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Analytical solution of a variational inequality for a cutted bar[1]

by

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Abstract: The equilibrium problem of a clamped elastic bar with a vertical crack is considered. A nonpenetration condition of the crack banks leads to the restriction imposed upon the solution established by Khludnev (1992;1994) as the inequality. This model is described by the elliptic variational inequality. We construct the analytical solution of the problem using the projection of the initial space onto the set of solutions with the restriction.

Some approximation methods for variational inequalities were suggested, for instance, by Barbu, Korman (1991), Glowinski, Lions, Tremolieres (1976), Kovtunenko (1994a;b;c). Some exact solutions for the problem of contact between an elastic bar and an obstacle were found by Cimatti (1973).

1. Introduction

Let the middle line of a bar coincide with the real interval $[0, 1]$ and the bar have a vertical crack in x_0, $0 < x_0 < 1$. Let us denote

$$\Omega = (0, x_0) \cup (x_0, 1).$$

We have to find the vector $u = (u_1, u_2)$ of horizontal displacements $u_1 = u_1(x)$ and vertical displacements $u_2 = u_2(x)$ of the bar points $x \in \Omega$ under the action of external load $f = (f_1, f_2)$ (Fig.1).

The jam condition

$$u_1 = u_2 = Du_2 = 0 \quad \text{in } x = 0, 1 \tag{1}$$

should hold. Here D is the derivative operator.

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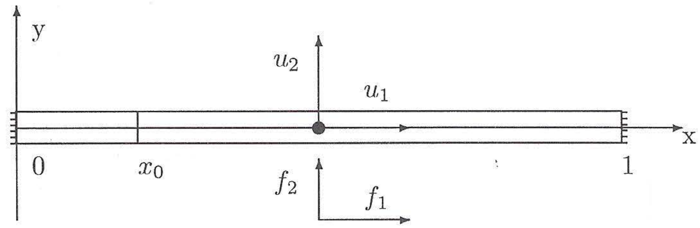

Figure 1.

In accordance with the Kirchhoff kinematic hypothesis, the displacements field alond the thickness $y \in [-h, h]$ of a bar is given by the following dependence of the bar middle line on displacements:

$$u_1(x, y) = u_1(x) - yDu_2(x), \quad u_2(x, y) = u_2(x).$$

The condition providing for nonpenetration of the cut edges along the cut thickness is

$$u_1(x_0 + 0, y) - u_1(x_0 - 0, y) \geq 0 \quad \forall |y| \leq h.$$

Substituting here the function value, one gets

$$[u_1] \geq y[Du_2] \quad \forall |y| \leq h,$$

where $[s]$ denotes the jump of the function $s(x)$ at x_0, i.e. $[s] = s(x_0 + 0) - s(x_0 - 0)$. Obviously, the last inequality is equivalent to

$$[u_1] \geq h|[Du_2]|.$$

Thus, we obtain the nonpenetration condition of the crack banks, as given by Khludnev (1992,1994). Later on, we consider the model assuming for simplicity $h = 1$:

$$[u_1] \geq |[Du_2]|. \tag{2}$$

Let us construct linear functions

$$\phi(u) = [u_1] + [Du_2], \quad \psi(u) = [u_1] - [Du_2],$$

Then, (2) is equivalent to

$$\phi(u) \geq 0, \quad \psi(u) \geq 0. \tag{3}$$

Let us define the basic Hilbert space

$$X = \{u \in H^1(\Omega) \times H^2(\Omega) \text{ and (1) holds } \},$$

its dual space $X^\star$ and a closed and convex subset

$$K = \{u \in X \text{ and } (3) \text{ holds }\}.$$

We introduce the inner product in X through

$$(u, v) = \int_\Omega Du_1 \cdot Dv_1 dx + \int_\Omega D^2u_2 \cdot D^2v_2 dx, \quad v = (v_1, v_2)$$

and the corresponding norm

$$(u, u) = \|u\|^2$$

in agreement with the obvious estimates

$$\int_\Omega (u_1)^2 dx \leq \int_\Omega (Du_1)^2 dx,$$

$$\int_\Omega (u_2)^2 dx \leq \int_\Omega (Du_2)^2 dx \leq \int_\Omega (D^2u_2)^2 dx.$$

The equilibrium problem for the clamped elastic bar with the crack induced by the action of the load $f \in X^\star$ is formulated as the following variational inequality, Khludnev (1992;1994):

$$u \in K, \quad (u, v - u) \geq \langle f, v - y \rangle \quad \forall v \in K. \tag{4}$$

Here the brackets $\langle \cdot, \cdot \rangle$ imply duality between X and $X^\star$. It is easy to see that the solution of (4) is unique.

2. Construction of the solution

Let $J^{-1} : X^\star \to X$ be the inverse duality injection, then

$$(J^{-1}f, v) = \langle f, v \rangle, \quad \forall v \in X. \tag{5}$$

We denote

$$w = J^{-1}f.$$

Let $f = (f_1, f_2) \in (L_2(\Omega))^2$ be given. By integrating (5), we can see that the vector $w = (w_1, w_2)$ is the solution of the following boundary problem

$$\begin{aligned} &-D^2w_1 = f_1, \quad D^4w_2 = f_2, && \text{in } \Omega \\ &[Dw_1] = [D^2w_2] = [D^3w_2] = 0, && \text{in } x_0 \\ &Dw_1 = D^2w_2 = D^3w_2 = 0. \end{aligned} \tag{6}$$

In the above notation the variational inequality (4) is equivalent to

$$u \in K, \quad (w - u, u - v) \geq 0 \quad \forall v \in K. \tag{7}$$

Let P be the projection operator of X onto K, i.e. for any $s \in X$ the unique projection $Ps \in K$ exists such that

$$(s - Ps, Ps - v) \geq 0 \quad \forall v \in K. \tag{8}$$

Comparing (7) and (8), it can be seen that (7) is equivalent to the following equation, Minoux (1989):

$$u = Pw. \tag{9}$$

To construct this projection, we introduce the function $\alpha(x) \in X \cap C^\infty(\Omega)$ in the following manner

$$\alpha = 0.5 \begin{cases} x^2, & x \in [0, x_0-] \\ (x-1)^2, & x \in [x_0+, 1] \end{cases}$$

and define the vector $\theta(w) = (\theta_1(w), \theta_2(w))$ by

$$\theta_1(w) = 0.5(\phi^-(w) + \psi^-(w)) \cdot D\alpha,$$

$$\theta_2(w) = 0.5(\phi^-(w) - \psi^-(w)) \cdot \alpha.$$

Here the superscript minus means the negative part of the number, i.e. $a = a^+ - a^-, \quad a^+, a^- \geq 0$. We mark the following properties of the constructed function

$$\begin{aligned} D^2\theta_1(w) &= D^3\theta_2(w) = 0, \text{ in } \Omega, & (10)\\ [\theta_1(w)] &= [D^2\theta_2(w)] = 0, & (11)\\ D\theta_1(w) &= 0.5(\phi^-(w) + \psi^-(w)), & (12)\\ D^2\theta_2(w) &= 0.5(\phi^-(w) - \psi^-(w)), \text{ in } x_0, & (13)\\ \phi(\theta(w)) &= -\phi^-(w), \quad \psi(\theta(w)) = -\psi^-(w). & (14) \end{aligned}$$

THEOREM 2.1

$$u = w - \theta(w)$$

is the solution of variational inequality (4) .

Proof. Taking into account (9), we have to prove that

$$Pw = w - \theta(w).$$

First, Pw belongs to K. Actually, in view of the linearity of ϕ and ψ, (14) gives

$$\phi(Pw) = \phi(w) - \phi(\theta(w)) = \phi^+(w) - \phi^-(w) + \phi^-(w) = \phi^+(w) \geq 0$$

and, similarly, $\quad \psi(Pw) = \psi^+(w) \geq 0$.

Second, let us verify (8), i.e.

$$(\theta(w), w - \theta(w) - v) \geq 0, \quad \forall v \in K.$$

In view of the smoothness of $\theta(w)$, the following integration holds for every $\xi = (\xi_1, \xi_2) \in X$

$$(\theta(w), \xi) = -\int_\Omega (D^2\theta_1(w) \cdot \xi_1 + D^3\theta_2(w) \cdot D\xi_2)dx -$$

$$[D\theta_1(w) \cdot \xi_1 + D^2\theta_2(w) \cdot D\xi_2].$$

Relations (10)-(13) give

$$(\theta(w), \xi) = -\frac{1}{2}\Big(\Big(\phi^-(w) + \psi^-(w)\Big) \cdot [\xi_1] + \Big(\phi^-(w) - \psi^-(w)\Big) \cdot [D\xi_2]\Big) =$$

$$-\frac{1}{2}\Big(\phi^-(w) \cdot \phi(\xi) + \psi^-(w) \cdot \psi(\xi)\Big).$$

Taking into account (14), we get

$$(\theta(w), w - \theta(w) - v) = -\frac{1}{2}\Big(\phi^-(w) \cdot \Big(\phi^+(w) - \phi(v)\Big) +$$

$$\psi^-(w) \cdot \Big(\psi^+(w) - \psi(v)\Big)\Big) = \frac{1}{2}\Big(\phi^-(w) \cdot \phi(v) + \psi^-(w) \cdot \psi(v)\Big) \geq 0$$

due to $v \in K$. The proof is complete. ■

REMARK 2.1 *It follows from (5) and (6), that if $f \in H^n(\Omega) \times H^m(\Omega)$, $n, m \geq 0$, then $u \in H^{n+2}(\Omega) \times H^{m+4}(\Omega)$ (here $H^0(\Omega) = L_2(\Omega\ )$. If $f \in C^n(\Omega) \times C^m(\Omega)$, $n, m \geq 0$, then $u \in C^{n+2}(\Omega) \times C^{m+4}(\Omega)$.*

REMARK 2.2 *The function $u = w - \theta(w)$ is the solution of the following boundary problem*

$$-D^2u_1 = f_1, \quad D^4u_2 = f_2, \qquad in\ \Omega,$$

$$[Du_1] = [D^2u_2] = [D^3u_2] = 0,$$

$$Du_1 = -0.5(\phi^-(w) + \psi^-(w)),$$

$$D^2u_2 = -0.5(\phi^-(w) - \psi^-(w)),$$

$$D^3u_2 = 0, \qquad in\ x_0,$$

$$[u_1] = 0.5(\phi^+(w) + \psi^+(w)),$$

$$[Du_2] = 0.5(\phi^+(w) - \psi^+(w)).$$

REMARK 2.3 *Let function u belong to $X \cap (H^2(\Omega) \times H^4(\Omega))$ and the following boundary conditions be fulfilled in x_0*

$$[Du_1] = [D^2u_2] = [D^3u_2] = 0,$$

$$(Du_1 + D^2u_2)\phi(u) = 0,$$

$$(Du_1 - D^2u_2)\psi(u) = 0,$$

$$D^3u_2 = 0,$$

$$\phi(u) \geq 0, \quad \psi(u) \geq 0,$$

$$-Du_1 \geq |D^2u_2|.$$

Then u is the solution of variational inequality (4) with the right-hand side part $f = (-D^2u_1, D^4u_2)$. For instance, this holds when $u \in X \cap (H_0^2(\Omega) \times H_0^4(\Omega))$.

3. Examples of solutions

EXAMPLE 3.1 *Let $f_1 \equiv a, a > 0, f_2 \equiv 0$, then $u_2 \equiv 0$. There are two cases:*

- *if $0 < x_0 \leq 0.5$, then*
$$u_1 = -0.5a \begin{cases} x^2 - 2x_0x, & x \in [0, x_0-] \\ (x-1)^2 - 2(x_0 - 1)(x-1), & x \in [x_0+, 1] \end{cases},$$
$$[u_1] = (0.5 - x_0)a \geq 0,$$
- *if $0.5 \leq x_0 < 1$, then*
$$u_1 = -0.5a \cdot x(x-1), \quad [u_1] = 0.$$

EXAMPLE 3.2 *Let $f_1 = \begin{cases} a_1, & x \in [0, 0.5-] \\ a_2, & x \in [0.5+, 1] \end{cases}, x_0 = 0.5, f_2 \equiv 0$. Then $u_2 \equiv 0$ and*

- *if $a_2 \leq a_1$, then*
$$u_1 = -0.125 \begin{cases} 4a_1x^2 - (3a_1 + 1_2)x, & x \in [0, 0.5-] \\ 4a_2x^2 - (5a_2 - a_1)x - a_1 + a_2, & x \in [0.5+, 1] \end{cases},$$
$$[u_1] = 0,$$
- *if $a_2 \geq a_1$, then*
$$u_1 = -0.5a \cdot x(x-1) \cdot f_1, \quad [u_1] = 0.125(a_2 - a_1) \geq 0.$$

EXAMPLE 3.3 *Let $f_1 \equiv 0, x_0 = 0.5, f_2 = \begin{cases} b_1, & x \in [0, 0.5-] \\ b_2, & x \in [0.5+, 1] \end{cases}$. If $b_1 + b_2 \geq 0$, then*

$$u_1 = -\frac{b_1 + b_2}{96} \begin{cases} x, & x \in [0, 0.5-] \\ x - 1, & x \in [0.5+, 1] \end{cases}, \quad [u_1] = \frac{b_1 + b_2}{96} \geq 0,$$

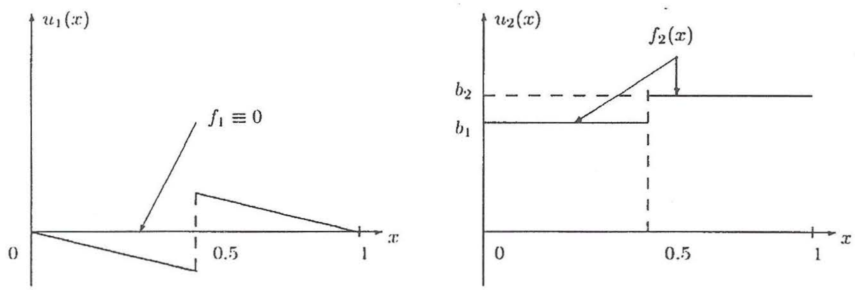

Figure 2.

$$u_2 = \frac{1}{192}\begin{cases} 8b_1x^4 - 16b_1x^3 + (11b_1 - b_2)x^2 \\ 8b_2(x-1)^4 + 16b_2(x-1)^3 + (11b_2 - b_1)(x-1)^2 \end{cases},$$

$[u_2] = (b_2 - b_1)/128$, $[Du_2] = -(b_1 + b_2)/96 < 0$ *(Fig.2).*

We may come to the following conclusions:

1. The equations ensure that $f_2 \equiv 0$ entails $u_2 \equiv 0$ (Examples 1,2).
2. $f_1 \equiv 0$ does not necessarily entail $u_1 \equiv 0$ (Example 3).
3. $[f_1] = 0$ or $[f_2] = 0$ do not guarantee $[u_1] = 0$, or $[u_2] = 0$, or $[Du_2] = 0$ (Examples 1,3).

4. Discussion of applications

In geophysics, the earth's solid surface is generally imagined as made of shells and plates. Plate faulting and their interactions are studied in the plate tectonics, see for example, Cox, Hart (1986). Such appearances are often illustrated by plates and bars with cuts. A certain version of two faulty plates is shown at a cross-section near the boundary in Fig.3a, Logatchev (1994). This figure is based on the assumption that plates are compressed. We can investigate this physical phenomenon by the mathematical model of a bar with a cut accounting for nonpenetration at the contact boundary. By omitting the nonpenetration condition, we may well obtain a crossed bar as shown in Fig.3b. Thus, the contact condition considered in the paper refers to the assumption of compression.

Certainly, the suggested mathematical model is the first approximation in describing the boundary contact. First, the friction phenomenon is of great importance. Second, the geometrical model of plate dislocation suggest the presence of slanting cuts as shown in Fig.3c, Cox, Hart (1986). These phenomena can be described by contact conditions in manner similar to the one presented here and we are engaged in the work on it.

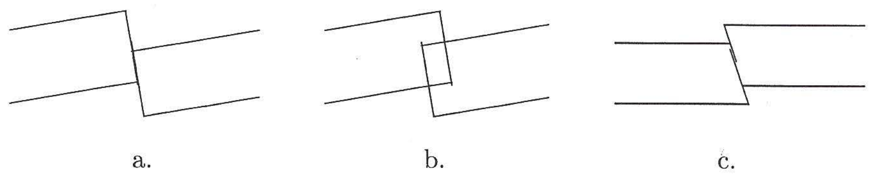

Figure 3.

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