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Approximate multidimensional polynomial factorization into general $m - D$ polynomial factors

by

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Abstract: In this paper, a solution to the approximate factorization problem of the $m - D$ polynomials is presented. More specifically, a non-factorizable multidimensional polynomial is approximately factorized into a product of general multidimensional polynomial factors in the sense of the least squares approach. The results are illustrated by means of a $2 - D$ numerical example.

Keywords: multidimensional systems, multidimensional polynomials, factorization, approximate factorization.

1. Introduction

In the recent years, much research effort has been devoted in the area of multidimensional $(m - D)$ signals and systems, Tzafestas (1986), Kaczorek (1985), Mastorakis (1988). The driving force behind this is the increasing mathematical interest and the various technical applications. A linear shift invariant multidimensional $(m - D)$ system is described by a transfer function which is a ratio of two $m - D$ polynomials.

$$G(z_1, \ldots, z_m) = \frac{q(z_1, \ldots, z_m)}{f(z_1, \ldots, z_m)} = \frac{\sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} b(i_1, \ldots, i_m) z_1^{i_1} \cdots z_m^{i_m}}{\sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} a(i_1, \ldots, i_m) z_1^{i_1} \cdots z_m^{i_m}}$$

where $N_1, \ldots, N_m$ are positive integers, and $a(i_1, \ldots, i_m)$, $b(i_1, \ldots, i_m) \in \mathbf{R}$.

If an $m - D$ polynomial can be written as a product of other lower degree polynomials, then it is said to be factorizable. Factorization of $m - D$ polynomials is one of the primary processes in the field of $m - D$ systems, since among others it helps in performing simpler realizations, applying simpler stability tests, and designing simpler controllers. More specifically, if the numerator and denominator of the transfer function $G = G(z_1, \ldots, z_m) = q/f = q(z_1, \ldots, z_m)/f(z_1, \ldots, z_m)$ are factorized as:

$$\begin{aligned} q(z_1, \ldots, z_m) &= q_1(z_1, \ldots, z_m) \ldots q_N(z_1, \ldots, z_m) \\ f(z_1, \ldots, z_m) &= f_1(z_1, \ldots, z_m) \ldots f_N(z_1, \ldots, z_m) \end{aligned}$$

where the q_i's and f_i's are obviously simpler than q and f, respectively, one has to realize the simpler $m - D$ transfer functions:

$$G_1(z_1, \ldots, z_m) = \frac{q_1(z_1, \ldots, z_m)}{f_1(z_1, \ldots, z_m)}, \ldots, G_N(z_1, \ldots, z_m) = \frac{q_N(z_1, \ldots, z_m)}{f_N(z_1, \ldots, z_m)}$$

As the stability tests are in the form: "check if $f(z_1, \ldots, z_m) = 0$ (in appropriate regions of $z_1, \ldots, z_m$)", it is important to factorize $f(z_1, \ldots, z_m)$ in factors $f_1(z_1, \ldots, z_m), \ldots, f_N(z_1, \ldots, z_m)$, because in this case the stability test is decomposed into simpler ones.

The factorization results of $m - D$ polynomials are also useful in the theory of distributed-parameter systems (DPS), which are described by partial differential equations, since the characteristic polynomials of DPS are actually $m - D$ polynomials.

Some properties of $m - D$ systems like controllability, observability, etc., are studied in a straightforward manner if $q(z_1, \ldots, z_m)$, $f(z_1, \ldots, z_m)$ are $m - D$ factorizable polynomials.

Since it is very difficult to obtain exact $m - D$ factorization, it is important to factorize approximately an $m - D$ polynomial. An appropriate norm is minimized. The minimization problem can be solved by several numerical techniques. The present paper refers to the approximation of an $m - D$ polynomial by a factorizable one of a certain type, Mastorakis (1994b), which is elucidated by a 2-D example.

2. Problem formulation and solution

It should be noted that, up to now, the general factorization problem, i.e. the factorization of any factorizable polynomial, has not yet been fully solved. For this reason, some more or less special types of $m - D$ polynomial factorization have been studied Theodorou and Tzafestas (1985a; b), Mastorakis (1990a; b; 1992a; b; 1994a; b), Chakrabarti, Bose and Mitra (1975), Chakrabarti and Mitra (1977), Collins (1973), Musser (1975), Wang (1976), Ekstrom and Woods (1976a; b), Ekstrom and Twogood (1977). Finally, there is a great interest from the pure mathematics standpoint, Mastorakis (1988).

In Theodorou and Tzafestas (1985a; b), the factorization in factors of one variable i.e. $f(z_1, \ldots, z_m) = f_1(z_1) \ldots f_m(z_m)$, or in factors with no common variables i.e. $f(z_1, \ldots, z_m) = f_1(\bar{z}_1) \ldots f_k(\bar{z}_k)$ where $\bar{z}_1, \ldots, \bar{z}_k$ are mutually disjoint groups of independent variables is analytically presented. In Mastorakis

(1990b), the factorization is succeeded by considering the given polynomial as $(1-D)$ polynomial with respect to z_j and applying the well known formulas from $1-D$ algebra. In Mastorakis (1992a), the factorization of the state-space model is investigated. In Mastorakis (1992b), the factorization of an $m-D$ polynomial in linear $m-D$ factors is completely solved. In Mastorakis (1990a), the factorization of an $m-D$ polynomial in factors where at least one factor contains no more than $m-1$ variables is fully examined. In Mastorakis (1994a), the $m-D$ factorization is achieved by factorizing appropriate lower order $m-D$ polynomials. In Mastorakis (1994b), the general type of $m-D$ polynomial factorization:

$$f(z_1,\dots z_m)=\prod_{i=1}^{N_1}\left(z_1+\sum_{\substack{i_2=0 \\ (i_2,\dots,i_m)\neq(0,\dots 0)}}^{n_2}\cdots\sum_{i_m=0}^{n_m} a_{i;i_2\dots i_m}z_2^{i_2}\cdots z_m^{i_m}+c_i\right) \tag{1}$$

is considered. Necessary and sufficient conditions are stated for this type of factorization. Papers by Mastorakis (1990a; b; 1992a; b; 1994a; b) actually publish the material of Mastorakis (1988) which is investigated by the author.

Chakrabarti, Bose and Mitra (1975; 1977) examine the concepts of reducibility and separability. Reducibility is a contiguous concept of the factorizability. Practically, it is the "factorizability" without taking into account the coefficients but only the various powers appearing in the given $m-D$ polynomial.

Collins (1973), Musser (1975) and Wang (1976) consider various cases mostly on various rings and fields. Besides Ekstrom, Woods and Twogood (1976a; b; 1977) investigate the concept of Spectral Factorization with many engineering applications. Spectral factorization is also examined in Mastorakis (1995).

Generally, it is very difficult and rare to obtain exactly a certain type of factorization. For this reason , if one type of factorization does not hold though it is desirable, the (optimum) approximation of the original given polynomial $f(z_1,\dots,z_m)$ by a factorizable (of the considered type) one is attempted. So, an unknown factorizable polynomial $\tilde{f}(z_1,\dots,z_m)$, the coefficients of which fulfill some conditions, is considered

$$\tilde{f}(z_1,\dots,z_m)=\sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m}\tilde{a}(i_1,\dots,i_m)z_1^{i_1}\cdots z_m^{i_m}$$

and the norm $\|f-\tilde{f}\|_2$ is minimized, where

$$\|f-\tilde{f}\|_2^2=\|f(z_1,\dots,z_m)-\tilde{f}(z_1,\dots,z_m)\|_2^2=$$
$$=\sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m}\left(a(i_1,\dots,i_m)-\tilde{a}(i_1,\dots,i_m)\right)^2$$

If we are not interested in a certain type of factorization, we can select the type for which the approximation is better i.e. $\|f-\tilde{f}\|_2$ is minimum. Work in approximate $m-D$ polynomial factorization can be found in the papers Mastorakis(1992c; 1993; 1994c).

In the present paper, the factorization in question is the general type (1) examined in Mastorakis (1994b). z_1 is selected as the variable for which the only existing monomial including the maximum power N_1 of z_1 is $z_1^{N_1}$, that is $a(N_1, 0, \ldots, 0) \neq 0$ (= 1 without loss of generality) and $a(N_1, i_2, \ldots, i_m) = 0$ when $i_2 + \ldots + i_m > 0$. If this does not hold, another variable z_j of $\{z_2, \ldots, z_m\}$ can be selected as z_1 and the variables z_1 and z_j are interchanged. If none variable $z_1, z_2, \ldots, z_m$ satisfies this requirement, our approximation - as one can see carrying out numerical experiments - has a great error and this type of approximate factorization is not recommended. In that case, other types of factorization may be more succesful. On the other hand, if the property $a(0, \ldots, 0, N_j, 0 \ldots, 0) \neq 0$ and $a(i_1, \ldots, i_{j-1}, N_j, i_{j+1}, \ldots, i_m) = 0$ when $i_1 + \ldots + i_{j-1} + i_{j+1} + \ldots + i_m > 0$ can be attributed for more than one variables, for example for z_1 and z_2, we select as "z_1" the variable for which the corresponding $m-D$ polynomial approximation is better i.e. the error $\|f - \tilde{f}\|_2$ is minimum.

The $m-D$ polynomial $f = f(z_1, \ldots, z_m)$ is given

$$f = f(z_1, \ldots, z_m) = z_1^{N_1} + \sum_{i_1=0}^{N_1-1} \cdots \sum_{i_m=0}^{N_m} a(i_1, \ldots, i_m) z_1^{i_1} \ldots z_m^{i_m} \tag{2}$$

(or in other words the only restriction required for $f(z_1, \ldots, z_m)$ is that there exists at least one independent variable, say z_1, such that the only existing monomial including the maximum power of z_1 is $z_1^{N_1}$)

The problem is to write it in the following form:

$$f(z_1, \ldots z_m) = \prod_{i=1}^{N_1} \left(z_1 + \sum_{\substack{i_2=0 \\ (i_2, \ldots, i_m) \neq (0, \ldots 0)}}^{n_2} \cdots \sum_{i_m=0}^{n_m} a_{i;i_2 \ldots i_m} z_2^{i_2} \cdots z_m^{i_m} + c_i \right) \tag{3}$$

In Mastorakis (1994b), two theorems provide the values of the unknown coefficients $a_{i;i_2,\ldots,i_m}$, c_i, as well as the necessary and sufficient conditions for the existence of such a factorizability.

Suppose now that the $m-D$ polynomial, given in (2), is not factorized into a product of general $m-D$ factors i.e. the necessary and sufficient conditions formulated in Mastorakis (1994b), are not satisfied.

In attempt to "factorize" f approximately an unknown factorizable polynomial $\tilde{f}(z_1, \ldots, z_m)$ is considered:

$$\begin{aligned} \tilde{f}(z_1, \ldots, z_m) &= \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} \tilde{a}(i_1, \ldots, i_m) z_1^{i_1} \ldots z_m^{i_m} \\ &= \prod_{i=1}^{N_1} \left(z_1 + \sum_{\substack{i_2=0 \\ (i_2, \ldots, i_m) \neq (0, \ldots 0)}}^{n_2} \cdots \sum_{i_m=0}^{n_m} \tilde{a}_{i;i_2 \ldots i_m} z_2^{i_2} \cdots z_m^{i_m} + \tilde{c}_i \right) \end{aligned}$$

and the norm $\|f-\tilde{f}\|_2$(or equivalently $\|f-\tilde{f}\|_2^2$) is minimized, where

$$\begin{aligned}\|f-\tilde{f}\|_2^2 &= \|f(z_1,\ldots,z_m)-\tilde{f}(z_1,\ldots,z_m)\|_2^2 = \\ &= \Bigg\| f(z_1,\ldots,z_m)-\prod_{i=1}^{N_1}\Bigg(z_1+ \\ &\qquad \sum_{i_2=0}^{n_2}\cdots\sum_{i_m=0}^{n_m}{}_{(i_2,\ldots,i_m)\neq(0,\ldots 0)} \tilde{a}_{i;i_2\ldots i_m} z_2^{i_2}\cdots z_m^{i_m}+\tilde{c}_i\Bigg)\Bigg\|_2^2 \\ &= \sum_{i_1=0}^{N_1}\cdots\sum_{i_m=0}^{N_m}\left(a(i_1,\ldots,i_m)-\tilde{a}(i_1,\ldots,i_m)\right)^2\end{aligned}$$

where the symbol $\tilde{}$ is used for the corresponding quantities of the unknown factorizable polynomial $\tilde{f}(z_1,\ldots,z_m)$.

The first order conditions for the minimum of $\|f-\tilde{f}\|_2^2$ are

$$\frac{\partial}{\partial\tilde{a}_{i;i_2,\ldots,i_m}}\|f-\tilde{f}\|_2^2=0,\ i_k=0,\ldots,n_k\ (k=2,\ldots,m)\text{ and } i=1,\ldots,N_1$$

and

$$\frac{\partial}{\partial\tilde{c}_i}\|f-\tilde{f}\|_2^2=0\ \ i=1,\ldots,N_1$$

The solution(s) of the above first-order conditions can be obtained by various numerical methods. However, it is well known that, in the sequel, one should also check second-order conditions. Besides, the numerical methods based on the first-order conditions sometimes have convergence problems. For this reason, in the relevant literature, non-linear optimization methods are recommended. In this paper, the Levenberg-Marquardt routine for solving nonlinear least squares problems is used Brown and Dennis (1972), Brown (1972),Levenberg (1944), Marquardt (1963). The problem, for this routine, is stated as follows:

$$\text{minimize}_{over}\underline{x}\{f_1^2(\underline{x})+\ldots+f_m^2(\underline{x})\}\text{ where }\underline{x}=(x_1,\ldots,x_N).$$

A sequence of approximation to the minimum point is generated by

$$\underline{x}^{n+1}=\underline{x}^n-[a_nD_n+J_n^TJ_n]^{-1}\cdot J_n^T\underline{f}(\underline{x}^n) \tag{4}$$

where J_n is the numerical Jacobian matrix evaluated at $\underline{x}^n$. D_n is a diagonal matrix equal to the diagonal of $J_n^TJ_n$ and a_n is a positive scaling constant.

3. Example

The Example refers to a $2-D$ polynomial which can be, for example, the characteristic polynomial of a $2-D$ system.

$$f(z_1,z_2)=-1.2+1.1z_1+z_1^2+1.5z_2+8z_2^2+2.8z_1z_2+z_1z_2^2+1.7z_2^3-4z_2^4 \tag{5}$$

After the calculations, it is seen that the necessary and sufficient conditions for factorization into general factors, Mastorakis (1994b), are not satisfied. So, the approximation of $f(z_1, z_2)$ by the factorizable polynomial $\tilde{f}(z_1, z_2)$ is attempted

$$\tilde{f}(z_1, z_2) = \prod_{i=1}^{2} \left(z_1 + \sum_{i_2=1}^{2} \tilde{a}_{i;i_2} z_2^{i_2} + \tilde{c}_i \right) \tag{6}$$

or in a simpler notation

$$\tilde{f}(z_1, z_2) = (z_1 + pz_2 + qz_2^2 + r)(z_1 + sz_2 + tz_2^2 + w)$$

Therefore the minimum of $\|f - \tilde{f}\|_2^2$ is considered, where:

$$\begin{aligned} \|f - \tilde{f}\|_2^2 &= (-1.2 - rw)^2 + (1.1 - r - w)^2 + (1.5 - rs - pw)^2 \\ &+ (2.8 - p - s)^2 + (8 - ps - rt - qw)^2 + (1 - q - t)^2 \\ &+ (1.7 - qs - pt)^2 + (-4 - qt)^2 \end{aligned}$$

Using the Levenberg-Marquardt routine the following solution is selected. This solution is selected as a "global" minimum among other solutions which are simply local minima.

$$p = 1.57274, q = -1.55809, r = 1.71299, s = 1.48338, t = 2.75002,$$
$$w = -.190665$$

and

$$\|f - f\|_2^2 = 3.77475$$

So we can write: $f \sim \tilde{f}$ i.e.

$$\begin{aligned} f(z_1, z_2) &\cong (1.71299 + z_1 + 1.57274z_2 - 1.55809z_2^2) \\ &\quad (-0.190665 + z_1 + 1.48338z_2 + 2.75002z_2^2) \\ &= -0.326606 + 1.52232z_1 + z_1^2 + 2.24115z_2 + \\ &\quad 3.05613z_1z_2 + 7.3408z_2^2 + 1.19193z_1z_2^2 + \\ &\quad 2.01383z_2^3 - 4.28477z_2^4 \end{aligned}$$

In Fig.1, the amplitude of the transfer function $B(z_1, z_2) = f$ is sketched when $z_1 = e^{j\omega_1}$, $z_2 = e^{j\omega_2}$ and $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$. In Fig.2, the amplitude of the transfer function $B(z_1, z_2) = \tilde{f}$ is also sketched when $z_1 = e^{j\omega_1}$, $z_2 = e^{j\omega_2}$ and $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$. In Fig.3, the amplitude of the error $E(z_1, z_2) = f - \tilde{f}$ is sketched when z_1, z_2 belong to the same domains.

In Fig.4, 5 and 6, the amplitude of the transfer functions $B(z_1, z_2) = \frac{1}{f}$, $B(z_1, z_2) = \frac{1}{\tilde{f}}$ and $E(z_1, z_2) = \frac{1}{f} - \frac{1}{\tilde{f}}$ are also sketched when z_1 and z_2 belong to the same domains too.

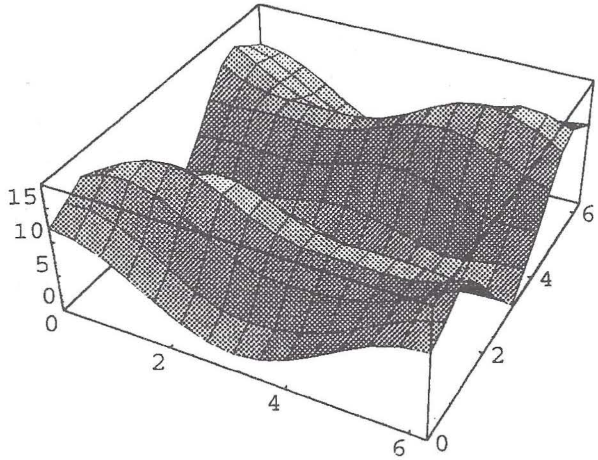

Figure 1. Abs$[f]$ versus ω_1, ω_2 : $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$

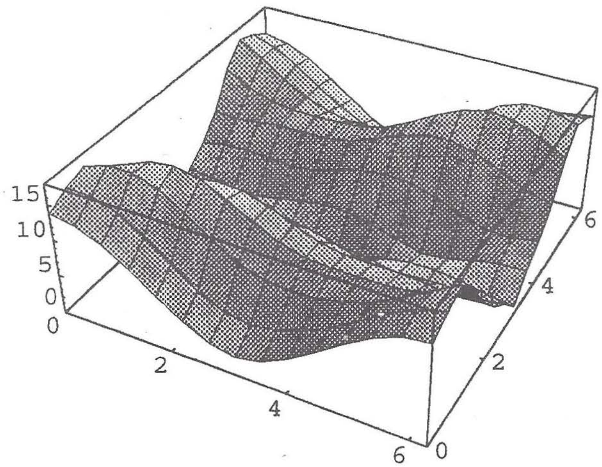

Figure 2. Abs$[\tilde{f}]$ versus ω_1, ω_2 : $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$

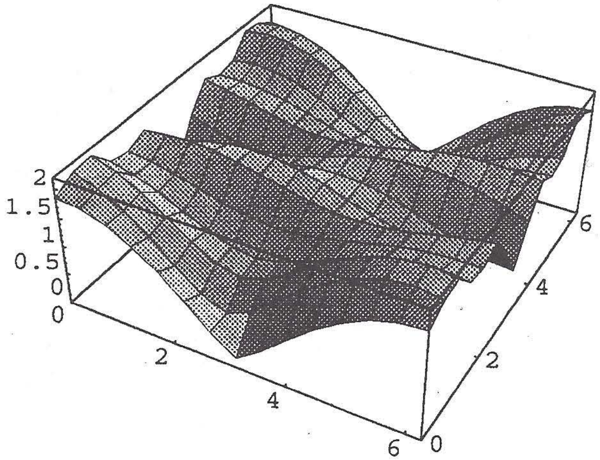

Figure 3. Abs$[f - \tilde{f}]$ versus ω_1, ω_2 : $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$

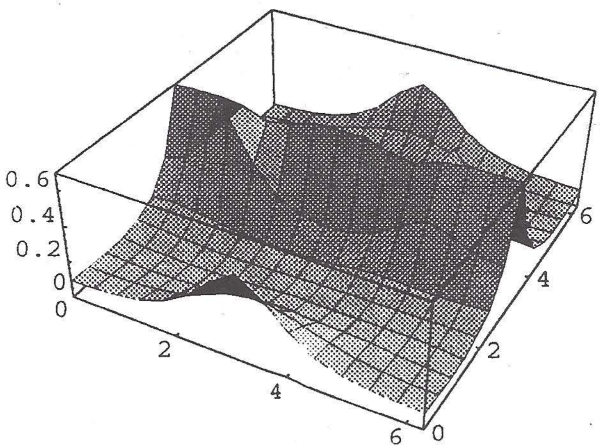

Figure 4. Abs$[1/f]$ versus ω_1, ω_2 : $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$

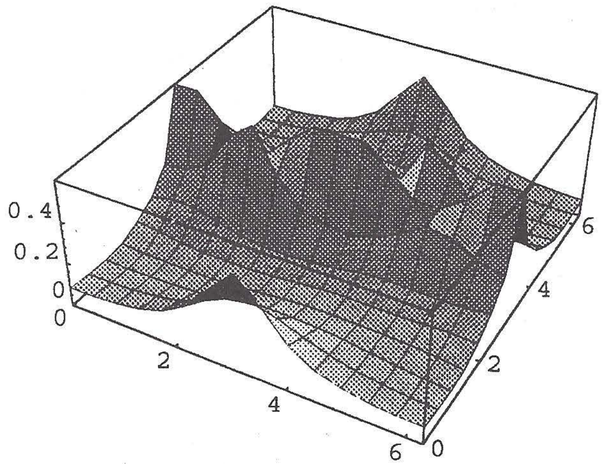

Figure 5. Abs$[1/\tilde{f}]$ versus ω_1, ω_2 : $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$

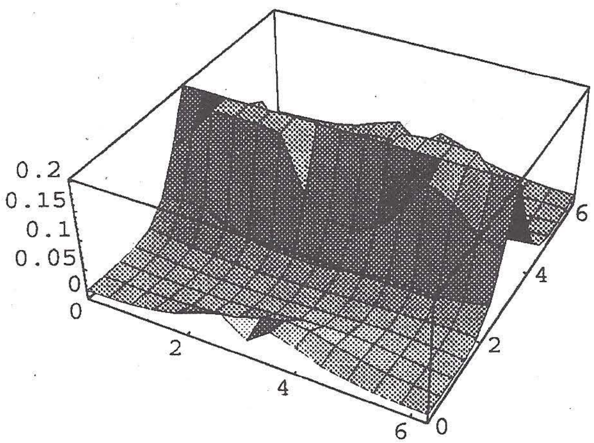

Figure 6. Abs$[(1/f) - (1/\tilde{f})]$ versus ω_1, ω_2 : $\omega_1 \in [0, 2\pi]$, $\omega_2 \in [0, 2\pi]$

4. Conclusion

An $m-D$ polynomial which is not exactly factorized into general $m-D$ polynomial factors is considered. This polynomial can be approximately factorized into general $m-D$ factors in the sense of the least square approach. This simple technique can be proved very useful in $m-D$ filters and $m-D$ networks design since, in the most cases, the exact $m-D$ factorization is impossible. An example is given in which, finally, we compare a transfer function having the original (unfactorizable) polynomial as numerator or denominator and a transfer function having the corresponding factorizable polynomial as numerator or denominator respectively.

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