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Higher order primal-dual interior point method for separable convex quadratic optimization[1]

by

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Abstract: This paper extends a higher order primal-dual method for linear programming to the case of separable convex quadratic optimization with linear constraints. The equivalents of separable objectives are considered. Computational results for modified Netlib problems are provided.
Keywords: Quadratic convex programming, interior point methods, higher order logarithmic barrier primal-dual method.

1. Introduction

The higher order primal-dual interior point method for linear programming (LP) was originally suggested by Mehrotra (1991). This method uses a higher order Taylor polynomial to approximate a primal-dual trajectory defined from an infeasible point. Mehrotra's method for LP has been efficiently implemented by many authors including Altman and Gondzio (1993).

Following other authors of efficient interior point implementations, see Carpenter et al. (1993) or Vanderbei (1992), we extended our software to handle separable convex quadratic programming.

The primary aim of this extension was to solve a particular class of the cost effective sulphur emission reduction problems (Altman et al. (1994)). A success with this modification led to creating a fairly general quadratic programming routine.

In the first section we develop, in particular, the general higher order primal-dual algorithm for convex quadratic problems. We give the theorem of existence and uniqueness of solution for quadratic problems, analogous as for linear ones.

In the second section we show how the method can be applied to problems with symmetric semi-definite quadratic matrix for which a factorization $F^T F$ is known.

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In the third section we describe a collection of test problems. Our problems are derived from the linear programs of the NETLIB suite. Numerical results of running the method on them are given.

Finally, we summarize our results and give the conclusions.

2. Higher order primal-dual method for quadratic programming

Let us consider the convex quadratic problem

$$\begin{aligned} &\text{minimize} && c^T x + \frac{1}{2} x^T Q x, \\ &\text{subject to} && Ax + s = b, \\ &&& x + s = u, \\ &&& x, s \geq 0, \end{aligned} \tag{1}$$

where $c, x, s, u \in \mathcal{R}^n, b \in \mathcal{R}^m, A \in \mathcal{R}^{m \cdot n}$ is presumed to have full row rank. The matrix $Q \in \mathcal{R}^{n \cdot n}$ is positive semi-definite. The dual of (1) is given by

$$\begin{aligned} &\text{maximize} && b^T y - u^T w - \frac{1}{2} x^T Q x, \\ &\text{subject to} && A^T y + z - w - Qx = c, \\ &&& x, z, w \geq 0, \end{aligned} \tag{2}$$

where $z, w \in \mathcal{R}^n, y \in \mathcal{R}^m$.

From the optimality conditions for convex quadratic programming, a point (x, s, y, z, w) is optimal if it satisfies:

$$\begin{aligned} Ax &= b, \\ x + s &= u, \\ A^T y + z - w - Qx &= c, \\ Xz &= 0, \\ Sw &= 0, \\ x, s, z, w &\geq 0, \end{aligned} \tag{3}$$

where X, S, Z and W are diagonal matrices with the diagonal elements x_j, s_j, z_j and w_j, respectively and $e \in \mathcal{R}^n$ is the vector of ones.

We shall apply the logarithmic barrier method (Fiacco and McCormick (1968)) to the solution of (1) - (2). We augment the objective by adding a logarithmic barrier term to it, which yields

$$\begin{aligned} &\text{minimize} && c^T x + \frac{1}{2} x^T Q x - \mu \sum_{j=1}^{n} (\ln x_j + \ln s_j), \\ &\text{subject to} && Ax + s = b, \end{aligned} \tag{4}$$

$$x + s = u,$$
$$x, s > 0$$

and an analogue of (2):

$$\begin{aligned}
\text{maximize} \quad & b^T y - u^T w - \frac{1}{2}x^T Qx + \mu \sum_{j=1}^{n} (\ln z_j + \ln w_j), \\
\text{subject to} \quad & A^T y + z - w - Qx = c, \\
& x \geq 0, z, w > 0.
\end{aligned} \tag{5}$$

The first order optimality conditions for (4) and (5) are

$$\begin{aligned}
Ax &= b, \\
x + s &= u, \\
A^T y + z - w - Qx &= c, \\
Xz &= \mu e, \\
Sw &= \mu e, \\
x, s, z, w &> 0.
\end{aligned} \tag{6}$$

For solving (6) we use a quadratic analogue of Mehrotra's higher order method (Mehrotra (1991)). This method computes the Taylor approximation of the optimal trajectory which starts at a given point and leads to the optimum of (1) and (2).

Let $x^{(1)}, s^{(1)}, z^{(1)}, w^{(1)} > 0$ and $y^{(1)}$ be the current estimate of the solution of (1) and (2). Then

$$\begin{aligned}
\xi_b &= Ax^{(1)} - b, \\
\xi_u &= x^{(1)} + s^{(1)} - u, \\
\text{and } \xi_c &= A^T y^{(1)} + z^{(1)} - w^{(1)} - Qx^{(1)} - c,
\end{aligned}$$

are the resulting residuals in the primal and dual constraints. Next, we consider the parametric system of equations

$$\begin{aligned}
Ax(\alpha) &= b + g(\alpha)\xi_b, \\
x(\alpha) + s(\alpha) &= u + g(\alpha)\xi_u, \\
A^T y(\alpha) + z(\alpha) - w(\alpha) - Qx(\alpha) &= c + g(\alpha)\xi_c, \\
X(\alpha)z(\alpha) &= g(\alpha)X^{(1)}z^{(1)} + f(\alpha)\mu e, \\
S(\alpha)w(\alpha) &= g(\alpha)S^{(1)}w^{(1)} + f(\alpha)\mu e, \\
x(\alpha), s(\alpha), z(\alpha), w(\alpha) &> 0,
\end{aligned} \tag{7}$$

where f and g are nonnegative functions determined on interval $[0, 1]$ such that $g(0) = 0, g(1) = 1$ and $g(\alpha) \in (0, 1)$ for $\alpha \in (0, 1)$. The definition of $f(\alpha)$ depends on whether the problems (1) and (2) have the interior feasible solutions or not. If a problem is known to have an interior feasible solution, we require $f(1) = 0$ and $f(0) = 1$. Then, for $\alpha = 0$, the problems (6) and (7) are identical. If we do not assume that a problem has primal and dual interior feasible solution, the requirement for function f are $f(0) = f(1) = 0$. Then for, $\alpha = 0$, the systems (3) and (7) are identical.

Let $\Gamma(\alpha) = (x(\alpha), s(\alpha), y(\alpha), z(\alpha), w(\alpha))$ be the solution of (7) for a given parameter α. We let $\Gamma(1) = (x(1), s(1), y(1), z(1), w(1)) = (x^{(1)}, s^{(1)}, y^{(1)}, z^{(1)}, w^{(1)})$, and so (7) is satisfied for $\alpha = 1$. We can construct trajectories that guide us from the current point $\Gamma(1)$ to a solution of (3) or (6) - $\Gamma(0)$.

We can generalize the Theorem 1.1 of Mehrotra (1991) for quadratic case.

THEOREM 2.1 *If the system (3) has a solution, then (7) has a solution for all $\alpha \in [0, 1]$. For any $\alpha \in (0, 1]$ the solution is unique.*

We omit here the proof, because it is done in the same way as for the linear case (see Mehrotra (1991) and Mehrotra (1992)).

The key point of Mehrotra's approach is to use local higher order information available at point $\Gamma(1)$ to construct a direction that approximates well the first point of trajectory Γ. In our implementation $f(\alpha) = \alpha(1-\alpha)^2$ and $g(\alpha) = \alpha$, which refers to Mehrotra's Algorithm II.

Since $\Gamma(\alpha)$ is a solution of (7) for a given α, the appropriate higher order terms of Taylor polynomial approximation of correction $(\Delta x, \Delta s, \Delta y, \Delta z, \Delta w)$ to the current estimate (x, s, y, z, w) result from the recursive differentiation of (7). The i-th order term of the correction vector can be obtained from

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ -Q & A^T & 0 & I & -I \\ I & 0 & I & 0 & 0 \\ Z & 0 & 0 & X & 0 \\ 0 & 0 & W & 0 & S \end{bmatrix} \begin{bmatrix} \Delta x^{(i)} \\ \Delta y^{(i)} \\ \Delta s^{(i)} \\ \Delta z^{(i)} \\ \Delta w^{(i)} \end{bmatrix} = \begin{bmatrix} \eta_1^{(i)} \\ \eta_2^{(i)} \\ \eta_3^{(i)} \\ \eta_4^{(i)} \\ \eta_5^{(i)} \end{bmatrix}, \tag{8}$$

where

$$\Delta x^{(i)} = \frac{x^{(i)}}{i!},$$
$$\Delta s^{(i)} = \frac{s^{(i)}}{i!},$$
$$\Delta y^{(i)} = \frac{y^{(i)}}{i!},$$
$$\Delta z^{(i)} = \frac{z^{(i)}}{i!},$$
$$\Delta w^{(i)} = \frac{w^{(i)}}{i!}$$

and

$$\begin{aligned}
\eta_1^{(i)} &= \frac{1}{i!} g^{(i)}(1)\xi_b, \\
\eta_2^{(i)} &= \frac{1}{i!} g^{(i)}(1)\xi_g, \\
\eta_3^{(i)} &= \frac{1}{i!} g^{(i)}(1)\xi_u, \\
\eta_4^{(i)} &= \frac{1}{i!}[g^{(i)}(1)X^{(1)}z^{(1)} + f^{(i)}(1)\mu e] - \sum_{l=1}^{i-1} \Delta x^{(l)} \Delta z^{(i-l)}, \\
\eta_5^{(i)} &= \frac{1}{i!}[g^{(i)}(1)S^{(1)}w^{(1)} + f^{(i)}(1)\mu e] - \sum_{l=1}^{i-1} \Delta S^{(l)} \Delta w^{(i-l)}.
\end{aligned} \tag{9}$$

For every i, the matrix involved in the linear systems (8) is the same, so its factorization is to be computed only once. For the linear case we can compute, see Altman and Gondzio (1993), the search directions in the primal and dual spaces as

$$\begin{aligned}
d_x &= -\sum_{l=1}^{l_p} (-\alpha_p)^l \Delta x^{(l)}(1), \\
d_s &= -\sum_{l=1}^{l_p} (-\alpha_p)^l \Delta s^{(l)}(1), \\
d_y &= -\sum_{l=1}^{l_d} (-\alpha_d)^l \Delta y^{(l)}(1), \\
d_z &= -\sum_{l=1}^{l_d} (-\alpha_d)^l \Delta z^{(l)}(1), \\
d_w &= -\sum_{l=1}^{l_d} (-\alpha_d)^l \Delta w^{(l)}(1),
\end{aligned} \tag{10}$$

where l_p and l_d are orders of Taylor polynomials in the primal and dual spaces, respectively. The parameters α_p and α_d in (10) are the largest numbers in $[0,1]$ for which

$$\begin{aligned}
x - d_x &\geq 0, \\
s - d_s &\geq 0, \\
z - d_z &\geq 0, \\
w - d_w &\geq 0.
\end{aligned}$$

Unfortunately, for quadratic problems the use of (10) can cause a loss of the dual feasibility. Note, that even if we have a feasible solution (x, s, y, z, w), the new point $(x - d_x, s - d_s, y - d_y, z - d_z, w - d_w)$ need not be dual feasible (it is always the case in linear programming). This is because in quadratic programming, the primal variable x appears also in dual constraints, hence

$$A^T(y - d_y) + (z - d_z) - (w - d_w) - Q(x - d_x) = c + Qd_x.$$

To overcome this disadvantage, we could follow Vanderbei and Carpenter (1993) and use $\alpha_p = \alpha_d$ and $l_p = l_d$ for primal and dual spaces. Unfortunately, it would imply $\alpha = \min(\alpha_p, \alpha_d)$ and $l = \min(l_p, l_d)$ and could considerably slow the method down.

In our computations we decided to use different α and l for the primal and the dual spaces, similarly to the linear case. Our experiments show that with formulas (10) the algorithm works much faster.

After computing the search directions we define step factors f_p and f_d as in Mehrotra (1991) and we define new approximations of the optimal point

$$\begin{aligned} x &:= x - f_p d_x, \\ s &:= s - f_p d_s, \\ y &:= y - f_d d_y, \\ z &:= z - f_d d_z, \\ w &:= w - f_d d_w. \end{aligned}$$

Elimination of $\Delta s^{(i)}, \Delta z^{(i)}$ and $\Delta w^{(i)}$ reduces (8) to

$$H \cdot \begin{bmatrix} \Delta x^{(i)} \\ \Delta y^{(i)} \end{bmatrix} = \begin{bmatrix} -\Theta^{-1} & A^T \\ A & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x^{(i)} \\ \Delta y^{(i)} \end{bmatrix} = \begin{bmatrix} h^{(i)} \\ \eta_1^{(i)} \end{bmatrix}, \tag{11}$$

where

$$h^{(i)} = \eta_2^{(i)} - (X^{(1)})^{-1}\eta_4^{(i)} + (S^{(1)})^{-1}(\eta_5^{(i)} - W^{(1)}\eta_3^{(i)})$$

and

$$\Theta = (Q + (X^{(1)})^{-1}Z^{(1)} + (S^{(1)})^{-1}W^{(1)})^{-1}, \tag{12}$$

where $\eta_1^{(i)}, \eta_2^{(i)}, \eta_3^{(i)}, \eta_4^{(i)}, \eta_5^{(i)}$ are defined by (9).

Further, we reduce (11) to the normal equation form

$$(A\Theta A^T)\Delta y^{(i)} = A\Theta h^{(i)},$$

and we compute sparse Cholesky factorization of the positive definite matrix $A\Theta A^T$.

If the matrix Q were diagonal, then all terms in (12) would be diagonal making the computation of Θ particularly easy. Furthermore the sparsity pattern for $A\Theta A^T$ would be the same for every Θ and we could use the same factorization techniques as in the linear programming case, see Duff et al. (1986), Gondzio (1993). Additionally, we could also use the same techniques for finding the starting point.

In the next Section, we shall show that a wide class of practical quadratic programming problems can be transformed to such a desirable form.

3. Separable equivalents

We explained in Section 2. (c.f. (12)) the advantages of having a diagonal matrix Q. Clearly, in general, a matrix Q does not have to be diagonal. However, a wide class of quadratic problems can be transformed to separable tasks. This is true, in particular, for such matrices Q that can be factorized to the form $Q = F^T F$.

We assume further in this section that we know such a factorization of matrix Q. Under this assumption we can rewrite (1) in another form

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x, \\ \text{subject to} & Ax = b, \\ & Q = F^T F, \\ & x + s = u, \\ & x, s \geq 0. \end{array}$$

The above system is further transformed to the following form

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} v^T v, \\ \text{subject to} & Ax = b, \\ & Fx = v, \\ & x + s = u, \\ & x, s \geq 0. \end{array} \tag{13}$$

The variable v, obtained in that way, is free. We exploit standard simplex device (used also widely for interior point method) of representing free variables as a difference of two nonnegative variables. That is, we split free variable v into $v = v_+ - v_-$, where v_+ and v_- are both nonnegative variables. If the free variables are expressed as a difference of nonnegative variables and substituted directly into (13), the resulting equivalent quadratic program is nonseparable. Our goal is to be able to solve (13) via a separable quadratic program in nonnegative variables. This is proved, c. f. Carpenter et al. (1993) that an optimal solution to (13) is as an optimal solution to

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} v_+^T v_+ + \frac{1}{2} v_-^T v_-, \\ \text{subject to} & Ax = b, \\ & Fx = v_+ - v_-, \end{array} \tag{14}$$

$$x + s = u,$$
$$x, s \geq 0.$$

To simplify the formulas we define a new variable

$$\tilde{x} = \begin{bmatrix} x \\ v_+ \\ v_- \end{bmatrix},$$

matrices

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ -F & I & -I \end{bmatrix} \text{ and } \tilde{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

and vectors

$$\tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} \text{ and } \tilde{c} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}.$$

Finally, our problem can be written as a simple separable quadratic problem, i.e., as a problem with a diagonal quadratic matrix

$$\begin{aligned} \text{minimize} \quad & \tilde{c}^T\tilde{x} + \frac{1}{2}\tilde{x}^T\tilde{I}\tilde{x}, \\ \text{subject to} \quad & \tilde{A}\tilde{x} = \tilde{b}, \\ & x + s = u, \\ & \tilde{x}, s \geq 0. \end{aligned}$$

4. Numerical results

4.1. Test problem description

Unfortunately, we are not aware of any publicly available set of positive semidefinite quadratic test programs. We follow Vanderbei and Carpenter (1993) and employ the NETLIB (Gay (1985)) set of linear programs to generate quadratic test examples. We create objective terms as a product based on a subset of the rows of constraint matrix A. We let F denote the 0-1 mask of the first 5% rows of A; an element of F is set to 1 whenever the corresponding element of A is nonzero and 0 otherwise. Next we define $Q = F^TF$, the quadratic objective term. Formed that way, Q is typically of low rank, but it may be quite dense.

For our purposes we modified the MPS input file. In ROWS section we added new lines indicated by letter "F", which means that these rows create quadratic part of the objective, i.e. $Q = F^TF$. Other sections are written in standard MPS format. We obviously add some new lines in COLUMNS section associated with "F" rows, but they are written in the same way as others. The rest of the MPS file remains unaltered. We call files written in this way QPS format files. We add a small program which generate a QPS file from a given MPS file by adding a given number of "F" rows in the way described above. Names of these new added rows are changed.

4.2. Computational results

The method presented in this paper has been incorporated into HOPDM code and resulted in creating its quadratic variant QHOPDM. The program is written in standard FORTRAN 77, which ensures portability. To date, the QHOPDM library was installed on IBM PC/AT and SUN SPARCstation.

We chose for test problems second order of approximation. QP codes iterate until the relative duality gap is reduced to a predetermined optimality tolerance 10^{-6}. Table 1 collects problems statistics: number of rows (M), columns (N), "F"-type rows (F) and results of their solution with the QHOPDM code: number of iterations (iters), objective value (obj), linear part of objective (Lobj) and quadratic part of objective (Qobj).

5. Conclusions

The numerical results demonstrate that higher order primal-dual interior point method is viable for solving separable quadratic problems. Both methods for linear and quadratic optimization are strictly related, hence every refinement in LP code gives better results in QP.

Unfortunately, this is not the case for nonseparable quadratic programming. A further goal of our work on interior point method for quadratic programming is then to develop an efficient method for solving nonseparable problems whose Hessian matrix yields dense Θ matrices.

Problem	M	N	F	iters	obj	Lobj	Qobj
AFIRO	27	51	1	10	-8.75e+00	-1.75e+01	8.75e+00
ADLITTLE	56	138	3	11	3.45e+05	2.33e+05	1.11e+05
SCSD1	77	760	4	9	8.67e+00	8.67e+00	4.35e-06
RECIPE	91	204	5	12	-2.67e+02	-2.67e+02	8.62e-05
SHARE2B	96	162	5	19	2.72e+03	-3.63e+02	3.08e+03
SHARE1B	117	253	6	24	8.92e+04	-7.66e+04	1.66e+05
SCAGR7	129	185	6	14	2.95e+06	-2.16e+06	5.10e+06
GROW7	140	301	7	16	-4.28e+07	-4.28e+07	2.05e+02
SCSD6	147	1350	7	12	5.06e+01	5.05e+01	6.25e-02
FORPLAN	161	466	8	31	2.56e+09	-5.93e+01	2.56e+09
BEACONFD	173	295	9	14	4.71e+04	3.40e+04	1.31e+04
ISRAEL	174	326	9	29	5.74e+06	-1.47e+05	5.89e+06
SC205	205	316	10	12	-3.57e-02	-7.14e-02	3.57e-02
BRANDY	220	292	11	21	9.61e+03	3.69e+03	5.92e+03
E226	223	472	11	22	1.06e+02	-1.85e+01	1.24e+02
BORE3D	233	334	12	20	1.58e+03	1.37e+03	2.07e+02
CAPRI	271	496	14	54	1.26e+07	3.02e+03	1.26e+07
GROW15	300	645	15	17	-1.02e+08	-1.02e+08	9.17e+01
SCTAP1	300	660	15	21	1.41e+03	1.41e+03	3.61e-01
BANDM	305	472	15	17	2.60e+03	-1.58e+02	2.76e+03
SCFXM1	330	600	17	35	8.83e+06	1.88e+04	8.81e+06
STAIR	356	620	18	48	3.19e+06	-2.09e+02	3.19e+06
STANDATA	359	1273	18	17	2.76e+03	1.26e+03	1.50e+03
SCORPION	388	466	19	11	1.88e+03	1.88e+03	4.06e-01
SCSD8	397	2750	20	11	9.12e+02	9.05e+02	7.16e+00

Table 1. Problem statistics and numerical results (a)

Problem	M	N	F	iters	obj	Lobj	Qobj
ETAMACRO	400	816	20	26	2.79e+04	-7.40e+02	2.86e+04
SHIP04S	402	1506	20	13	2.00e+06	1.81e+06	1.95e+05
SHIP04L	402	2166	20	14	2.00e+06	1.80e+06	1.95e+05
GROW22	440	946	22	22	-1.50e+08	-1.50e+08	8.17e+02
STANDMPS	467	1273	23	20	3.14e+03	1.41e+03	1.73e+03
SCAGR25	471	671	23	16	2.91e+07	-1.40e+07	4.31e+07
SCRS8	490	1275	24	23	9.04e+02	9.04e+02	2.65e-02
SEBA	515	1091	26	22	1.53e+07	2.23e+04	1.53e+07
FFFFF800	524	1029	26	46	6.17e+05	5.74e+05	4.28e+04
SHELL	536	1777	27	36	4.22e+11	1.23e+09	4.21e+11
GFRDPNC	616	1160	31	17	1.96e+10	7.61e+06	1.96e+10
SCFXM2	660	1200	33	37	1.47e+07	3.92e+04	1.47e+07
SHIP08S	778	2467	39	15	2.14e+06	1.93e+06	2.09e+05
SHIP08L	778	4363	39	15	2.13e+06	1.92e+06	2.10e+05
25FV47	821	1876	41	35	6.08e+06	1.09e+04	6.07e+06
PILOTNOV	975	2446	49	102	6.59e+05	-4.35e+03	6.64e+05
SCFXM3	990	1800	50	37	1.53e+07	5.75e+04	1.52e+07
SCTAP2	1090	2500	51	15	1.73e+03	1.72e+03	1.22e+00
SHIP12S	1151	2869	58	17	2.21e+06	1.49e+06	7.19e+05
SHIP12L	1151	5533	58	19	2.19e+06	1.47e+06	7.15e+05
SIERRA	1227	2735	61	22	1.72e+07	1.60e+07	1.24e+06
SCTAP3	1480	3340	74	16	1.43e+03	1.42e+03	1.80e+00

Table 2. Problem statistics and numerical results (b)

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