tag{8}$$

Such a function represents the state of an agent's knowledge about variables from the set $\mathbf{A}$ when it is assumed that the variables from the set $\mathbf{B}$ take the values described by the element $\mathbf{b}$.

Knowing a set of conditional belief functions we may attempt to recover a joint belief function. Although this problem has no unique solution we may apply the method of *conditional embedding* proposed by Smets (1994). Due to this method a mass function $m(\cdot|\mathbf{B} = \mathbf{b})$, defined over $\Theta(\mathbf{A})$, is extended to the mass function $m(\cdot|\mathbf{B} = \mathbf{b})^{\Rightarrow X}$ with focal elements of the form

$$C = A \times \{\mathbf{b}\} \cup \Theta(\mathbf{A}) \times (\Theta(\mathbf{B}) - \{\mathbf{b}\}) \text{ for all } A \in \mathbf{F}(m(\cdot|\mathbf{B} = \mathbf{b})) \tag{9}$$

According to Almond (1991), that conditionally embedded mass function is equivalent to the statement

> "If $\mathbf{B} = \mathbf{b}$ then we are sure to the extent $m(A|\mathbf{B} = \mathbf{b})$ that the (joint) variable $\mathbf{A}$ takes a value $\mathbf{a}$ which belongs to the set $A \subseteq \Theta(\mathbf{A})$; if not, our knowledge about $\mathbf{A}$ is vacuous".

The join belief function is obtained now by combining the belief functions $\mathrm{Bel}(\cdot|\mathbf{B} = \mathbf{b})^{\Rightarrow X}$ over all $\mathbf{b}$'s. Conditioning such a function (i.e. combining it with a categorical belief function) we obtain so-called generalized Bayes theorem. It can be shown that such a procedure leads to a counterpart of the *Modus Tollens* logical scheme - see Wierzchoń (1996) for details.

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[1]Let $\mathbf{B} = \{X_{i1}, X_{i2}, \ldots, X_{ik}\}$. Then the symbol $\mathbf{B} = \mathbf{b}$ should be read as $X_{i1} = b_{i1}, X_{i2} = b_{i2}, \ldots, X_{ik} = b_{ik}$, where $b_ij \in \Theta(\{X_{ij}\})$, $j = 1, \ldots, k$, and $\mathbf{b} = (b_{i1}, b_{i2}, \ldots, b_{ik})$.

3. Dynamical systems under uncertain restrictions

We assume there are given three discrete sets: $\Theta(\{U\}) = \{u_1, \ldots, u_m\}$, $\Theta(\{X\}) = \{x_1, \ldots, x_n\}$, and $\Theta(\{Y\}) = \{y_1, \ldots, y_k\}$, which are the domains of the three variables U, X, and Y, called respectively *control*, *state*, and *output*.

A dynamical system is said to be *deterministic* and *stationary* if its behaviour is described by the set of equations

$$\begin{cases} X_{t+1} = f(X_t, U_t) \\ Y_t = h(X_t, U_t) \end{cases} \quad t = 0, 1, 2, \ldots \tag{10}$$

where $f : X \times U \to X$ is a *state transition function* and $h : X \times U \to Y$ is an *output function*. The parameter t represents discrete time. When the output equation is of the form $Y_t = h(U_t)$ then the system is said to be *memoryless*.

In the sequel we shall write $x_j = f(x_i, u_k)$ to express the fact that if the system was at the moment t in a state $x_i \in \Theta(\{X\})$ and a control $u_k \in \Theta(\{U\})$ was applied then, in the moment $t+1$, it proceeded to a new state $x_j \in \Theta(\{X\})$.

A dynamical system is said to be *nondeterministic* and stationary if its behaviour is described by the next set of equations

$$\begin{cases} X_{t+1} = \mathbf{f}(X_t, U_t) \\ Y_t = \mathbf{h}(X_t, U_t) \end{cases} \quad t = 0, 1, 2, \ldots \tag{11}$$

where the mappings $\mathbf{f}$ and $\mathbf{h}$ are of the form $\mathbf{f} : 2^{X \times U \times X} \to [0, 1]$ and $\mathbf{h} : 2^{X \times U \times Y} \to [0, 1]$. Particular and well-known examples of nondeterministic systems are: Schweppe's uncertain systems, stochastic systems, and fuzzy systems. In the next two subsections we demonstrate how such systems (including deterministic systems) may be modelled in the framework of belief functions theory.

3.1. Naive representation of dynamical systems

Consider first a deterministic transition fuction. It can be represented by a categorical belief function Bel_S with one focal element $A \subseteq \Theta(\{X\}) \times \Theta(\{U\}) \times \Theta(\{X'\})$ of the form

$$A = \{\{x_i, u_k, f(x_i, u_k)\} | x_i \in \Theta(\{X\}),\ u_k \in \Theta(\{U\})\} \tag{12}$$

where $\Theta(\{X'\})$ represents the new set of possible states (at the moment $t+1$). The set A consists simply of all triplets of the form $(x_i, u_k, f(x_i, u_k))$.

Quite similarly we define the belief functions Bel_U and Bel_X representing our knowledge about current control and actual state of the system, respectively. That is Bel_U is a categorical belief function with one focal element $B = \{u_0\}$, where $u_0 \in \Theta(\{U\})$ and Bel_X has one focal element $C = \{x_0\}$, where $x_0 \in \Theta(\{X\})$.

Now, the next state of the system is computed according to the equation

$$\mathrm{Bel}_{X'} = (\mathrm{Bel}_S \oplus \mathrm{Bel}_U^{\uparrow X \times U \times X'} \oplus \mathrm{Bel}_X^{\uparrow X \times U \times X'})^{\downarrow X'} \tag{13}$$

Indeed. The focal element of the function $\mathrm{Bel}_U^{\uparrow X\times U\times X'}$ equals now $B^{\uparrow X\times U\times X'} = \Theta(\{X\}) \times \{u_0\} \times \Theta(\{X'\})$ and the focal element of the function $\mathrm{Bel}_X^{\uparrow X\times U\times X'}$ can be expressed as $C^{\uparrow X\times U\times X'} = \{x_0\} \times \Theta(\{U\}) \times \Theta(\{X'\})$.

According to the Dempster's rule of combination, (4), the focal element, D, of the function $\mathrm{Bel}_{X'}$ is computed as

$$\begin{aligned} D &= (A \cap B^{\uparrow X\times U\times X'} \cap C^{\uparrow X\times U\times X'})^{\downarrow X'} = (\{x_0, u_0, f(x_0, u_0)\})^{\downarrow X'} = \\ &\quad \{f(x_0, u_0)\} \end{aligned} \tag{14}$$

Similarly, if the transition function $\mathbf{f}$ represents uncertain system, i.e. $\mathbf{f}(x_i, u_k) \subseteq \Theta(\{X'\})$ then the focal element of the *categorical* belief function Bel_S can be expressed as the set of all triplets $\{x_i, u_k, x_j\}$ where x_j's vary over the range of $\mathbf{f}(x_i, u_k)$ for all $x_i \in \Theta(\{X\})$, $u_k \in \Theta(\{U\})$. Note that in this case the equation (14) produces a set of values, and not a single value.

Consider now a more complicated situation. We assume that the transition function f is deterministic but both a current state and control are represented by simple support functions with focal elements $\mathbf{F}(m_X) = \{\{x_0\}, \Theta(\{X\})\}$ and $\mathbf{F}(m_U) = \{\{u_0\}, \Theta(\{U\})\}$. In other words, our knowledge about a current state of the system is represented by a simple support m-function of the form

$$m_X(A) = \begin{cases} \alpha_X & \text{if } A = \{x_0\} \\ 1 - \alpha_X & \text{if } A = \Theta(\{X\}) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

The equation (15) should be read as follows: we are certain to the degree α_X, where $\alpha_X \in [0,1]$, that the current state of the system is x_0 and with the degree $1-\alpha_X$ we state that the current state locates somewhere in $\Theta(\{X\})$, i.e. $1-\alpha_X$ is the extent to which the state of the system is uncertain.

Similarly, our belief that the correct control, u_0, was applied is expressed by the next m-function:

$$m_U(B) = \begin{cases} \alpha_U & \text{if } B = \{u_0\} \\ 1 - \alpha_U & \text{if } B = \Theta(\{U\}) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

Equation (13) decomposes now into four parts defining four (not necessarily different) focal elements of the mass function $m_{X'}$ corresponding to $\mathrm{Bel}_{X'}$:

1. Current state and control are known to be x_0 and u_0, respectively. Hence the system proceeds to the state $X^1_{t+1} = \{x_t + 1\} = f(x_0, u_0)$ and the degree of belief $m_{X'}(X^1_{t+1}) = \alpha_X \alpha_U$.
2. Current state is known to be x_0 and control value is uncertain. The new state of the system becomes uncertain and it equals to the set $X^2_{t+1} = \cup\{f(x_0, u_t) | u_t \in \Theta(\{U\})\}$ (i.e it is computed by projecting all the triplets of the form $(x_0, u_t, f(x_0, u_t)), u_t \in U$ onto the space X'). The degree of belief associated to X^2_{t+1} equals to $\alpha_X(1 - \alpha_U)$.

3. Current state is uncertain and the value of control parameter is known to be u_0. Again, system comes to an uncertain state $X^3_{t+1} = \cup\{f(x_t, u_0))|x_t \in \Theta(\{X\})\}$. We assign to such a subset the degree of belief $m_{X'}(X^3_{t+1}) = (1-\alpha_X)\alpha_U$.
4. Both control and state are uncertain. The system comes to an uncertain state defined by the set $X^4_{t+1} = \cup\{f(x_t, u_t)), x_t \in \Theta(\{X\}), u_t \in \Theta(\{U\})\}$. Obviously $m_{X'}(X^4_{t+1}) = (1-\alpha_X)(1-\alpha_U)$.

Obviously if we are certain that the current state of the system is x_0 and that the control u_0 was applied we put $\alpha_X = \alpha_U = 1$ and we obtain the equation (14) itself.

3.2. Conditional belief functions

The naive representation is too vast in a general case. More efficient is a representation using conditional belief functions, that is the system is represented by the set $\{\text{Bel}_{X'}(\cdot|x_i, u_k), x_i \in \Theta(\{X\}),\ u_k \in \Theta(\{U\})\}$ of belief functions.

With such a representation, to find next state of the system we proceed along two steps:

- first we create a joint belief function Bel_S defined as the orthogonal sum of conditionally embedded belief functions $\text{Bel}_{X'}(\cdot|x_i, u_k)$:
$$\text{Bel}_S = \oplus\{\text{Bel}_{X'}^{\Rightarrow X\times U\times X'}(\cdot|x_i, u_k)|x_i \in \Theta(\{X\}),\ u_k \in \Theta(\{U\})\} \quad (17)$$
- next we apply the rule (13)

Let us investigate the formula for a focal element of the belief function Bel_S. According to the equation (9) the focal elements of the conditional belief function $\text{Bel}_{X'}^{\Rightarrow X\times U\times X'}(\cdot|x_i, u_k)$ are of the form

$$A \times \{x_i, u_k\} \cup [\Theta(\{X'\}) \times (\Theta(X \times U) - \{x_i, u_k\})],$$
$$A \in \Theta(\{X'\}),\ x_i \in \Theta(\{X\}),\ u_k \in \Theta(\{U\}) \quad (18)$$

Denote by A_j, where $j = (i-1)\cdot m + k$, $m = \text{Card}(\Theta(\{U\}))$, a focal element of the conditional belief function $\text{Bel}_{X'}^{\Rightarrow X\times U\times X'}(\cdot|x_i, u_k)$. Then the focal element A of Bel_S computed as the intersection of A_j's equals

$$A = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(A_{(i-1)\cdot m+k} \times \{x_i, u_k\}) \quad (19)$$

Below we give three propositions which demonstrate basic properties of the new representation.

PROPOSITION 3.1 *If Bel_U and Bel_X are categorical belief functions focused on singletons $\{u_0\}$ and $\{x_0\}$, respectively then*

$$Bel_S \oplus Bel_U^{\uparrow X\times U\times X'} \oplus Bel_X^{\uparrow X\times U\times X'} = Bel_{X'}(\cdot|x_0, u_0).$$

Proof: Intersection of a focal element of Bel_S of the form described by the equation (19) with a focal element $X' \times \{x_0, u_0\}$ of the function $\mathrm{Bel}_U^{\uparrow X\times U\times X'} \oplus \mathrm{Bel}_X^{\uparrow X\times U\times X'}$ produces exactly the set $A_j \times \{x_0, u_0\}$ which is a focal element of the conditional belief function $\mathrm{Bel}_{X'}(\cdot|x_0, u_0)$. ■

From this proposition it follows that when all conditional belief functions $\mathrm{Bel}_{X'}(\cdot|x_i, u_k)$ are categorical functions focused on singletons then we obtain the deterministic system. The next proposition shows how to find next state of the system under a categorical control.

PROPOSITION 3.2 *If Bel_U is a categorical belief functions focused on singleton $\{u_0\}$ then*

$$
\begin{aligned}
& Bel_S \oplus Bel_U^{\uparrow X\times U\times X'} \oplus Bel_X^{\uparrow X\times U\times X'} \\
&= \oplus\{Bel_{X'}^{\Rightarrow X\times U\times X'}(\cdot|x_t, u_0) \oplus Bel_X^{\uparrow X\times U\times X'}(\cdot)|x_t \in \Theta(\{X\})\}
\end{aligned}
$$

Proof: Assume that u_0 is the K-th value in $\Theta(\{U\})$. The intersection of a focal element A of Bel_S with the only focal element $X' \times \{u_K\} \times X$ of the belief function $\mathrm{Bel}_U^{\uparrow X\times U\times X'}$ leads to a focal element of the form

$$
\begin{aligned}
A \cap (X' \times \{u_K\} \times X) &= \left(\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_{(i-1)\cdot m+k} \times \{x_i, u_k\}) \right) \cap \\
& (\Theta(\{X'\}) \times \{u_K\} \times \Theta(\{X\}) \\
&= \bigcup_{i=1}^{n} (A_{(i-1)\cdot m+K} \times \{x_i, u_K\})
\end{aligned}
$$

which includes only focal elements of the conditional belief functions $\mathrm{Bel}_{X'}^{\Rightarrow X\times U\times X'}(\cdot|x_t, u_K)$. Thus the remaining conditional belief functions can be ignored. ■

PROPOSITION 3.3 *If Bel_U and Bel_X are Bayesian belief functions with masses p_U and p_X, respectively then the formula (13) reduces to*

$$
\begin{aligned}
m_{X'}(A) &= \kappa^{-1} \cdot \sum\{m_{X'}(A|x_i, u_k) \cdot p_X(x_i) \cdot p_U(u_k) | x_i \in \Theta(\{X\}), \\
u_k &\in \Theta(\{U\})\}, A \subseteq \Theta(\{X'\})
\end{aligned}
$$

where κ is a normalizing constant.

Proof: Focal elements of the function $\mathrm{Bel}' = \mathrm{Bel}_U^{\uparrow X\times U\times X'} \oplus \mathrm{Bel}_X^{\uparrow X\times U\times X'}$ are singletons $\{x_i, u_k\}$ with masses $p_X(x_i) \cdot p_U(u_k)$. Combining this function with Bel_S we obtain the joint belief function characterized by mass function

$$
\begin{aligned}
& m_{X'\times U\times X}(A \times \{x_t, u_t\}) = m_{X'}(A \times \{x_t, u_t\} \cup \\
& [X' \times ((X \times U) - \{x_t, u_t\})]) \cdot p_X(x_t) \cdot p_U(u_t)
\end{aligned}
$$

Projecting $m_{X' \times U \times X}$ onto X' we obtain the result. ■

From this proposition it follows that under the Bayesian control and Bayesian information about the current state we obtain the Bayesian-like behaviour of the system. Particularly if all conditionals are Bayesian belief functions then our representation reduces to the stochastic system. Finally, when all $\mathrm{Bel}_{X'}$'s are consonant belief function we obtain fuzzy dynamical system (in this case we use rather plausibilities than beliefs: $m_{X'}(s_{t+1}|s_t, c_t) = \mathrm{Pl}_{X'}(\{s_{t+1}\}|s_t, c_t)$).

4. Controlling uncertain system

Assume that a dynamical system is described by a set of conditional belief functions $\{\mathrm{Bel}_{X'}(\cdot|x_i, u_k),\ x_i \in \Theta(\{X\}),\ u_k \in \Theta(\{U\})\}$. Assume further that at each stage t, $t = 0, 1, \ldots, N-1$ there is given a fuzzy constraint $C(t)$, represented by a membership function $\mu_{C(t)} : \Theta(\{U_t\}) \to [0, 1]$. As usual, the value $\mu_{C(t)}(u_k)$ where $u_k \in \Theta(\{U_t\})$, represents the degree to which the control u_k is considered as admissible at the stage t. Further with the final state X_N a fuzzy goal G, represented by a membership function $\mu_G : \Theta(\{X_N\}) \to [0, 1]$, is associated. This problem was firstly posed by Bellman and Zadeh (1970). The task is to find optimal sequence of controls $(u_0^*, u_1^*, \ldots, u_{N-1}^*)$ which maximize the next functional

$$\Phi(u_0^*, \ldots, u_{N-1}^*) = \bigwedge_{(u_0, \ldots, u_{N-1}) \in \Theta(U_0) \times \cdots \times \Theta(U_{N-1})} (\mu_{C(0)}(u_0), \ldots, \mu_{C(N-1)}(u_{N-1}), \mathbf{E}[\mu_G]) \quad (20)$$

where the symbol $\bigwedge$ stand for the "min" operator, and $\mathbf{E}[\mu_G(x_i)]$ denotes the expected value of fuzzy event which, in the Bayesian case, it is defined by the equation (21):

$$\mathbf{E}[\mu_G] = \sum \{p(x_N|x_{N-1}, u_{N-1}) \cdot \mu_G(x_i) | x_i \in \Theta(\{X_N\})\} \quad (21)$$

To find the optimal sequence $(u_0^*, \ldots, u_{N-1}^*)$ in this fuzzy-stochastic case we use dynamical programming approach which can be summarized in the next pseudo code - see Kacprzyk (1986) for details.

```
Program FindSequence;
const
        N {mumber of stages},
        noX {cardinality of the space X}
        noU {cardinality of the space U}
var
        constraints: array [1..N,1..noU] of real; {matrix of fuzzy constraints}
        goal: array [1..noX] of real; {fuzzy goal}
        transition: array [1..noU,1..noX,1..noX] of real; {transition function}
        expectation:array [1..noU,1..noX] of real; {computed due to (15)}
```

```
    policies: array [1..N,1..noX] of integer; {optimal strategy}

Procedure FindExpectation;
begin
    expectation := 0;
    for i:=1 to noU do
    for j:=1 to noX do
    for k:=1 to noX do
    expectation[i,j]:=expectation[i,j]+transition[i,j,k]*goal[k];
end; {FindExpectation}

Procedure FindStrategy;
var temp: real; g: array [1..noX] of real;
begin
    g := 0;
    for i:=1 to N do
    begin
        FindExpectation;
        for j:=1 to noX do
        for k:=1 to noU do
        begin
            temp:=min(constraints[N-i,k],e[k,j]);
            if temp>g[j] then
            begin g[j]:=temp; policies[N-i+1,j]:=k; end;
        end;
        goal:=g; g:=0;
    end;
end; {FindStrategy}
```

The procedure *FindStrategy* finds optimal policies, i.e. a function $\pi : X \to U$ determining best control for intermediate states.

When dealing with uncertain systems, we are faced with the problem of appropriate definition of the matematical expectation. The role of a probability measure plays now a capacity that is a belief or plausibility measure, and the expectation is generalized as a Choquet integral (see Nguyen and Walker, 1994 for details). Under the discrete case we computed the upper expectation defined in the equation (22) below

$$\mathbf{E}^*[\mu_G] = \sum_{A \subseteq \Theta(\{X\})} m(A|x_i, u_k) \cdot \max\{\mu_G(x_i)|x_i \in A\} \tag{22}$$

and the lower expectation which equals

$$\mathbf{E}_*[\mu_G] = \sum_{A \subseteq \Theta(\{X\})} m(A|x_i, u_k) \cdot \min\{\mu_G(x_i)|x_i \in A\} \tag{23}$$

X_t	$U_t = u_1$						
	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\{x_1\}$	0	0	0	0.8	0	0	0.2
$\{x_2\}$	0	0	0	0	0	0.7	0.3
$\{x_3\}$	0.7	0.2	0.1	0	0	0	0

Table 1.

X_t	$U_t = u_2$						
	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\{x_1\}$	0	0	0.6	0	0	0	0.4
$\{x_2\}$	0	0	0	0.9	0	0	0.1
$\{x_3\}$	0	0	0	0	0	1	0

Table 2.

The closed interval $[\mathbf{E}_*[\mu_G], \mathbf{E}^*[\mu_G]]$ can be treated as an interval estimate of unknown value of $\mathbf{E}[\mu_G]$. In practice, we can use only one endpoint of this interval, or a point from this interval (being a convex combination of its endpoints) - see e.g. Smets (1994). Problems concerned with the ordering of such intervals are discussed in Wierzchoń (1987).

To illustrate these notions consider a simple example. The uncertain transition function, expressed in terms of a conditional mass function is given below.

In the first table the evolution of the system under control for $U_t = u_1$ is described, and its evolution under $U_t = x_2$ is shown in the second table. The table should be read as follows: Under control $U_t = u_1$ the system *usually* comes from a state, say x_1, to an uncertain state (x_1 or x_2) and sometimes its state is fully indetermined. Similarly, when the system was in the state x_3 and $U_t = c_1$ was applied, then it behaved in strictly probabilistic way.

Assume for simplicity that we consider a one-step strategy only, and the fuzzy constraint and fuzzy goal are as below:

$$\begin{aligned} \mu_{C(0)}(u_1) &= 1.0,\ \mu_{C(0)}(u_2) = 0.6, \\ \mu_G(x_1) &= 0.3,\ \mu_G(x_2) = 1.0,\ \mu_G(x_3) = 0.8. \end{aligned}$$

Thus, the expectations calculated according to the equations (22) and (23) are listed in the table 3, and the matrix g, defined in *FindStrategy* equals

$$\begin{aligned} g^*(x_1) &= 1.00,\ g^*(x_2) = 1.00,\ g^*(x_3) = 0.60 \\ g_*(x_1) &= 0.36,\ g_*(x_2) = 0.65,\ g_*(x_3) = 0.60. \end{aligned}$$

Lastly we compute the upper and lower policies

	Upper expectation			Lower expectation		
	x_1	x_2	x_3	x_1	x_2	x_3
u_1	1.00	1.00	0.49	0.30	0.65	0.49
u_2	0.88	1.00	1.00	0.60	0.30	0.80

Table 3.

$$\begin{aligned} \pi^*(x_1) &= u_1,\ \pi^*(x_2) = u_1 \text{ or } u_2,\ \pi^*(x_3) = u_3 \\ p_*(s_1) &= c_2,\ p_*(s_2) = c_1,\ p_*(s_3) = c_3 \end{aligned}$$

5. Concluding remarks

In this paper a first attempt to modelling uncertain systems was presented. A number of problems concerned with finding an optimal sequence of controls satisfying certain restrictions - discussed in Kacprzyk (1986) - still wait for satisfactory solutions. To solve these problems we can use graph theoretic concepts - cf. Kacprzyk (1986), Sect. 10.4.2. Under such a setting the idea of message propagation, formulated in Shenoy and Shafer (1990), can be applied. This problem is discussed in Wierzchoń (1996).

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