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Dispersion of a radioactive tracer in the blood flow

by

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Abstract: Some approximations to the problem of convection-diffusion of a radioactive tracer in the blood stream are presented. In particular we deal with the problem of retrieving the tracer concentration at the entrance of a vessel from measurements performed at some distance from the heart. The convergence of a numerical scheme with time discretization for the computation of the direct diffusion-convection process is proved.

1. Introduction

The quantity measuring the blood flow to the brain, called "brain perfusion" and defined as the volume in ml of blood crossing the unit mass (gram) of brain tissue in one second, has been the object of many neurological studies (see Bacciottini, 1990, Iida, Kanno, Mirua, Murakami, Takahashi, Uemura, 1986, Quarles, Mintun, Larson, Markham, 1993 and related references). Radioactive tracers and the Single Photon Emission Tomography (SPECT), a technique which enables us to measure the tracer concentration even in very small portion of brain tissue, are often employed in its analysis. A radioactive tracer, in the specific here considered hexamethyl-propylene-aminoxine (HMPAO), is injected in a patient's vein; following the circulatory system it reaches the heart and then various body districts, including the Central Nervous System (CNS). There the tracer crosses the capillary wall, it diffuses and is distributed in the brain tissue where its concentration, $C_i(t)$, can be measured by means of the SPECT. If the concentration $C_a(t)$ of the tracer in the capillaries which are in direct contact with the considered part of brain tissue were known, it would be possible, due to a compartmental model described in Bacciottini (1990), to retrieve the absorption rate of the tracer in brain, a measurement which is closely related to perfusion. Since it is not possible to measure $C_a(t)$ directly, at the moment the usual procedure is to sample the concentration ($C_p(t)$) of the tracer at the end of a peripheral artery, usually in an arm, relating $C_a(t)$ to it through the formula:

$$C_a(t) = \int_{Dt}^{t+Dt} C_p(\eta) \cdot d(t-\eta) d\eta$$

where Dt is the time delay between the instants at which HMPAO is detected in the brain and at the peripheral exit, $d(t)$ is the dispersion function $\frac{1}{\tau}e^{-t/\tau}$, whose justification is experimental (see Iida, Kanno, Mirua, Murakami, Takahashi, Uemura, 1986), and τ is a constant. This paper is in the framework of a project aimed at the construction of methods for the computation of $C_a(t)$ with an acceptable theoretical basis. The project is developed in cooperation with the research group directed by Prof. Alberto Pupi of the Nuclear Medicine Department at the University of Florence.

Here we deal first with the problem of the determination of $C_a(t)$, simply assuming that the tracer diffusion can be neglected compared to the convection due to the blood flow (see section 3). Concerning the direct set-up, i.e. the combined diffusion-convection of the tracer in the blood stream, it is well known to give rise to a well-posed parabolic initial-boundary value problem. Here we prove the convergence of a numerical method for the computation of its solution, based on time discretization and on the step by step computation of a sequence of purely diffusive and purely convective processes (see section 4).

2. The matematical model

We focus our attention on a single blood vessel, which for our purpose will be considered as a rigid tube of uniform circular cross-section. We assume that blood can be modelled as a Newtonian fluid (although it is known that the reality is much more complicated) and that its flow through the vessel is laminar and steady. In a system of cylindrical coordinates, the equation governing the tracer diffusion and transport is:

$$D_R\left(u_{rr} + \frac{1}{r}u_r\right) + D_A u_{xx} - v(r)u_x = u_t, \tag{1}$$

where $u(x, r, t)$ is the tracer concentration, D_R and D_A are the radial and axial diffusion coefficients of the tracer in blood, respectively, and $v(r)$ is the blood velocity (a steady Pojseulle flow):

$$v(r) = \frac{2Q}{\pi R^2}\left(1 - \frac{r^2}{R^2}\right), \tag{2}$$

where R is the vessel inner radius and Q the prescribled flow rate.

It is to be observed that the quantity $C_p(t)$ introduced in section 1 is the average tracer concentration over the vessel section (this is what we really can measure). Obviously the "real" concentration u is a function of r too. If we put the origin of x just at the exit from heart, it seems reasonable to consider the tracer homogeneously distributed on the section $x = 0$, because of the mixing action occurring in the heart. Moreover, we impose a zero flux condition at the vessel wall. Thus the initial and boundary conditions associated to (1) are:

$$u(x,r,0) = 0,\ u(0,r,t) = C(t),\ u_r(x,0,t) = 0,\ u_r(x,r,t) = 0.$$

This model is of course largely approximated. First af all, blood vessels are not rigid, on the contrary they must be able to meet the pressure fluctuations they are liable to; moreover, blood is not a Newtonian fluid, but more precisely it is a suspension, and in any case the flow is only approximately laminar (hemodynamic models which include the elasticity of the walls as well as pressure and flow fluctuations have been considered in the physiological literature since 1950*s* by J. R. Womersley and others; these models were already included in books like Attinger, 1964, Bergel, 1972 and McDonald, 1960). However, here we are not concerned with the study of the blood flow, hence we assume (2) to simplify calculus. In order to further simplify the problem, since blood takes only a few seconds to go from heart to brain, or to the peripheral sampling station, it seems reasonable to assume that axial diffusion can be neglected with respect to convective transport.

The above assumptions lead us to the following initial-boundary value problem:

$$\begin{cases} D_R\left(u_{rr} + \frac{1}{r}u_r\right) - v(r)u_x = u_t & \text{in } \Omega = (0,L] \times [0,R) \times (0,\infty), \\ u(x,r,0) = 0, & x \in (0,L),\ r \in [0,R), \\ u_r(x,0,t) = u_r(x,r,t) = 0, & x \in (0,L),\ t > 0, \\ u(0,r,t) = C(t), & r \in [0,R),\ t > 0, \end{cases} \tag{3}$$

with the differential equation satisfied in the classical sense and u bounded, u and u_r continuous in $\overline{\Omega}$, with the exception of the points lying on $x = 0$ for $t = 0$. If we introduce the nondimensional variables: $\xi = \frac{x}{L}$, $\rho = \frac{r}{R}$, $\tau = \frac{t}{T}$, $U = \frac{u}{\|u\|}$, $V(\rho) = \frac{v(R\rho)}{v_{max}} = 1 - \rho^2$, $C_0 = \frac{C}{\|C\|}$, $\alpha = \frac{D_R T_0}{R^2}$, where $v_{max} = \frac{2Q}{\pi R^2}$, $T_0 = \frac{L}{v_{max}}$, $\|C\| = sup_{t\in(0\infty)}|C|$ and $\|u\| = sup_{\Omega}|u|$,
We get:

$$\begin{cases} \alpha\left(U_{\rho\rho} + \frac{1}{\rho}U_\rho\right) - (1-\rho^2)U_\xi = U_\tau & \text{in } \Omega_0 = (0,1] \times [0,1) \times (0,\infty), \\ U(\xi,\rho,0) = 0, & \xi \in (0,1),\ \rho \in [0,1), \\ U_\rho(\xi,0,\tau) = U_\rho(\xi,1,\tau) = 0, & \xi \in (0,1),\ \tau > 0, \\ U(0,\rho,\tau) = \frac{C_0(\tau)}{2}, & \rho \in [0,1),\ \tau > 0. \end{cases} \tag{4}$$

It is known that such a problem is well posed for any continuous function $C_0(\tau)$ such that $C_0(0) = 0$. About notation: with $\|\cdot\|$ we denote the usual norm, i.e. the supremum of the absolute value of a function in its domain.

3. Purely convective problem

If we neglect diffusion completely, then the problem (3) is reduced to:

$$\begin{cases} u_t + v(r)u_x = 0, & \text{in}\,\Omega \\ u(x,r,0) = 0, & x \in [0,L],\, r \in [0,R], \\ u(0,r,t) = C(t), & r \in [0,1],\, t \in [0,\infty), \end{cases} \tag{5}$$

whose solution can easily be found:

$$u(x,r,t) = C\left(t - \frac{x}{v(r)}\right), \tag{6}$$

where $C(t) \equiv 0$ for $t < 0$. In this section we want to solve the problem of determining the boundary value $C(t)$, once the average $C_p(t)$ of u at $x = L$ is prescribed as a continuously differentiable function. If we denote by $\overline{C}(x,t)$ the average of concentration over the vessel section x, we have:

$$\overline{C}(x,t) = \frac{2}{R^2}\int_0^R \rho \cdot C\left(t - \frac{x}{v(\rho)}\right) d\rho = 2\int_0^1 \rho \cdot C\left(t - \frac{x}{v_{max}(1-\rho^2)}\right) d\rho. \tag{7}$$

Setting $\eta = t - \frac{x}{v_{max}(1-\rho^2)}$ and $T(x) = \frac{x}{v_{max}}$, we have:

$$\overline{C}(x,t) = T(x)\int_{-\infty}^{t-T(x)} \frac{1}{(t-\eta)^2} \cdot C(\eta)d\eta = T(x)\int_0^{t-T(x)} \frac{1}{(t-\eta)^2} \cdot C(\eta)d\eta. \tag{8}$$

Recalling that $C_p(t)$ is the average of u sampled at the section $x = L$ of the vessel and setting $T_0 = T(L)$ and $\tau = t - T_0$, we obtain:

$$\int_0^\tau \frac{1}{[T_0 + \tau - \eta]^2} \cdot C(\eta)d\eta = \frac{1}{T_0}C_p(\tau + T_0). \tag{9}$$

Differentiating with respect to τ we arrive at the following Volterra equation of the second kind for the unknown $C(t)$:

$$C(\tau) - \int_0^\tau \frac{2{T_0}^2}{[T_0 + \tau - \eta]^3} \cdot C(\eta)d\eta = T_0\dot{C}_p(\tau + T_0), \tag{10}$$

that is

$$C(\tau) - \int_0^\tau K(\tau,\eta) \cdot C(\eta)d\eta = Y(\tau)$$

where $K(\tau,\eta) = \frac{2{T_0}^2}{[T_0+\tau-\eta]^3}$, $Y(\tau) = T_0\dot{C}_p(\tau + T_0)$.

This problem is well posed (see Tricomi, 1957) and the solution is given by

$$C(\tau) = Y(\tau) + \sum_{r=1}^{\infty}\int_0^\tau k_r(\tau,\eta) \cdot Y(\eta)d\eta \tag{11}$$

with $k_1 = K$, $k_{r+1} = \int_\eta^\tau K(\tau,\xi) \cdot k_r(\xi,\eta)d\xi$.

Another way of calculating $C(\tau)$ from (9) is to use Laplace Transforms. This technique leads to

$$c(s) \cdot k(s) = \frac{1}{T_0} exp(sT_0)c_p(s),$$

where $c(s)$ and $c_p(s)$ are the Laplace trasforms of $C(\tau)$ and $C_p(\tau)$, respectively, and $k(s)$ is the Laplace transform of $\frac{1}{(T_0+\tau)^2}$, i.e. $k(s) = -s \cdot exp(sT_0) \cdot Exp_i(sT_0) + \frac{1}{T_0}$, with $Exp_i(x) = \int_x^\infty \frac{e^u}{u} du$. Hence

$$C(s) = \frac{1}{T_0 k(s)} exp(sT_0)c_p(s).$$

A delicate aspect in the calculation of $C(\tau)$ is to check that it is positive for all times. Generally speaking, it is difficult to characterize the set of admissible data $C_p(\tau)$. For instance it is clear that it cannot decay at infinity faster than τ^{-2}, corresponding to $C(\tau)$ positive almost everywhere and with compact support. Finally, let us restrict our interest to a finite time interval, say for instance $[0,T]$, and consider the linear operator $L : C^1[0,T] \to C^0[0,T]$ (in $C^1[0,T]$ and $C^0[0,T]$ we adopt the usual norms, i.e. $sup_{[0,T]}|f(\tau)| + sup_{[0,T]}|\dot{f}(\tau)|$ and $sup_{[0,T]}|f(\tau)|$ respectively) which gives $C(\tau)$ in terms of $C_p(\tau)$ as follows $L(f(\tau)) = T_0 \left(\dot{f}(\tau) + \int_0^\tau R(\tau,\eta)\dot{f}(\eta)d\eta\right)$, where $f \in C^1([0,T])$ and $R(\tau,\eta)$ is the resolvent kernel $\sum_{r=1}^\infty k_r(\tau,\eta)$; it is easily seen that L is bounded for any $T < +\infty$. In fact (see Tricomi, 1957), $|R(\tau,\eta)| \le hA(\tau)B(\eta)$, where h is a constant,

$$A(\tau) = \left(\int_0^\tau K^2(\tau,\eta)d\eta\right)^{1/2} = \left(\frac{4}{5T_0}\left(1 - \frac{T_0^5}{(T_0+\tau)^5}\right)\right)^{1/2}$$

and

$$B(\eta) = \left(\int_\eta^\infty K^2(\tau,\eta)d\tau\right)^{1/2} = \left(\frac{4}{5T_0}\right)^{1/2}.$$

It follows $\|L\| \le T_0 \left(1 + \int_0^T R(\tau,\eta)d\eta\right) \le T_0 \left(1 + \frac{4T}{5T_0}\left(1 - \frac{T_0^5}{(T_0+T)^5}\right)^{1/2}\right)$. Hence $C(\tau)$ depends continuously by $C_p(\tau)$ in any finite time interval $[0,T]$, in the sense specified above.

4. A numerical scheme for the problem with radial diffusion

4.1. Time discretization

We divide the time interval $(0,T]$ into n intervals $(t_i, t_{i+1}]$, $i = 0,1,...,n-1$, of width $\Delta_n t = \frac{T}{n}$. In each interval $(t_i, t_{i+1}]$ we assume that for $t = t_i$

the concentration is known from the previous time step and with this initial condition $(v(\xi,\rho,t_i^+) = w(\xi,\rho,t_i^-))$ we solve the purely convective problem, determining a function $v(\xi,\rho,\tau)$; then we use the value $v(\xi,\rho,t_{i+1})$ as the initial data for a purely radially diffusive problem in the same time interval $(t_i,t_{i+1}]$, whose solution will be denoted by $w(\xi,\rho,\tau)$.

Thus we construct the following scheme:

$$\begin{cases} v_\tau + (1-\rho^2)v_\xi = 0, \quad \xi > 0,\ \rho \in [0,1), & \tau \in (t_i,t_{i+1}], \\ v(0,\tau) = C_0(\tau), & \rho \in [0,1),\ \tau \in (t_i,t_{i+1}], \\ v(\xi,\rho,t_i^+) = w(\xi,\rho,t_i^-), & \xi > 0,\ \rho \in [0,1), \end{cases} \tag{12}$$

$i = 0,1,\ldots,n-1$,

where ρ is a parameter and $w(\xi,\rho,t_0) = w(\xi,\rho,0) = 0$ for $\xi \geq 0$ and $\rho \in [0,1]$;

$$\begin{cases} \alpha\Delta_\rho w = w_\tau, & \xi > 0,\ \rho \in [0,1),\ \tau \in (t_i,t_{i+1}], \\ w_\rho(\xi,0,\tau) = w_\rho(\xi,1,\tau) = 0, & \xi > 0,\ \tau \in (t_i,t_{i+1}], \\ w(\xi,\rho,t_i^+) = v(\xi,\rho,t_{i+1}^-), & \xi > 0,\ \rho \in [0,1), \end{cases} \tag{13}$$

$i = 0,1,\ldots,n-1$,

where ξ is considered a parameter and Δ_ρ is the differential operator $\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}$. Further we set $U_n(\xi,\rho,t_i) = w(\xi,\rho,t_i^-)$, $i = 1,2,3,\ldots,n$.

We want to demonstrate that U_n converges to a function U defined over the whole domain and solving (4) in the classical sense. Concerning notation, we remark that we write v instead of $v^{(n)}$, t_i instead of $t_i^{(n)}$, etc. when there is no ambiguity. Moreover we introduce the symbols $v_i(\xi,\rho) = v(\xi,\rho,t_i^-)$ and $w_i(\xi,\rho) = w(\xi,\rho,t_i^-)$.

4.2. Solution of (4.1) and (4.2)

First of all, let us write down the solutions of (12) and of (13), namely

$$v(\xi,\rho,\tau) = \begin{cases} C_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right), & \xi \leq (1-\rho^2)(\tau - t_i), \\ w_i\left(\xi - (1-\rho^2)(\tau-t_i),\rho\right), & \xi > (1-\rho^2)(\tau-t_i), \end{cases} \tag{14}$$

$$w(\xi,\rho,,\tau) = \int_0^1 rN(\rho,\tau,r,t_i)v_{i+1}(\xi,r)dr, \tag{15}$$

where $N(\rho,\tau,r,t)$ is the Neumann's function for the heat equation in a circle (see Crank, 1975, p. 81): $N(\rho,\tau,r,t) = \frac{2}{R^2}\left\{1 + \sum_{k=1}^{\infty} e^{-\alpha a_n(\tau-t)} \cdot \frac{J_0(a_n\rho)J_0(a_n r)}{J_0^2(R\rho)}\right\}$. Here R is the radius of the circle (in our case $R=1$), J_0 is Bessel's function of rank 0, a_n are the roots of $J_1(Ra_n) = 0$, with J_1 Bessel's function of rank 1. It is not difficult to demonstrate the continuity of v and of its first derivates in the intervals $[t_i,t_{i+1})$, $i = 0,1,\ldots,n-1$.

4.3. Convergence of the method

The demonstration of the convergence of the above described method, follows the pattern of a similar study by Fasano (1968), where the convergence of the so-called Rothe's method for a parabolic problem with non-compatible initial and boundary data is proved. In Fasano (1968) the results of Pucci (1954) relative to the compactness of sequences of functions defined on sequences of sets, are exploited. The following sections will be devoted to proving the theorem stated below.

THEOREM 4.1 *If $C_0(\tau) \in C^3(R)$ and $C_0(\tau) = 0$ for $\tau \leq 0$, then the sequence $\{U_n(\xi,\rho,\tau)\}$ biconverges in $A_\sigma \equiv [\sigma,1] \times [0,1] \times [0,T]$, $\forall \sigma > 0$ small enough, to a continuous function $U(\xi,\rho,\tau)$, satisfying the system:*

$$\begin{cases} \alpha\Delta_\rho U - (1-\rho^2)U_\xi = U_\tau & in\,[\sigma,1]\times[0,1)\times(0,T] \\ U(\xi,\rho,0) = 0, & \xi\in[\sigma,1],\ \rho\in[0,1) \\ U_\rho(\xi,0,\tau) = U_\rho(\xi,1,\tau) = 0, & \xi\in[\sigma,1),\ \tau\in[0,T] \\ \lim_{\sigma\to 0} U(\sigma,\rho,\tau) = C_0(\tau), & \rho\in[0,1),\ \tau\in[0,T]. \end{cases}$$

The definitions of biconvergence, pseudocontinuity, etc. are to be found in the appendix.

4.4. Proof scheme

First we prove the existence of the limit function U and its regularity. Then, we will show that it really meets the requirements of theorem 4.1. To this end, we will exploit the following formula which comes easily from the definitions of v_{i+1}, w_{i+1} and U_n:

$$\begin{aligned} \Delta_t U_n &= \frac{U_n(\xi,\rho,t_{i+1}) - U_n(\xi,\rho,t_i)}{\Delta_n t} = \frac{w_{i+1}^{(n)}(\xi,\rho) - w_i^{(n)}(\xi,\rho)}{\Delta_n t} = \\ &= \frac{w_{i+1}^{(n)}(\xi,\rho) - v_{i+1}^{(n)}(\xi,\rho)}{\Delta_n t} + \frac{v_{i+1}^{(n)}(\xi,\rho) - w_i^{(n)}(\xi,\rho)}{\Delta_n t} = \\ &= \frac{w^{(n)}(\xi,\rho,t_{i+1}) - w^{(n)}(\xi,\rho,t_i)}{\Delta_n t} + \\ &+ \frac{v^{(n)}(\xi,\rho,t_{i+1}) - v^{(n)}(\xi,\rho,t_i)}{\Delta_n t} = \\ &= \alpha w_\tau^{(n)}(\xi,\rho,\bar{t}_i) + v_\tau^{(n)}(\xi,\rho,\bar{\bar{t}}_i) = \\ &= \alpha\Delta_\rho w^{(n)}(\xi,\rho,\bar{t}_i) - (1-\rho^2)\frac{\partial}{\partial\xi}v^{(n)}(\xi,\rho,\bar{\bar{t}}_i) \end{aligned} \tag{16}$$

with $\bar{t}_i$, $\bar{\bar{t}}_i \in (t_i, t_{i+1}]$.

Namely we will find that U is continuously differentiable with respect to time in the domain we are interested in, and that $\{\Delta_t U_n\}$ converges to U_τ. We further demonstrate that $\Delta_\rho w_i^{(n)}$ and $\frac{\partial}{\partial\xi}v_i^{(n)}$ biconverge, for $n \to \infty$, to $\Delta_\rho U$ and U_ξ, respectively, and that $\Delta_\rho w^{(n)}$ and $\frac{\partial}{\partial\xi}v^{(n)}$ are pseudoequicontinuous, for $n \in N$, in $\{(t_i, t_{i+1}]\}_{i=0,1,..,n-1}$; hence, the biconvergence of $\Delta_\rho w^{(n)}(\xi,\rho,\bar{\bar{t}}_i)$ and of $\frac{\partial}{\partial\xi}v^{(n)}(\xi,\rho,\bar{t}_i)$ to $\Delta_\rho U$ and U_ξ is proved.

4.5. Existence of the limit function

We define:

$A_\sigma \subset R^3$, $A_\sigma \equiv \{(\xi,\rho,\tau) : \xi \in [\sigma,1],\ \rho \in [0,1],\ \tau \in [0,T]\}$,

$A_n^\sigma \subset R^3$, $A_n^\sigma \equiv \{(\xi,\rho,t_i) : \xi \in [\sigma,1],\ \rho \in [0,1],\ t_i = \frac{iT}{n} i = 0,1,2,...,n\}$.

Theorem VII of Pucci (1954), ensures that it is possible to extract from $\{U_n\}$ a subsequence biconvergent in A to a continuos U function, provided we can demonstrate the following properties:

1) the sequence of sets $\{A_n^\sigma\}$ converges to the set A_σ;
2) the functions $U_n(\xi,\rho,\tau)$ are uniformly bounded;
3) the functions U_n are pseudoequicontinuous.

We will now demonstrate the above properties:

1) Convergence of $\{A_n^\sigma\}$:

the proof is trivial. If we consider any point belonging to A_σ and an ϵ-neighborhood of it, for $n > \frac{T}{\epsilon}$, A_n^σ certainly has points in that neighborhood, while for any point outside A_σ there is a neighborhood having no intersection with all A_n^σ; this proves that $Liminf_{n\to\infty} A_n^\sigma = A_\sigma$ and that $Limsup_{n\to\infty} A_n^\sigma \subseteq A_\sigma$. Since it is clear that $Liminf_{n\to\infty} A_n^\sigma \subseteq Limsup_{n\to\infty} A_n^\sigma$, our statement has been shown (the definitions of *Limsup* and *Liminf* are to be found in the appendix).

2) Uniform bounds for U_n:

for $v_1^{(n)}$ we have $\|v_1^{(n)}\| \le |C_0(t_1)| \le 1$. The maximum principle relative for heat equation, implies: $\|w_{i+1}^{(n)}\| \le \|v_{i+1}^{(n)}\|$.

From 14 it follows that $\|v_{i+1}^{(n)}\| \le max\{|C_0(t_{i+1})|, \|w_i^{(n)}\|\}$ with $i = 0,1,...,n-1$. Hence, by induction, $\|w_{i+1}^{(n)}\| \le 1$, that is $\|U_n(\xi,\rho,t_1)\| \le 1$ with $i = 0,1,...,n$ and for any $n \in N$.

3) Equicontinuity of U_n:

we want to show that $\forall\epsilon > 0$ there exist two numbers $\delta_\epsilon > 0$ and $\eta_\epsilon \in N$ such that $|U_n(\xi,\rho,\tau) - U_n(\xi',\rho',\tau')| < \epsilon$ for $n > \eta_\epsilon$ and $|\xi-\xi'|+|\rho-\rho'|+|\tau-\tau'| < \delta_\epsilon$. This is the most delicate of the three properties, and it is a consequence of a theorem we will demonstrate later:

THEOREM 4.2 *$\frac{\partial}{\partial\rho}w^{(n)}$ and $\frac{\partial}{\partial\xi}v^{(n)}$ are equibounded in A_σ, $\frac{\partial}{\partial\tau}w^{(n)}$ and $\frac{\partial}{\partial\tau}v^{(n)}$ are equibounded in $A_\sigma \backslash \{\tau = t_i, i = 0,1,...,n\}$, for $n \in N$.*

Due to the above theorem, and since:

$$|U_n(\xi,\rho,t_{i+1}) - U_n(\xi',\rho',t_i)| = |w_{i+1}^{(n)}(\xi,\rho) - w_i^{(n)}(\xi',\rho')| \le$$
$$\le |w_{i+1}^{(n)}(\xi,\rho) - w_{i+1}^{(n)}(\xi,\rho')| + |w_{i+1}^{(n)}(\xi,\rho') - v_{i+1}^{(n)}(\xi,\rho')| +$$

$$+|v_{i+1}^{(n)}(\xi,\rho') - v_{i+1}^{(n)}(\xi',\rho')| + |v_{i+1}^{(n)}(\xi',\rho') - w_i^{(n)}(\xi',\rho')| =$$
$$= |w_{i+1}^{(n)}(\xi,\rho) - w_{i+1}^{(n)}(\xi,\rho')| + |w^{(n)}(\xi,\rho',t_{i+1}^-) - w^{(n)}(\xi,\rho',t_i^+)| +$$
$$+|v_{i+1}^{(n)}(\xi,\rho') - v_{i+1}^{(n)}(\xi',\rho')| + |v^{(n)}(\xi',\rho',t_{i+1}^-) - v^{(n)}(\xi',\rho',t_i^+)|$$

the assetion follows. ■

Let us point out that nothing is changed if we consider the functions $V_n(\xi,\rho,t_i) = v_i(\xi,\rho)$; we can thus find, in this case as well, a limit V of a subsequence $\{V_{\lambda_n}\}$ extracted from $\{V_n\}$. It is easily demonstrated that U and V coincide; indeed:

$$\begin{aligned} \|U-V\| &= \lim_{n\to\infty} \|w_i^{(n)} - v_i^{(n)}\| = \\ &= \lim_{n\to\infty} \|w^{(n)}(\xi,\rho',t_{i+1}^-) - w^{(n)}(\xi,\rho',t_i^+)\| \le \\ &\le \lim_{n\to\infty} \Delta_n t \|\frac{\partial}{\partial\tau} w^{(n)}\| = 0 \end{aligned}$$

We then choose $\{U_{\lambda_n}\}$ so that $\{V_{\lambda_n}\}$ converges as well.

4.6. U as a solution of the problem

First of all we have to show that U can be differentiated at least once with respect to ξ and τ and at least twice with respect to ρ and that $\Delta_\rho U$ is continous. To do this we state the following theorem, which will be demonstrated later:

THEOREM 4.3 *$\frac{\partial}{\partial\rho}U_n$, $\frac{\partial}{\partial\xi}V_n$ and $\Delta_\rho U_n$ are pseudoequicontinous and equibounded for $n \in N$.*

Thanks to this theorem, following a technique parallel to the one already used, we demonstrate that there exists a subsequence $\{U_{k_n}\}$ of $\{U_{\lambda_n}\}$ such that $\{\frac{\partial}{\partial\rho}U_{k_n}\}$, $\{\frac{\partial}{\partial\xi}V_{k_n}\}$ and $\{\Delta_\rho U_{k_n}\}$ converge to three respective functions U^*, U^{**} e U^{***}. Further, the following is valid:

THEOREM 4.4 *Functions U^*, U^{**} and U^{***} coincide with the derivatives U_ρ, U_ξ and $\Delta_\rho U$ of U, respectively.*

Proof The technique to be employed is basically the one of Fasano (1968). We give the proof of $U^{**} = U_\xi$. The other two cases are completely analogous. Since $\{V_{k_n}\}$ biconverges to U, for theorem III of Pucci (1954), we have that $\forall\epsilon > 0$ there exist two positive numbers δ_ϵ and n_ϵ such that $\forall(\xi,\rho,\tau),(\xi',\rho,\tau) \in A_\sigma$, $(\xi,\rho',\tau'),(\xi',\rho',\tau') \in A_n^\sigma$, $\xi \ne \xi'$,

$|U(\xi,\rho,\tau) - V_{k_n}(\xi,\rho',\tau')| < \epsilon/3$, $|U(\xi',\rho,\tau) - V_{k_n}(\xi',\rho',\tau')| < \epsilon/3$

with $n > n_\epsilon$ and $|\tau-\tau'| + |\rho-\rho'| < \delta_\epsilon$.

Hence

$$|U(\xi,\rho,\tau) - U(\xi',\rho,\tau) - [V_{k_n}(\xi,\rho',\tau') - V_{k_n}(\xi',\rho',\tau')]| < 2\epsilon/3$$

with $n > n_\epsilon$ and $|\tau - \tau'| + |\rho - \rho'| < \delta_\epsilon$.

For the continuity of $\frac{\partial}{\partial \xi} V_{k_n}$ it follows

$$|U(\xi,\rho,\tau) - U(\xi',\rho,\tau) - (\xi - \xi')\frac{\partial}{\partial \xi} V_{k_n}(\xi - \theta(\xi - \xi'), \rho', \tau')\,| < \epsilon/3.$$

Thanks again to theorem III in Pucci (1954), applied to sequence $\{\frac{\partial}{\partial \xi} V_{k_n}\}$, biconvergent to U^{**}, there exist n'_ϵ and δ'_ϵ such that:

$$\begin{aligned} &|\xi - \xi'| \cdot |\frac{\partial}{\partial \xi} V_{k_n}(\xi - \theta(\xi - \xi'), \rho', \tau') - U^{**}(\xi - \theta(\xi - \xi'), \rho, \tau)\,| < \\ &< \; |\frac{\partial}{\partial \xi} V_{k_n}(\xi - \theta(\xi - \xi'), \rho', \tau') - U^{**}(\xi - \theta(\xi - \xi'), \rho, \tau)\,| < \epsilon/3 \end{aligned}$$

with $n > n_\epsilon$ and $|\tau - \tau'| + |\rho - \rho'| < \delta_\epsilon$, $|\xi - \xi'| < 1$.

From all this it follows

$$|U(\xi,\rho,\tau) - U(\xi',\rho,\tau) - (\xi - \xi') \cdot U^{**}(\xi - \theta(\xi - \xi'), \rho, \tau)\,| < \epsilon,$$

that is:

$$\frac{U(\xi,\rho,\tau) - U(\xi',\rho,\tau)}{(\xi - \xi')} = U^{**}(\xi - \theta(\xi - \xi'), \rho, \tau),$$

which, by the continuity of U^{**}, implies the existence of U_ξ and the equality $U_\xi = U^{**}$. ■

Finally, using theorem XVIII of Pucci (1954), we establish that from the sequence

$$\left\{ \frac{U_{\lambda_n}(\xi,\rho,\tau) - U_{\lambda_n}(\xi,\rho,\tau - \delta)}{\delta} \right\}$$

it is possible to extract a subsequence convergent to U_τ, which exists and is continuous by the same theorem. Because of (16) and of all the results obtained in this section we can say that U really meets the requirements of Theorem 4.1. The only thing still to be done, in order to complete the demonstration of Theorem 4.1, is to identify the subsequence $\{U_{\lambda_n}\}$ with the whole sequence $\{U_n\}$; this is a straightforward consequence of the uniqueness of solution of our initial problem, as observed in Pucci (1953); if $\{U_n\}$ were not be convergent to U, there would exist a subsequence $\{U_{\mu_n}\}$ that, in a point $h_0 \equiv (\xi_0, \rho_0, \tau_0) \in A_\sigma$, biconverges to a value different from $U(h_0)$. For the pseudoequicontinuity of $\{U_{\mu_n}\}$, there exists a subsequence $\{U_{s_n}\}$ biconvergent to a function $\overline{U}$ which turns out to be another solution of the considered problem, contradicting the uniqueness.

4.7. Proof of theorems 4.2 and 4.3

We start with a lemma which shows some properties useful in the sequel:

LEMMA 4.1 *The function* $H(\xi,\rho,\tau) = C_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right)$ *and its derivatives up to the tird order and* $\Delta_\rho H$ *with its first derivatives are equibounded in* A_σ *and converge to zero as* ρ *converges to 1.*

Proof Let us start with the last assertion: it follows immediately observing that $C_0(\tau) \equiv 0$ for $\tau \leq 0$, hence $H \equiv 0$ for $\sqrt{1-\frac{\xi}{\tau}} \leq \rho < 1$ and a fortiori for $\sqrt{1-\frac{\sigma}{T}} \leq \rho < 1$. It is evident that $\|H\| \leq \|C_0\|$. Moreover:

$H_\tau = \dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right)$, hence $\|H_\tau\| \leq \|C^{\cdot}{}_0\|$;

$H_\xi = -\frac{1}{(1-\rho^2)}\dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right)$, hence $\|H_\xi\| \leq \|\dot{C}_0\|$;

$H_\rho = -\frac{2\xi\rho}{(1-\rho^2)^2}\dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right)$,

hence $\|H_\rho\| \leq \frac{2T^2}{\sigma^2}\|\dot{C}_0\|$ and $\|\frac{1}{\rho}H_\rho\| \leq \frac{2T^2}{\sigma^2}\|\dot{C}_0\|$;

$\Delta_\rho H = \left(\frac{2\xi\rho}{(1-\rho^2)^2}\right)^2 \ddot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right) + \frac{4\xi(1+\rho^2)}{(1-\rho^2)}\dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right)$,

hence $\|\Delta_\rho H\| \leq \frac{4T^2}{\sigma^4}\|\dot{C}_0\| + \frac{8T^3}{\sigma^3}\|\ddot{C}_0\|$. ■

The proof of theorem 4.2 is entirely based on lemmas 4.2, 4.4 and 4.6. Theorem 4.3 follows, as already said, by the same kind of the proof of the pseudoequicontinuity of U_n and needs the equiboundedness of sufficiently many derivatives of $w^{(n)}$ and $v^{(n)}$. We are going to derive the required estimates in the following lemmas.

LEMMA 4.2 $\frac{\partial}{\partial\xi}v^{(n)}$ *and* $\frac{\partial}{\partial\tau}v^{(n)}$, $n \in N$, *are equibounded in* A_σ *and in* $A_\sigma\backslash\{\tau = t_i,\ i = 0,1,...,n\}$ *respectively.*

Proof From (14) and (15) it follows:

$$\frac{\partial}{\partial\xi}v^{(n)}(\xi,\rho,\tau) = H_\xi = -\frac{1}{(1-\rho^2)}\dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right) \quad \text{for } \tau \in (0,t_1],$$

$$\frac{\partial}{\partial\xi}v^{(n)}(\xi,\rho,\tau) = -\frac{1}{(1-\rho^2)}\dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right) \text{ for } \xi \leq (1-\rho^2)(\tau - t_i),$$

$$\frac{\partial}{\partial\xi}v^{(n)}(\xi,\rho,\tau) = \begin{cases} -\frac{1}{(1-\rho^2)}\dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right) \\ \qquad \text{for } \xi > (1-\rho^2)(\tau - t_i) \\ \int_0^1 rN(\rho,t_i,r,t_{i-1})\frac{\partial}{\partial\xi}v_i^{(n)}(\xi - (\tau - t_i)(1-\rho^2), r)dr \\ \qquad \text{for } \xi > (1-\rho^2)(\tau - t_i), \end{cases}$$

for $\tau \in (t_i, t_{i+1}]$, $i \geq 1$.

By induction, and by the maximum principle, applied to $w_i^{(n)}$, it is now easily seen that $\|\frac{\partial}{\partial\xi}v^{(n)}\| \leq \|H_\xi\|$ in A_σ. Again from 14 and 15 we have:

$$\frac{\partial}{\partial\tau}v^{(n)}(\xi,\rho,\tau) = \dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right) \text{ for } \tau \in (0,t_1],$$

$$\frac{\partial}{\partial\tau}v^{(n)}(\xi,\rho,\tau) = \begin{cases} \dot{C}_0\left(\tau - \frac{\xi}{(1-\rho^2)}\right) \\ \qquad \text{for } \xi \leq (1-\rho^2)(\tau - t_i), \\ -(1-\rho^2)\int_0^1 rN(\rho,t_i,r,t_{i-1}) \\ \qquad \frac{\partial}{\partial\xi}v_i^{(n)}(\xi-(\tau-t_i)(1-\rho^2),r)dr \\ \qquad \text{for } \xi > (1-\rho^2)(\tau-t_i), \end{cases}$$

for $\tau \in (t_i, t_{i+1}]$, $i \geq 1$.

Still by induction in i and by the maximum principle we obtain:

$$\|\frac{\partial}{\partial\tau}v^{(n)}\| \leq \|\dot{C}_0\| \quad \text{in } A_\sigma\backslash\{\tau = t_i, i = 0,1,...,n\}.$$

■

Incidentally, we note that $\lim_{\rho\to 1}\frac{\partial}{\partial\tau}v_i^{(n)} = 0$, $i = 0,1,...,n$.

Uniform bounds for $v_{\xi\tau}^{(n)}$, $v_{\xi\xi}^{(n)}$ and $v_{\xi\xi\xi}^{(n)}$ can be obtained in the same way and it is also easy to deduce similar bounds for $w_\xi^{(n)}$, $w_{\xi\xi}^{(n)}$ and $w_{\xi\xi\xi}^{(n)}$, thus proving the following

LEMMA 4.3 *$w_\xi^{(n)}$, $w_{\xi\xi}^{(n)}$, $v_{\xi\xi}^{(n)}$, $v_{\xi\xi\xi}^{(n)}$ and $w_{\xi\xi\xi}^{(n)}$, $n \in N$, are equibounded in A_σ and $v_{\xi\tau}^{(n)}$, $n \in N$, are equibounded in $A_\sigma\backslash\{\tau = t_i,\ i = 0,1,...,n\}$.*

LEMMA 4.4 *$w_\rho^{(n)}$, $n \in N$, are uniformly bounded in A_σ.*

Proof Here we use Cartesian coordinates denoting the Laplacian operator by the usual symbol Δ. $w^{(n)}$ fulfills (13), i.e. the heat equation $\alpha\Delta w = w_\tau$ in the domain $D \times (t_i, t_{i+1}]$, $i = 0,1,...,n$, where $D \equiv \{0 \leq \sqrt{x^2+y^2} < 1, \xi \in (0,1]\}$, with the conditions:

$$\begin{cases} \frac{\partial w}{\partial n} = 0 & \text{on } \partial D\ \tau \in (t_i,t_{i+1}], \\ w = v_{i+1} \text{ in } D & \text{for } \tau = t_i. \end{cases}$$

The tangential derivative of w is zero at the boundary because of radial symmetry. Therefore, if we set $u = \frac{\partial w^{(n)}}{\partial x_k}$, it fullfils, for $\tau \in (t_i, t_{i+1}]$, $i = 0,1,...,n-1$,

$$\begin{cases} \alpha\Delta u = u_\tau, & \text{in } D \times (t_i,t_{i+1}], \\ u = 0, & \text{on } \partial D,\ \tau \in (t_i,t_{i+1}], \\ u = \frac{\partial}{\partial x_k}v_{i+1}, & \text{in } D \text{ for } \tau = t_i, \end{cases}$$

for k$= 1,2$, $n \in N$.

We now want to find a uniform bound of $\frac{\partial}{\partial x_k}v_{i+1}^{(n)}$, so that the maximum principle yields the equiboundedness of u (hence the same of $w_\rho^{(n)}$). First we note that $\|\frac{\partial}{\partial\rho}v_1^{(n)}\| \le \|H_\rho\|$, so that we can found a uniform bound for $\|\frac{\partial}{\partial x_k}v_1^{(n)}\|$, hence for u for $\tau \in [0, t_1]$. Then, from (14) we deduce

$$\frac{\partial}{\partial\rho}v_{i+1} = \begin{cases} H_\rho(\xi,\rho,t_{i+1}), & \text{for } 0 \le \rho \le \sqrt{1-\frac{\xi}{\tau}}, \\ \frac{\partial}{\partial\rho}w_i\left(\xi - \Delta_n t(1-\rho^2),\rho\right), & \text{for } \sqrt{1-\frac{\xi}{\tau}} \le \rho < 1. \end{cases}$$

Hence, noting that:

$$\begin{aligned} \frac{\partial}{\partial\rho}w_i\left(\xi - \Delta_n t(1-\rho^2),\rho\right) &= w_{i\rho}\left(\xi - \Delta_n t(1-\rho^2),\rho\right) + \\ &+ 2\rho\Delta_n t\frac{\partial}{\partial\xi}w_i\left(\xi - \Delta_n t(1-\rho^2),\rho\right) \end{aligned}$$

and that

$$\begin{aligned} &\frac{\partial}{\partial\xi}w_i\left(\xi - (\tau - t_i)(1-\rho^2),\rho\right) = \\ &= \int_0^1 rN(\rho,t_i,r,t_{i-1})\frac{\partial}{\partial\xi}v_i\left(\xi - (\tau-t_i)(1-\rho^2),r\right)dr\,, \end{aligned}$$

by induction in i, we get $\|\frac{\partial}{\partial\rho}v_{i+1}\| \le \|H_\rho\| + 2T\|\frac{\partial}{\partial\xi}v_i\| \le \|H_\rho\| + 2T\|H_\xi\|$. From the above estimates we obtain the equiboundedness of $\frac{\partial}{\partial x_k}v_{i+1}^{(n)}$, consequently the same of $\frac{\partial}{\partial x_k}w^{(n)}$, $k = 1, 2$ and, eventually, of $\frac{\partial}{\partial\rho}w^{(n)}$. ∎

Incidentally, we have also obtained the equiboundedness of $v_\rho^{(n)}$. Similar bounds for $v_{\xi\rho}^{(n)}$ and for $w_{\xi\rho}^{(n)}$ are easily obtained in an analogous way:

LEMMA 4.5 *$v_\rho^{(n)}$, $v_{\xi\rho}^{(n)}$ and $w_{\xi\rho}^{(n)}$, $n \in N$, are uniformly bounded in A_σ.*

LEMMA 4.6 *$w_\tau^{(n)}$, $n \in N$, are equibounded in $A_\sigma \backslash \{\tau = t_i, i = 0, 1, ..., n\}$, i.e. $\Delta_\rho w^{(n)}$ are equibounded in A_σ.*

Proof From (13), keeping in mind that $\Delta w^{(n)} = \frac{\partial}{\partial\tau}w^{(n)}$ and differentiating with respect to time, we get, for $\tau \in (0, t_1]$:

$$\begin{cases} \alpha\Delta_\rho\Phi = \Phi_\tau, & \text{in } [\sigma,1]x[0,1)x(0,t_1], \\ \Phi_\rho(\xi,0,\tau) = \Phi_\rho(\xi,1,\tau) = 0, & \xi \in [\sigma,1],\ \tau \in (0,t_1], \\ \Phi(\xi,\rho,0) = \Delta_\rho v_1, & \text{for } \xi \in [\sigma,1],\ \rho \in [0,1). \end{cases}$$

where $\Phi = \Delta_\rho w^{(n)}$, $n \in N$. We can do it since $\lim_{\rho\to 1}\frac{\partial}{\partial\rho}\Delta_\rho v_1 = 0$; hence, because of the maximum principle, we obtain $|w_\tau^{(n)}| \le \|\Delta_\rho H\|$ for $\tau \in (0, t_1]$.

Unfortunately, for $\tau \in (t_i, t_{i+1}]$ with $i > 0$, we have:

$$\frac{\partial}{\partial\rho}\Delta_\rho v_{i+1} = \frac{\partial \Delta_\rho H}{\partial\rho}|_{(\xi,\rho,t_i)} \text{ for } \xi \leq (1-\rho^2)\Delta_n t,$$
$$\frac{\partial \Delta_\rho w}{\partial\rho}|_{(\xi-\Delta_n t(1-\rho^2),\rho,t_i)} + 8\rho(\Delta_n t)^2[\rho^2\Delta_n t w_{\xi\xi\xi} + 2w_{\xi\xi}]|_{(\xi-\Delta_n t(1-\rho^2),\rho,t_i)}$$
$$\text{for } \xi > (1-\rho^2)\Delta_n t,$$

and we cannot be sure that $\lim_{\rho\to 1}\frac{\partial}{\partial\rho}\Delta_\rho v_{i+1} = 0$, but it is easy to see that:

$$|\lim_{\rho\to 1}\frac{\partial}{\partial\rho}\Delta_\rho v^{(n)}_{i+1}| \leq J(\Delta_n t)^2, \text{ where } J \text{ is a constant.} \tag{17}$$

Let us slightly alter the form of the problem solved by $w^{(n)}$, modifying it in the following way: we choose a value ϵ_n which must be $O(\Delta_n t)$, for example $\epsilon_n = \frac{\Delta_n t}{2}$; then we require that $w^{(n)}$ solves the problems:

$$\begin{cases} \alpha\Delta_\rho w = w_\tau & \text{in } \Omega, \\ w(\xi,\rho,t_i) = v_{i+1} & \xi \in [\sigma,1],\ \rho \in [0,1), \\ w_\rho(\xi,0,\tau) = 0 & \xi \in [\sigma,1],\ \tau \in (t_i,t_{i+1}], \\ w_\rho(\xi,1,\tau) = 0 & \text{for } \tau \in (t_i+\epsilon_n, t_{i+1}],\ \xi \in [\sigma,1], \\ w_\rho(\xi,1,\tau) = \phi_i^{(n)}(\tau) & \text{for } \tau \in (t_i, t_i+\epsilon_n],\ \xi \in [\sigma,1], \end{cases}$$

where $\phi_i^{(n)}(\tau)$ is a C^∞ function chosen so that the boundary and initial data of this problem are compatible, i.e. such that

$$\lim_{\tau\to t_i}\phi_i'(\tau) = \lim_{\rho\to 1}\frac{\partial}{\partial\rho}\Delta_\rho v_{i+1} \text{ and } \lim_{\tau\to t_i+\epsilon_n}\phi_i'(\tau) = 0.$$

It is not difficult to check that formula (17) remains still valid.

(17) shows that the boundary data for $\Delta_\rho w^{(n)}$ are uniformly bounded, hence, thanks to the maximum principle, the assertion has been proved. ■

If we choose ϕ_i so that

$$\lim_{\tau\to t_i}\phi_i''(\tau) = \lim_{\rho\to 1}\frac{\partial}{\partial\tau}\Delta_\rho v_{i+1} =$$
$$\lim_{\rho\to 1}\Delta_\rho\Delta_\rho v_{i+1} \text{ and } \lim_{\tau\to t_i+\epsilon_n}\phi_i''(\tau) = 0,$$

then the argument can be repeated for $\Delta^2 w^{(n)}$, i.e. $\frac{\partial}{\partial\tau}\Delta w^{(n)}$.

$\frac{\partial}{\partial\rho}\Delta w^{(n)}$ can be treated in the same way as $\frac{\partial}{\partial\rho}w^{(n)}$, while for $\frac{\partial}{\partial\xi}\Delta w^{(n)}$ everything is easier, since it solves a problem obtained by differentiating w.r.t. ξ the problem satisfied by $\Delta w^{(n)}$. Finally, we note that the uniform bounds for $\frac{\partial}{\partial\rho}w^{(n)}$ and for $\Delta_\rho w^{(n)}$ provide a uniform bound for $\frac{\partial^2}{\partial\rho^2}w^{(n)}$.

From these results, we get Theorem 4.2 and, proceeding as in point 3 of the proof of the existence of the limit function U, Theorem 4.3 has been proved.

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Appendix

For the sake of completeness, we recall some definitions from Pucci (1954).

Let E_n be a set of R^n and $f_n(x)$ a real-valued function defined for $x \in E_n$.

We say that a point x belongs to $Limsup_{n\to\infty}E_n$ if $Liminf_{n\to\infty}d(x,E_n)=0$, where $d(x,E_n)$ is the usual distance of x from the set E_n.

In analogous way we define $Liminf_{n\to\infty}E_n$.

We say that $\{f_n(x)\}$ *biconverges* in x_o to l if $x_o \in Liminf_{n\to\infty}E_n$ and $\forall\epsilon>0$ it is possible to find two positive numbers n_ϵ and δ_ϵ such that: $|l-f_n(x)|<\epsilon$ for $x\in E_n$, $d(x,x_o)<\delta_\epsilon$, $n>n_\epsilon$.

We say that $\{f_n(x)\}$ *biconverges* in the set A to $f(x)$ if it biconverges in x_o to $f(x_o)$ $\forall x_o\in A$.

Finally, we say that the functions $f_n(x)$ are *pseudoequicontinuous* if $\forall\epsilon>0$ it is possible to find two positive numbers n_ϵ and δ_ϵ such that: $|f_n(x)-f_n(x')|<\epsilon$ for $n>n_\epsilon$, x, $x'\in E_n$ and $d(x,x')<\delta_\epsilon$.