Control and Cybernetics

vol. 25 (1996) No. 4

Time-optimal stabilization of a discontinuous and non-autonomous dynamic object

by

Piotr Kulczycki

Cracow University of Technology, Faculty of Electrical Engineering, ul. Warszawska 24, PL-31-155 Cracow, Poland

Abstract: The present paper deals with the time-optimal control of systems described by a discontinuous and non-autonomous differential inclusion, principally useful in the field of robotics. The existence and characteristics of that control are shown. The paper also contains corollaries for practical applications, particularly concerning the random approach to an optimal control. Empirical examinations confirmed numerous advantages of the control system worked out here, especially in the area of robustness.

1. Introduction

In various kinds of applications in contemporary engineering, from manufacturing to space exploration, the objects of time-optimal control very frequently prove to be devices whose dynamics are described by the differential inclusion

$$\ddot{y} \in H + u, \tag{1}$$

where u is a bounded control function, y denotes a position, and the multivalued (set-valued) function H represents the model of motion resistances. (Note, that for $H \equiv 0$ formula (1) expresses the second law of Newtonian mechanics; therefore, such physical objects are even known as natural systems (see Kreutz, 1989)). As the simplest example one might mention the wide class of industrial plants which realize their technological cycles mainly through changes of the positions in particular mechanisms, e.g. saddles of machine tools, rollers of reversing mills, as well as, especially, industrial automata and robots (see Hejmo, 1990; Slotine and Lee, 1991). The control yielding minimum operation time directly influences the efficiency of such plants. Other examples are various security and failure devices: the shortest possible time of reaction is a basic element of their reliability. An optimal control task usually consists, then, in time- -optimal stabilization, i.e. reaching the equilibrium state $\dot{y}(t) = y(t) = 0$ in a minimal and finite time.

In many practical problems, the form of the applied model of motion resistances has great influence on the complexity or even the feasibility of a successful analysis. The classical theory of dynamic optimization (see Athans and Falb, 1966), especially Pontriagin's maximum principle, allows a synthesis of the time-optimal control, but under the assumption that the function H is univalued and belongs to the class C^1, which due to the physical character of friction phenomena presents a considerable restriction of the usefulness of such models. In a very important case from a practical point of view, in which the function H includes only dependence on velocity (i.e. this function is of the form $F(\dot{y}(t))$, where F is univalued and C^1), the time-optimal control takes on the extreme values of the admissible set and has at most one discontinuity point.

These facts were also shown by Hejmo and Kloch (1981) for the case where the function F is piecewise continuous.

In the present paper, this thesis will be generalized to non-autonomous objects described by the differential inclusion

$$\ddot{y}(t) \in v(t)F(\dot{y}(t)) + u(t), \tag{2}$$

where v and F are real piecewise continuous functions, and additionally F can be multivalued in a finite number of points. Such a model can be applied during the design of modern time-optimal control structures, especially in the random approach, where the function v may be treated as the realization of a stochastic process (see Kulczycki, 1992; 1993). This concept will be considered with great care in Conclusions.

2. Theorem

Let T be an interval with nonempty interior and $G : \mathbf{R}^n \times T \to \mathcal{P}(\mathbf{R}^n)$, where $\mathcal{P}(B)$ denotes the set of subsets of B. A function $x : T \to \mathbf{R}^n$ is a solution in the Caratheodory sense (C-solution) of the differential inclusion

$$\dot{x}(t) \in G(x(t), t), \tag{3}$$

if it is absolutely continuous on every compact subinterval of T, and fulfills inclusion (3) almost everywhere in T.

THEOREM 2.1 *Assume:*

(A) $t_0 \in \mathbf{R}$ *and* $T = [t_0, \infty)$*;*

(B) $x_0 \in \mathbf{R}^2$ *represents the initial state as well as* $U_a = \{u : T \to [-1, 1]\}$ *a set of admissible controls;*

(C) $f : \mathbf{R} \to [-1, 1]$ *denotes a piecewise continuous function fulfilling locally a Lipschitz condition except points of discontinuity, and* $z \cdot f(z) \geq 0$ *for every* $z \in \mathbf{R}$*; as well as* $F : \mathbf{R} \to \mathcal{P}([-1, 1])$ *is such that*

$$F(z) = \begin{cases} f(z) & \text{if } z \neq z_i \\ F_i & \text{if } z = z_i \end{cases}, \tag{4}$$

where z_i *denotes any real number,* F_i *any real subset, and* $i = 1, 2, \ldots, k$*;*

(D) $v : T \to [v_-, v_+]$, *where* $[v_-, v_+] \subset (-1, 1)$, *is a piecewise continuous function;*

(E) a differential inclusion

$$\dot{x}_1(t) = x_2(t) \tag{5}$$

$$\dot{x}_2(t) \in u(t) - v(t)F(x_2(t)) \tag{6}$$

with an initial condition

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = x_0 \tag{7}$$

describes the dynamics of the system submitted to the control u.

Then, there exists a time-optimal control, which takes on the values $+1, -1$, *has at most one discontinuity point, and brings the state of the system to the origin of coordinates along a unique* C*-solution.*

Proof. In the following, the notation will be adopted whereby the lower indexes "1" and "2" denote the coordinate of the point in $\mathbf{R}^2$, e.g. $x = [x_1, x_2]^T$, or the component of a function taking on values in $\mathbf{R}^2$, e.g. $x_+(t) = [x_{+1}(t), x_{+2}(t)]^T$.

Suppose that x_{++} and x_{+-} are unique C-solutions of system (5)-(6) with the terminal condition $x(0) = [0, 0]^T$, defined on the interval $(-\infty, 0]$, and generated by the control $u \equiv +1$, when $v \equiv v_+$ or $v \equiv v_-$, respectively.

Formula (6) implies

$$\dot{x}_{++2}(t) \geq 1 - v_+ > 0 \text{ for } t \in (-\infty, 0]. \tag{8}$$

Let:

$$K_{++} = \{[x_1, x_2]^T \in x_{++}(t) \text{ for } t \in (-\infty, 0)\} \tag{9}$$

$$K_{+-} = \{[x_1, x_2]^T \in x_{+-}(t) \text{ for } t \in (-\infty, 0)\}. \tag{10}$$

These are the sets of all states which can be brought to the origin by the control $u \equiv +1$, when $v \equiv v_+$ or $v \equiv v_-$, respectively.

Consider the function $k_{++} : (-\infty, 0] \to \mathbf{R}$ defined by

$$x_1 = k_{++}(x_2) \text{ if } [x_1, x_2]^T \in (K_{++} \cup \{[0, 0]^T\}). \tag{11}$$

Inequality (8) ensures the existence of the function x_{++2}^{-1}, so k_{++} is well defined, because it constitutes the composition $x_{++1} \circ x_{++2}^{-1}$. Moreover, x_{++2} as a continuous function, and also thanks to (8) being open (i.e. maps open sets to open sets) and invertible, is a homeomorphism; so, as a composition of continuous functions, k_{++} is continuous. Due to inequality (8) it is also piecewise C^1, because in the continuity and univalence areas of the functions F and v, x_{++1} is of class C^1, while x_{++2} as a regular function (i.e. class C^1 and with the Jacobian different from zero in every point of domain) and invertible, is a diffeomorphism.

Let the function $k_{+-} : (-\infty, 0] \to \mathbf{R}$ be given analogously. It is well defined, continuous, and piecewise C^1, too.

From dependencies (5) and (6) the following can be received, thanks to inequality (8):

$$\frac{dk_{++}(x_2)}{dx_2} = \frac{x_2}{1 - v_+ F(x_2)} \text{ almost everywhere in } (-\infty, 0] \tag{12}$$

and similarly

$$\frac{dk_{+-}(x_2)}{dx_2} = \frac{x_2}{1 - v_- F(x_2)} \text{ almost everywhere in } (-\infty, 0]. \tag{13}$$

Equations (12), (13) and the assumption $z \cdot f(z) \geq 0$ yield

$$\frac{dk_{+-}(x_2)}{dx_2} \leq \frac{dk_{++}(x_2)}{dx_2} < 0 \text{ almost everywhere in } (-\infty, 0], \tag{14}$$

which in connection with the equality $k_{+-}(0) = 0 = k_{++}(0)$ implies

$$0 \leq k_{++}(x_2) \leq k_{+-}(x_2) \text{ for } x_2 \in (-\infty, 0]. \tag{15}$$

Now, denote by x_{--} and x_{-+} unique C-solutions of system (5)-(6) with the condition $x(0) = [0,0]^T$, defined on $(-\infty, 0]$, and generated by the control $u \equiv -1$, when $v \equiv v_-$ or $v \equiv v_+$, respectively. Let sets K_{--} and K_{-+}, as well as functions k_{--} and $k_{-+} : [0, \infty) \to \mathbf{R}$, respectively, be defined similarly to the ones above. The dependence

$$k_{--}(x_2) \leq k_{-+}(x_2) \leq 0 \text{ for } x_2 \in [0, \infty), \tag{16}$$

analogous to inequality (15), is thus true.

Finally, define the following sets:

$$Q_+ = \{[x_1, x_2]^T \in \mathbf{R}^2 \text{ such that there exist } [x_1', x_2]^T \in K_{++} \text{ and } [x_1'', x_2]^T \in K_{+-} \text{ with } x_1' \leq x_1 \leq x_1''\} \tag{17}$$

$$Q_- = \{[x_1, x_2]^T \in \mathbf{R}^2 \text{ such that there exist } [x_1', x_2]^T \in K_{--} \text{ and } [x_1'', x_2]^T \in K_{-+} \text{ with } x_1' \leq x_1 \leq x_1''\} \tag{18}$$

$$R_+ = \{[x_1, x_2]^T \in \mathbf{R}^2 \setminus Q \text{ such that there exists } [x_1', x_2]^T \in Q \text{ with } x_1 < x_1'\} \tag{19}$$

$$R_- = \{[x_1, x_2]^T \in \mathbf{R}^2 \setminus Q \text{ such that there exists } [x_1', x_2]^T \in Q \text{ with } x_1' < x_1\}, \tag{20}$$

where $Q = Q_+ \cup \{[0,0]^T\} \cup Q_-$. By virtue of this, the state space has been subdivided into disjoint, non-empty sets: $\{[0,0]^T\}$, Q_+, Q_-, R_+, and R_- (Fig. 1).

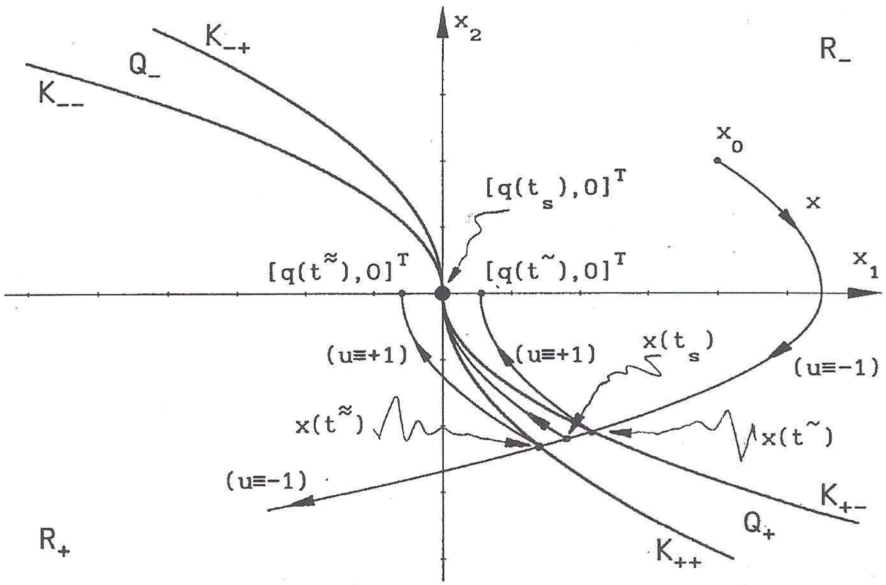

Figure 1. Subdivision of state space into sets $\{[0,0]^T\}$, Q_+, Q_-, R_+, R_- and illustration of proof in case $x_0 \in R_-$.

(In the rest of the proof, v is the fixed function given in point (D) of the assumptions.)

Let $x_0 \in R_-$ (Fig. 1). First, the C-solution x of system (5)-(7), defined on $[t_0, \infty)$, and generated by the control $u \equiv -1$, will be considered.

Similarly to the proof of inequality (15), it can be shown that this C-solution crosses the set K_{+-} in the finite time marked with $t^{\sim}$, and also the set K_{++} in the finite time $t^{\approx}$, with $t^{\sim} \leq t^{\approx}$.

At this point, a change in the control value will be introduced. Denote $t' \in [t^{\sim}, t^{\approx}]$ defined as the time of the control change from the previous value -1 to the new one $+1$. The dependencies $x_2(t') < 0$ and

$$\dot{x}_2(t) \geq 1 - v_+ > 0 \text{ if } u \equiv +1, \tag{21}$$

true by formula (6), yield that the C-solution x being considered now, with the change of the control value, crosses the x_1-axis in the finite time t''.

Suppose the function $q : [t^{\sim}, t^{\approx}] \to \mathbf{R}$ expressed as $q(t') = x_1(t'')$, i.e. assigning the coordinate of the x_1-axis crossing point along the C-solution x to the time of change of the control value. That function is continuous by the form of equation (5) and the continuity of an integral with parameter. And now, considering the boundary conditions of the function q, similarly to formula (15), it can be proved that $q(t^{\approx}) \leq 0 \leq q(t^{\sim})$. Because a continuous function defined on a connected set takes on all intermediate values, there exists such $t_s \in [t^{\sim}, t^{\approx}]$ that $q(t_s) = 0$, or $x(t'') = [0, 0]^T$, and then $t_f = t''$ is the finite time of reaching the origin along the C-solution x.

To summarize, if $x_0 \in R_-$, then there exists t_s such that the C-solution generated by the control

$$u^o(t) = \begin{cases} -1 & \text{for } t \in [t_0, t_s) \\ +1 & \text{for } t \in [t_s, \infty) \end{cases} \tag{22}$$

reaches the origin in the finite time t_f, with $t_0 < t_s < t_f$ and $x(t_s) \in Q_+$ (Fig. 1).

It can be proved analogously that if $x_0 \in R_+$, there exists t_s such that the C-solution generated by the control

$$u^o(t) = \begin{cases} +1 & \text{for } t \in [t_0, t_s) \\ -1 & \text{for } t \in [t_s, \infty) \end{cases} \tag{23}$$

reaches the origin in the finite time t_f, with $t_0 < t_s < t_f$ and $x(t_s) \in Q_-$.

Consider now $x_0 \in Q_+$. The C-solution x generated by the control $u \equiv +1$ reaches the x_1-axis in the finite time t''. If $x_1(t'') = 0$, or $x(t'') = [0, 0]^T$, the desired control is as follows:

$$u^o(t) = +1 \text{ for } t \in [t_0, \infty), \tag{24}$$

and then $t_f = t''$ constitutes a finite time for this C-solution to reach the origin. However, if $0 < x_1(t'')$, a consideration analogous to the case $x_0 \in R_-$ can be carried out, proving the existence of the control (22), while the role of the point $x(t^{\sim})$ is taken by x_0. If finally $x_1(t'') < 0$, then $x(t'') \in R_+$, so, by prolongation of the control of +1 value, the consideration proper for the case $x_0 \in R_+$ can be continued for $t \geq t''$, giving an adequate control of the form (23).

The case $x_0 \in Q_-$ can be considered similarly. Then, the counterpart of the control (24) is

$$u^o(t) = -1 \text{ for } t \in [t_0, \infty). \tag{25}$$

Hereby the control u^o of the form (22), (23), (24), or (25) has been assigned to every initial state $x_0 \in \mathbf{R}^2 \setminus \{[0,0]^T\}$. At this point, it will be proved that in every case this control is time-optimal (see also Hejmo and Kloch, 1981).

First, the initial state x_0 to which the control (24) was assigned is considered. The optimality of this control will be proved by contradiction. Therefore, assume the existence of the control $u^* \in U_a$ which brings the initial state under consideration along the C-solution x^* to the origin in the time t_f^* such that $t_f^* < t_f$.

It results from formula (6) that for the control (24) the function $\dot{x}_2$ is positive, so, the absolutely continuous function x_2 is strongly increasing; especially

$$x_2(t_f^*) < x_2(t_f) = 0. \tag{26}$$

However, the form of the set U_a implies

$$\begin{aligned} \dot{x}_2^*(t) &= f(x_2^*(t))v(t) + u^*(t) \leq f(x_2^*(t))v(t) + u^o(t), \\ &\quad \text{for } t \in [t_0, \infty). \end{aligned} \tag{27}$$

From this and from the dependence $x_2^*(t_0) = x_{02} = x_2(t_0)$, on the basis of differential inclusion (56) quoted in Appendix, the following is true:

$$x_2^*(t) \leq x_2(t) \text{ for } t \in [t_0, \infty), \tag{28}$$

or especially

$$0 = x_2^*(t_f^*) \leq x_2(t_f^*). \tag{29}$$

Inequalities (26) and (29) constitute a contradiction which proves the optimality of the control (24) for the initial state under consideration.

The initial state to which the control (25) was assigned can be analogously considered by proving the optimality of this control.

Now, the initial state to which the control (22) was assigned is considered. The optimality of this control will be also proved by contradiction, assuming - as previously - the existence of u^*, x^*, t^*, respectively.

First, the following inequalities are shown:

$$x_2(t) \leq x_2^*(t) \text{ for } t \in [t_0, \min(t_s, t_f^*)] \tag{30}$$

$$x_2(t) < x_2^*(t) \text{ for } t \in (t_s, t_f^*]. \tag{31}$$

Formula (30) is true on the basis of differential inequality (59), whose assumptions are fulfilled thanks to the condition $x_2^*(t_0) = x_{02} = x_2(t_0)$ and the form of the set U_a. For the needs of the proof of inequality (31) carried out through a contradiction, let $t_* \in (t_s, t_f^*]$ such that

$$x_2(t_*) \geq x_2^*(t_*) \tag{32}$$

exist. Formulas (6) and (22) imply that for $t \in [t_*, t_f^*]$ the function x_2 is strongly increasing; therefore

$$x_2(t_f^*) < x_2(t_f) = 0. \tag{33}$$

However, with reference to inequality (56), thanks to (32) and the form of the set U_a, the following is true:

$$x_2(t) \geq x_2^*(t) \text{ for } t \in [t_*, \infty), \tag{34}$$

or especially

$$x_2(t_f^*) \geq x_2^*(t_f^*) = 0. \tag{35}$$

Dependencies (33) and (35) are contradictory, which has finally proved the truthfulness of formulas (30) and (31).

It results from them directly that

$$x_2(t_f^*) \leq x_2^*(t_f^*) = 0. \tag{36}$$

Hence, because $x_2(t_f) = 0$ and the function x_2 is first strongly decreasing and later strongly increasing, one obtains:

$$x_2(t) < 0 \text{ for } t \in (t_f^*, t_f), \tag{37}$$

so, equation (5) yields

$$x_1(t_f^*) > x_1(t_f) = 0. \tag{38}$$

However, formulas (5), (30) and (31) imply

$$x_1(t) \leq x_1^*(t) \text{ for } t \in [t_0, t_f^*], \tag{39}$$

so, especially

$$x_1(t_f^*) \leq x_1^*(t_f^*) = 0, \tag{40}$$

which is contradictory to dependence (38). The optimality of the control (22) has therefore been proved.

Finally, for the initial state to which the control (23) was assigned, the proof of its optimality is analogous to the above.

This concludes the proof that the control u^o of the forms (22)-(25), assigned to any initial state $x_0 \in \mathbf{R}^2 \setminus \{[0,0]^T\}$, is time-optimal. By this Theorem 2.1 has been proved. ■

3. Conclusions and applications

The theorem shown in Section 2 provides a mathematical base for solving the time-optimal control problem investigated in this paper. The state space has been subdivided here into the sets R_-, R_+, Q_-, Q_+, and the origin, being a target (Fig. 1). The border lines are the sets K_{--}, K_{-+}, K_{+-}, K_{++}, defined constructively in the proof. Thus, if $x_0 \in R_-$, then the time-optimal control has the form of the sequence $(-1, +1)$, where the change of the value, i.e. switching of the control, occurs when the system state belongs to the set Q_+. Similarly if $x_0 \in R_+$, the sequence takes on the form $(+1, -1)$ and the switching exists when the state is included in the set Q_-. In the cases $x_0 \in Q_-$ and $x_0 \in Q_+$, both the above controls are possible and additionally also (-1) or $(+1)$, respectively. Because the switching of the control can appear only when the system state belongs to the closed area $Q = Q_+ \cup \{[0,0]^T\} \cup Q_-$, this set will be called a switching area. It constitutes the generalization of the switching curve γ, well known from the classical case (Section 7.2 of Athans and Falb, 1966), but $\gamma \subset Q$ only if $0 \in [v_-, v_+]$, which rarely occurs in practice. Namely, if in Theorem 2.1, $v_- = v_+$, implying that the function v is constant and the system is autonomous, then $K_{+-} = K_{++}$ and $K_{--} = K_{-+}$, therefore the switching area Q is reduced to a switching curve considered by Hejmo and Kloch (1981), which form is dependent on the above value $v_- = v_+$. Of course, the second additional condition $v_- = v_+ = 0$ implies that $Q = \gamma$.

The C-solutions occurring in the system according to Theorem 2.1 are of course K-solutions, i.e. in Krasovski sense (see Hajek, 1979), and thanks to Lemma 2.8 from Hajek (1979) also F-solutions, i.e. in Filippov sense (see Hajek, 1979); (in the proof of this part the following additional definition $F(z_i) = \lim_{z \to Z_1^+} f(z)$ for $i = 1, 2, \ldots, k$ should be accepted thanks to the obtained form of the time-optimal control). The requirement concerning the Lipschitz condition, assumed with respect to the function F, implies the uniqueness of K-solutions (Corollary 8.6 of Hajek, 1979), therefore also F-solutions. Finally, the time-optimal control elaborated in Theorem 2.1 generates unique and equal to each other solutions in the senses of Caratheodory, Filippov and Krasovski. This property is worth emphasizing, considering the substantial difficulties presented by the lack of a universal concept of a solution for differential equations and inclusions with discontinuous right-hand side.

Finally, an example of the application of the task considered here will be shown. Especially, the presented material will be adopted to the random concept of solving the time-optimal control problem.

Suppose that the function v is the realization of a given stochastic process V having almost all realizations continuous and bounded to the interval $[v_-, v_+]$. The random factor introduced by this process causes dynamic system (5)-(7) to take on the form of the following random differential inclusion:

$$\dot{X}_1(\omega, t) = X_2(\omega, t) \quad (41)$$

$$\dot{X}_2(\omega,t) \quad \in \quad U(\omega,t) - V(\omega,t)F(X_2(\omega,t)) \tag{42}$$

with the initial condition

$$\begin{bmatrix} X_1(\omega,t_0) \\ X_2(\omega,t_0) \end{bmatrix} = x_0 \text{ for almost all } \omega, \tag{43}$$

where X_1, X_2 and U are stochastic processes. Especially, it can be shown (see Kulczycki, 1992) that the family of the time-optimal control functions u^o obtained in Theorem 2.1 for particular realizations of V treated there as the functions v constitute also a stochastic process U^o, called an almost certain time-optimal control.

According to the above concept, the model of motion resistances, represented in formula (1) by the function H, has been decomposed into two factors: $F(\dot{y}(t))$ and $V(\omega,t)$. The former, a deterministic one, made it possible to incorporate the properties of discontinuity and multivalency of friction phenomena. The latter one, thanks to its probabilistic nature, includes among other things approximations and identification errors (of the first factor, too), the dependence of motion resistances on position, time and temperature, as well as perturbations and noise naturally occurring in real systems. (These elements are usually omitted in the approaches commonly applied, due to the necessity to simplify the model.) The result of Theorem 2.1 can now be easily interpreted that the switching curve which is implied by the first - deterministic - factor has been "blurred" by the second - random - one to the switching area.

The almost certain time-optimal control U^o obviously ensures realization of the minimum of expected value of the time to reach the target set; however, it depends on the random factor, in practice *a priori* unknown. Thus, the above control is difficult to apply directly, but constitutes a useful basis for the creation of technical constructions of suboptimal structures in which the direct dependency of the control function on the random factor is eliminated.

For example, in the case of open-loop systems, the expectation of the stochastic process U^o can be used in the construction of the suboptimal control. If the limits of the actuator accept only extreme values of the admissible controls set, it is possible to apply the control sequences $(-1,+1)$ or $(+1,-1)$, where the time of switching is the expectation of the sign changes in particular realizations of the stochastic process U^o. In both cases unique C-, F-, and K-solutions exist in the system.

Similarly to the classical case, the time-optimal control designed in this paper can be defined as a feedback controller by the following formula:

$$U^o(X(\omega,t)) = \begin{cases} +1 & \text{if } X(\omega,t) \in R_+ \\ -1 & \text{if } X(\omega,t) \in R_- \end{cases}, \tag{44}$$

and then for $X(\omega,t) \in Q_- \cup Q_+$ this function can be additionally defined, without direct dependence on the random factor, in a suboptimal way, as

$$U^s(X(\omega,t)) = \begin{cases} d & \text{if } X(\omega,t) \in Q_+ \\ -d & \text{if } X(\omega,t) \in Q_- \end{cases}, \tag{45}$$

where $0 < d \leq 1$. In practice the value of the parameter d can be obtained heuristically. Usually this value should be close to 1, but it can also vary in the area Q, taking on the value

$$d = 1 - v_+ + v_- \tag{46}$$

on the sets K_{+-}, K_{--}, and increasing continuously up to 1 on the sets K_{++}, K_{-+}. This makes it possible to achieve a result similar to the bicycle-racing track or bob-sleigh track, which are horizontal on the interior part, and become more vertical the farther they go to the outside. In particular, the value of the parameter d should be equal to 1 even in the neighborhood of the sets K_{++}, K_{-+}, due to neutralization of the most unfavorable realizations of the random factor. Unique F- and K-solutions occur in the system obtained. Condition (46) and the above postulated continuity of variation of the value of the parameter d have the goal of fulfilling also the existence of C-solutions.

Analogously to the first pair of examples, if constraints of an actuator limit the control to the extreme values of the admissible set, the results of Theorem 2.1 may be modified according to the physical observation that the influence of motion resistances in both periods of time - before and after the switching - can be averaged. Thus, after performing a rigorous analysis of the sensitivity of the control system to the values of motion resistances, one can use elements of statistical decision theory, where a loss function is connected with extending the time of reaching the target if the control switching has been too late or too early. A detailed description of such a concept of a feedback controller is presented by Kulczycki (1992; 1996). It is worth noticing that in the general case there are no C-solutions in the system obtained, whereas F- and K-solutions are nonunique.

Empirical examinations (see Kulczycki, 1992) have shown many advantages of the control system presented above, especially in the area of robustness.

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4. Appendix

We will show here Lemma 4.2, which was used in the proof of Theorem 2.1. For this purpose another auxiliary thesis will first be proved.

LEMMA 4.1 *Assume:*

(A) I denotes an interval with non-empty interior and $t_0^* \in I$;

(B) $E = [e_1, e_2]$ or $E = [-e_2, -e_1]$, where $0 < e_1 < e_2$;

(C) the function G fulfills one of the following conditions:

(a) $G : \mathbf{R} \times I \to \mathbf{R}$ is continuous,

(b) $G : \mathbf{R} \times I \to \mathcal{P}(E)$ takes on the form

$$G(y(t), t) = c - v(t)F(y(t)) \text{ for } y(t) \in \mathbf{R} \text{ and } t \in I, \tag{47}$$

where $c \in \mathbf{R}$, $v : I \to \mathbf{R}$ is a continuous function, while for the mapping $F : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ there exists a piecewise continuous function $f : \mathbf{R} \to \mathbf{R}$ such that

$$F(z) = \begin{cases} f(z) & \text{for } z \neq z_i \\ F_i & \text{for } z = z_i \end{cases}, \tag{48}$$

where z_i denotes any real number, F_i any real subset, and $i = 1, 2, \ldots, k$.

Then, a differential inclusion

$$\dot{y}(t) \in G(y(t), t) \tag{49}$$

with an initial condition

$$y(t_0^*) = y_0 \tag{50}$$

has a C-solution defined on the interval I.

Moreover, if

(D) the function F fulfills locally the Lipschitz condition except points of discontinuity and multivalency,

then that solution is unique.

Proof. In the case of assumption (C) for version (a), there exists a saturated classical solution, unique if requirement (D) is fulfilled (see Pelczar and Szarski, 1987; Hubbart and West, 1990). The above facts directly imply the existence of a C-solution, also unique under condition (D) (for details see Hajek, 1979).

For variant (C)-(b), however, an analogous line of reasoning can be followed in the areas of the simultaneous univalence and continuity of the function F, after which the obtained solutions may undergo "joining". It should be noted that, because of the inequalities $\dot{y}(t) \geq e_1 > 0$ or $\dot{y}(t) \leq -e_1 < 0$ for $t \in I$, resulting from the form of the set E, it suffices to consider only a finite number of such "joinings". ■

LEMMA 4.2 *Assume:*

(A) $t_0^* \in \mathbf{R}$ *and* $y_0 \in \mathbf{R}$;

(B) $I = [t_0^*, \infty)$ *or* $I = [t_0^*, t_*)$, *where* $t_0^* < t_*$;

(C) $E = [e_1, e_2]$ *or* $E = [-e_2, -e_1]$, *where* $0 < e_1 < e_2$;

(D) $G : \mathbf{R} \times I \to \mathcal{P}(E)$ *takes on the form*

$$G(y(t), t) = c - v(t)F(y(t)) \text{ for } y(t) \in \mathbf{R} \text{ and } t \in I, \tag{51}$$

where $c \in \mathbf{R}$, $v : I \to \mathbf{R}$ *is a piecewise continuous function, and* $F : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *denotes a piecewise continuous function which additionally can be multivalued in a finite number of points (i.e. in the sense of assumption (C)-(b) of Lemma 4.1);*

(E) $y : I \to \mathbf{R}$ *is a* C*-solution of a differential inclusion*

$$\dot{y}(t) \in G(y(t), t) \tag{52}$$

with an initial condition

$$y(t_0^*) = y_0; \tag{53}$$

(F) $z : I \to \mathbf{R}$ *denotes any fixed absolutely continuous function.*

If the following conditions are fulfilled

$$z(t_0^*) \leq y(t_0^*) \tag{54}$$

$$\dot{z}(t) \leq \inf G(z(t), t) \text{ almost everywhere in } I, \tag{55}$$

then

$$z(t) \leq y(t) \text{ for } t \in I. \tag{56}$$

However, if

$$z(t_0^*) \geq y(t_0^*) \tag{57}$$

$$\dot{z}(t) \geq \sup G(z(t), t) \text{ almost everywhere in } I, \tag{58}$$

then

$$z(t) \geq y(t) \text{ for } t \in I. \tag{59}$$

Proof. (See also Hejmo and Kloch, 1991). The existence of the solution y occurring in assumption (E) is guaranteed by Lemma 4.1.

The proof will be shown for equality (56). The proof of dependence (59) is analogous.

Because of the form of the interval E, the set of the points $t \in I$ of multivalency or discontinuity of the function F is finite. It is thus sufficient to prove the thesis under the assumption of the simultaneous univalence and continuity of that function, since after the "joining" of the relevant solutions the considered inequality will remain true.

Let a differential inclusion

$$\dot{p}(t) \in \begin{cases} G(p(t), t) & \text{if } p(t) \geq z(t) \\ G(z(t), t) & \text{if } p(t) \leq z(t) \end{cases} \tag{60}$$

with an initial condition

$$p(t_0^*) = y_0 \tag{61}$$

be given. The existence of the C-solution of that inclusion results from Lemma 4.1.

It will first be shown that

$$p(t) \geq z(t) \text{ for } t \in I. \tag{62}$$

For the contradiction of this inequality, let there exist $t^{\approx} \in I$ such that

$$p(t^{\approx}) < z(t^{\approx}). \tag{63}$$

The continuity of the functions z and p implies, on the basis of dependencies (54) and (61), the existence of such $t^{\sim} \in [t_0^*, t^{\approx}]$ that

$$p(t^{\sim}) = z(t^{\sim}) \tag{64}$$

$$p(t) < z(t) \text{ for } t \in (t^{\sim}, t^{\approx}]. \tag{65}$$

In turn, formulas (52) and (60) together with (65) yield

$$\dot{p}(t) - \dot{z}(t) \geq G(z(t), t) - G(z(t), t) = 0 \text{ almost everywhere in } [t^{\sim}, t^{\approx}]; \tag{66}$$

therefore, thanks to equality (64), one obtains

$$p(t) \geq z(t) \text{ for } t \in [t^{\sim}, t^{\approx}], \tag{67}$$

which is in contradiction to hypothesis (63). Inequality (62) has thus been shown.

From dependencies (60)-(62) and the uniqueness of the C-solutions of inclusion (52)-(53), guaranteed by 4.1, it results that $p \equiv y$, which, thanks to condition (62), finally proves inequality (56). ■