Control and Cybernetics

vol. **25** (1996) No. 5

Some basics in optimal control of domains

by

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Abstract: The present work is devoted to the theoretical and numerical presentation of the basic shape optimal design approach with, as a learning example, the harmonic linear acoustics model. Some numerical results are presented.

1. Introduction

The general problem is to find, amongst all of some *admissible* domains *i.e.* belonging to a prescribed class of domains, the one that minimizes a given cost $\jmath(\Omega)$.

The cost is usually obtained by means of an *observation* functional J_Ω:

$$\jmath(\Omega) = J_\Omega(u_\Omega)$$

where u_Ω is itself the solution of some *partial differential equation* posed over the domain Ω.

The bibliography related to this field is very large, see e.g. Céa (1986), Simon, Murat (1976), Masmoudi (1987), Sokolowski and Zolesio (1992), Pironneau (1984), Habbal (1992), etc.

In our case study, the optimal design problem is the following:

Let Ω be a bounded open subset of $\mathbb{R}^2$, with a regular enough boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. A time-harmonic acoustical source, located inside Ω, generates an acoustical pressure p_Ω, which depends on the shape of the domain Ω. Given a subset Z of Ω, the optimization problem is to find the shape of Γ_1 which minimizes an *objective function*, explicitly depending on p_Ω and, eventually, on Ω.

The problem set above is an approximation of real-world situations, where the source is machinery, air duct or any noise generating system, and Z is a region to be protected against the noise annoyance.

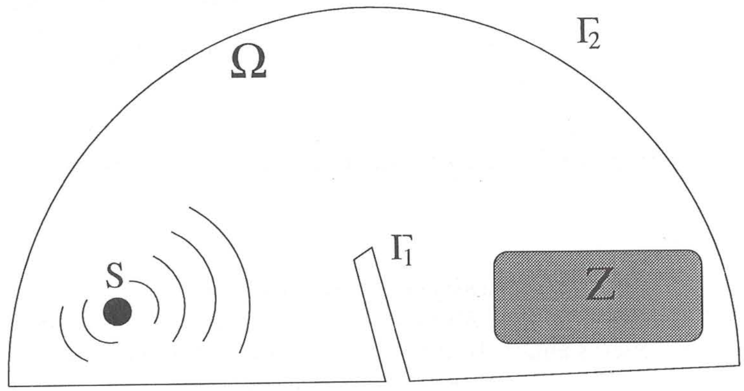

Figure 1. Example of domain geometry.

In Section 2., we recall the classical mathematical framework, where the *Helmholtz* equations are stated. Then, is Section 3., we define the basic mathematical tools that are needed in the framework of optimal domain design, with an emphasis on the *domain derivative concept*. Then, in Section 4., we apply the results of the Section 3. to the Helmholtz model. Afterwards, Section 5. describes with details the numerical implementation, with some remarks on the optimization software design.

Finally, in Section 6., we present some numerical optimization results, for two differentiable cost functions and a nondifferentiable one.

2. The Helmholtz equation

The propagation of acoustical waves of small amplitude, in a compressible, inviscid, isotropic homogeneous fluid, is governed by the *linearized Euler equations*, see e.g. Morand, Ohayon (1992).

When the emitting sources are time-harmonic, the resulting acoustical pressure is too. We assume, for simplification, that the domain occupied by the fluid is bounded, with a *sound-hard* boundary. Then, if the acoustical source is of the form

$$S(x,t) \;=\; S(x)cos(\omega t)$$

the pressure field, solution of the linearized Euler equation, is of the same form,

$$p(x,t) \;=\; p(x)cos(\omega t)$$

where x is the spatial variable, t the time variable, and ω the pulsation of the source.

The unknown of the problem is now $p(x)$, which is the solution to the Helmholtz equation:

$$\left\{ \begin{array}{rcl} -\Delta p - k^2 p & = & S \ in \ \Omega \\ \dfrac{\partial p}{\partial n} & = & 0 \ over \ \partial\Omega \end{array} \right. \tag{1}$$

where $k = \frac{\omega}{c}$ is the wave number, c is the speed of sound inside the fluid.

Since the domain Ω is bounded, the operator $-\Delta$ has real, positive eigenvalues depending on Ω. We assume that k^2 *is not one of them*, which implies that for any source $S \in L^2(\Omega)$, there exists one and only one solution p to the Helmholtz equation (1), known to belong to the Sobolev space $H^1(\Omega)$, see e.g. Trudinger, Gilbarg (1977).

The above equation has an equivalent *variational form*:

Find $p \in H^1(\Omega)$ such that

$$\forall q \in H^1(\Omega), \quad a_\Omega(p,q) = L_\Omega(q) \tag{2}$$

where

$$a_\Omega(p,q) = \int_\Omega \nabla p \nabla q d\Omega \quad - k^2 \int_\Omega p.q d\Omega$$

$$L_\Omega(q) = \int_\Omega S.q d\Omega$$

Remark 2.1 *It is shown in Rousselet (1982), that the mapping $\Omega \longmapsto \lambda_\Omega$, where λ_Ω is an eigenvalue of the Laplace operator, is* continuous. *Then, if k^2 is not an eigenvalue for a given Ω, it is not one for any domain sufficiently close to Ω.*

The optimization process will perturbate the initial domain Ω, giving a new domain Ω', and we have to check, numerically at least, that k^2 is not an eigenvalue of Ω'.

3. Optimal domain design

In the classical optimal control problems, the space of the control variables is generally a *Banach* or a *Hilbert* one, which allows to define and compute *derivatives* and *gradients* that can be used as *descent directions* in optimization procedures.

The problem in our context is that *sets of domains in $\mathbb{R}^2$ have no vector space structure!* making it impossible to define classical derivatives with respect to the domain variable...

In order to bypass this lack of structure, we shall use a *transport method*, see Simon, Murat (1976), which consists in choosing a *reference fixed domain* $\hat{\Omega}$, regular open bounded subset in $\mathbb{R}^2$, and then define *generic* admissible domains

Ω as the images of $\hat{\Omega}$, by generic mappings $\hat{T}$, elements of the Banach space $W^{1,\infty}(\hat{\Omega};\ \mathbb{R}^2)$[1].

The differentiability framework could now be set in the above Banach space, the transport method being the tool that makes the link between the classical derivative notion in a Banach space, and the domain derivative concept.

Remark 3.1 *The reason for the choice of the space $W^{1,\infty}$ instead of more regular spaces $W^{m,p}$ is twofold:*

- *it is* large enough *to not smooth corners (which are often present in acoustics problems).*
- *it is* restricted enough *to not allow creation of* cracks *(loss of regularity for the u_Ω variable).*

Remark 3.2 *The transport method does not permit* topological modifications *of the domain. All the admissible domains as defined below, are topologically similar to the reference domain. Therefore, within a coupled acoustic-structure framework for example, it is not possible, using the transport method, to create or remove* holes *in a radiating structure.*

When looking for such a modification, specific approaches must be used, see e.g. Kohn, Allaire (1993).

3.1. The domain derivative

Let us make some notations more precise:

$\hat{\Omega}$ is a fixed open bounded subset of $\mathbb{R}^2$, with a piecewise C^1 boundary,

$\hat{W} \ = \ W^{1,\infty}(\hat{\Omega};\ \mathbb{R}^2)$

$\hat{U} \ = \ \{\hat{T} \in \hat{W},\ \hat{T} \textit{ is bijective},\ \hat{T}^{-1} \in W^{1,\infty}(\hat{T}(\hat{\Omega});\ \mathbb{R}^2)\}$

$\hat{U}$ is an open subset of $\hat{W}$.

The *set of domains* that will be considered is:

$$D(\hat{\Omega}) \ = \ \{\Omega \ = \ \hat{T}(\hat{\Omega}) \ / \ \hat{T} \in \hat{U}\}$$

Let now be a generic domain functional, j, such that:

$$\begin{array}{rcl} j \ : \ D(\hat{\Omega}) & \longrightarrow & \mathbb{R} \\ \Omega & \longmapsto & j(\Omega) \end{array}$$

we then associate to j, a functional $\hat{\jmath}$ defined by:

$$\begin{array}{rcl} \hat{\jmath} \ : \ \hat{U} & \longrightarrow & \mathbb{R} \\ \hat{T} & \longmapsto & \hat{\jmath}(\hat{T}) = j(\Omega) \\ & & \textit{where } \Omega \ = \ \hat{T}(\hat{\Omega}) \end{array} \tag{3}$$

[1] $W^{1,\infty}(\hat{\Omega};\ \mathbb{R}^2) = \left\{\hat{T} : \hat{\Omega} \mapsto \mathbb{R}^2 \ s.t.\ \hat{T} \in L^\infty(\hat{\Omega}),\ D\hat{T} \in L^\infty(\hat{\Omega})\right\}$ where $D\hat{T}$ is the Jacobian matrix of $\hat{T}$.

The functional $\hat{j}$ is now defined over an open subset of the Banach space $\hat{W}$, and we can define the *Frechet derivative* of $\hat{j}$, in any element $\hat{T}_0 \in \hat{U}$.

The *domain derivative* of j, in a given $\Omega_0 \in D(\hat{\Omega})$ is therefore defined as follows:

Definition 3.1 *Let be $\hat{T}_0 \in \hat{U}$ such that $\hat{T}_0(\hat{\Omega}) = \Omega_0$.*

The functional j is said to be differentiable in Ω_0, if the mapping $\hat{j}$ is Frechet differentiable in $\hat{T}_0$. We then set:

$$\frac{d}{d\Omega} j(\Omega_0).V = \frac{d}{d\hat{T}} \hat{j}(\hat{T}_0).(V \circ \hat{T}_0) \qquad \forall V \in W^{1,\infty}(\Omega_0;\ I\!R^2) \tag{4}$$

Remark 3.3 *It can be easily shown that the domain derivative is independent from the mapping which represents the domain Ω_0.*

Remark 3.4 *When a domain functional j is differentiable, its* directional derivative $\frac{d}{d\Omega} j(\Omega).V$ *as defined above corresponds to the intuitive* finite difference *approximation:*

$$\frac{d}{d\Omega} j(\Omega).V = \lim_{t \to 0} \frac{j(\Omega + tV) - j(\Omega)}{t}$$

From a numerical *point of view, this is a simple and good tool to check the* validity *of derivative values, computed using e.g. the* adjoint state method.

Remark 3.5 *It is proved, in Simon, Murat (1976), under some regularity assumptions on the vector field V (V of bounded second derivatives), that for* any *differentiable domain functional, the derivative depends only on the trace of V over the boundary.*

3.2. The adjoint state method

Using the same notations as above, our optimal design framework can be stated as follows:

- $\hat{W}$ is the Banach space of *controls*,
- $\hat{U}$ is an open subset of *admissible* controls of $\hat{W}$,
- For any $\hat{T} \in \hat{U}$, we associate a *Hilbert* space $\mathcal{V}(\Omega)$, where $\Omega = \hat{T}(\hat{\Omega})$, and a *state equation*:
  Find $u_\Omega \in \mathcal{V}(\Omega)$ such that
  $$\forall v \in \mathcal{V}(\Omega), \quad a_\Omega(u_\Omega, v) = L_\Omega(v) \tag{5}$$
  The solution u_Ω is called *the direct state variable.*
- a_Ω is a continuous bilinear form over $\mathcal{V}(\Omega) \times \mathcal{V}(\Omega)$. It is assumed symmetric and *elliptic*[2]

[2] The ellipticity condition is, in fact, sufficient but not necessary. The same results hold if the *associated linear operator* defined by $(Au, v) = a(u, v)$ is *an isomorphism* from $\mathcal{V}$ onto its dual $\mathcal{V}'$.

- L_Ω is a continuous linear form over $\mathcal{V}(\Omega)$
  It is demonstrated, in Rousselet (1982), that if the mappings $\Omega \longmapsto a_\Omega$ and $\Omega \longmapsto L_\Omega$ are differentiable, then the mapping $\Omega \longmapsto u_\Omega$ is also differentiable (in the sense of *definition 4*).
- Given an observation:
  $$\begin{array}{rcl} J_\Omega : \mathcal{V}(\Omega) & \longrightarrow & I\!R \\ v & \longmapsto & J_\Omega(v) \end{array}$$
  we define *the cost* j by : $j(\Omega) = J_\Omega(u_\Omega)$
  If we assume that the mappings $\Omega \longmapsto J_\Omega$ and $v \longmapsto J_\Omega(v)$ are continuously differentiable, then, using the composition theorem, we conclude that the cost j is differentiable with respect to the domain variable.
- We then define the *adjoint state* variable, as the unique solution of the *adjoint state equation*:
  Find $p_\Omega \in \mathcal{V}(\Omega)$ such that
  $$\forall w \in \mathcal{V}(\Omega), \quad a_\Omega(p_\Omega, w) = \frac{\partial J_\Omega}{\partial v}(u_\Omega).w \tag{6}$$
- The domain derivative of the cost j is then given by:
  $$\frac{d}{d\Omega} j(\Omega).V = -\frac{\partial a_\Omega}{\partial \Omega}(u_\Omega, p_\Omega).V + \frac{\partial L_\Omega}{\partial \Omega}(p_\Omega).V + \frac{\partial J_\Omega}{\partial \Omega}(u_\Omega).V$$
  $$\forall\, V \in W^{1,\infty}(\Omega; I\!R^2) \tag{7}$$

From a *numerical* point of view, the adjoint state variable can be computed, using e.g. the *same* finite element solver that computes the direct state.

The computation of the derivative of j requires, however, the implementation of the subroutines

$$(\Omega, u, p, V) \longrightarrow \frac{\partial a_\Omega}{\partial \Omega}, \frac{\partial L_\Omega}{\partial \Omega}, \frac{\partial J_\Omega}{\partial \Omega}$$

We give, hereafter, some formulas (without proof) that may be useful, at least when implementing domain derivatives for *second order elliptic* problems.

3.3. Formulas

- $\Omega \subset I\!R^2$ is an open domain, with a piecewise C^1 boundary,
- τ : unitary tangent vector to the boundary $\partial\Omega$,
- $f,\ g\ :\ \Omega \longrightarrow I\!R$
  $h\ :\ \partial\Omega \longrightarrow I\!R$
  are given functions,
- $V = (V_1,\ V_2) \in W^{1,\infty}(\Omega;\ I\!R^2)$,
- $V_{i,j} = \frac{\partial V_i}{\partial x_j}$
- $div(V) = V_{1,1} + V_{2,2}$
- $(DV)_{ij} = (V_{i,j})$ (DV) is the Jacobian matrix of V,
- $[DV] = (DV) + (DV)^T$

Using these notations, one gets:

$$\begin{aligned}
\frac{\partial}{\partial\Omega}\left(\int_\Omega f d\Omega\right).V &= \int_\Omega f.div(V)d\Omega \\
\frac{\partial}{\partial\Omega}\left(\int_\Omega \nabla f.\nabla g\, d\Omega\right).V &= \int_\Omega \nabla f.\nabla g\, div(V)d\Omega - \int_\Omega <[DV]\nabla f, \\
&\quad \nabla g > d\Omega \\
\frac{\partial}{\partial\Omega}\left(\int_{\partial\Omega} h d\Gamma\right).V &= \int_{\partial\Omega} <\tau,(DV)\tau> h\, d\Gamma
\end{aligned} \tag{8}$$

Remark 3.6 *Under regularity assumptions, the two first domain integrals above can be expressed as boundary integrals. Instead of being cost-less, this approach leads to* worse *numerical results, see Masmoudi (1987), which may cost more expensive finite element analyses, during the optimization process.*

4. The application to the Helmholtz model

Within the optimal design framework set in the previous section, we consider a given reference domain $\hat{\Omega}$, e.g. the one of Fig. 1 with a rectangular barrier Γ_1.

The space of controls $\hat{W}$, the open subset $\hat{U}$ of $\hat{W}$ and the set of admissible domains are the following:

- $\hat{W} = W^{1,\infty}(\hat{\Omega};\ I\!R^2)$
- $\hat{U} = \{\hat{T} \in \hat{W},\ \hat{T}\ is\ bijective,\ \hat{T}^{-1} \in W^{1,\infty}(\hat{T}(\hat{\Omega});\ I\!R^2)\}$
- $D(\hat{\Omega}) = \{\Omega = \hat{T}(\hat{\Omega})\ /\ \hat{T} \in \hat{U}\}$

4.1. The state equation

To any domain $\Omega \in D(\hat{\Omega})$, we associate a *direct state* equation:

$$\left\{\begin{aligned} -\Delta p_\Omega - k^2 p_\Omega &= S_\Omega \quad over\ \Omega \\ \frac{\partial p_\Omega}{\partial n} &= 0 \quad over\ \partial\Omega \end{aligned}\right. \tag{9}$$

If the acoustical source S_Ω belongs to the space $L^2(\Omega)$, then the solution p_Ω belongs to the space $\mathcal{H}(\Omega)$, defined by:

$$\mathcal{H}(\Omega) = \left\{ v \in H^1(\Omega) \cap H^2(\Omega');\ \frac{\partial v}{\partial n} = 0 \right\}$$

where $\Omega' \subset \overline{\Omega'} \subset \Omega$ is any open regular set, sufficiently far from the boundary of Ω, see e.g. Necas (1967).

The strong linear operator $A(\Omega)$ associated to the Helmholtz equation:

$$A(\Omega)\ : \mathcal{H}(\Omega) \longrightarrow L^2(\Omega) \tag{10}$$

$$p \longmapsto A(\Omega).p = -\Delta p - k^2 p \tag{11}$$

is then *an isomorphism* from $\mathcal{H}(\Omega)$ onto $L^2(\Omega)$.

As the mapping $\Omega \longmapsto A(\Omega)$ is differentiable, see Simon, Murat (1976), we conclude using the implicit function theorem, that, provided the mapping $\Omega \longmapsto S_\Omega$ is too, the mapping $\Omega \longmapsto p_\Omega$ is differentiable from $D(\hat{\Omega})$ onto $\mathcal{H}(\Omega)$, and hence, *a fortiori*, from $D(\hat{\Omega})$ onto the Sobolev space $H^1(\Omega)$. This allows us to consider the equivalent weak form:

$$\exists! p_\Omega \in H^1(\Omega), \quad \forall q \in H^1(\Omega), \quad a(\Omega; p_\Omega, q) = L(\Omega; q) \tag{12}$$

where:
- $a(\Omega; p, q) = \int_\Omega \nabla p \nabla q d\Omega - k^2 \int_\Omega p.q d\Omega$
- $L(\Omega; q) = \int_\Omega S.q d\Omega$

which will be subsequently used in order to derive the adjoint state equation, and to compute the cost gradient.

4.2. The observation

The *target zone* i.e. the region where we want to minimize noise, denoted Z, is an open, regular enough, subset of the whole domain Ω, which is assumed to be *far enough* from the acoustical source, and from the boundary of Ω.

We also make the assumption that Z is stable for any mapping $\hat{T} \in \hat{U}$ i.e. $\hat{T}(\hat{Z}) = \hat{Z} = Z$.

This is a natural and realistic assumption, since the *target element* i.e. the geometrical piece of the domain to re-design is the barrier Γ_1, and *not* the sensitive area Z!

Then, we consider a family of observations:

$$J_m(p) = \left(\int_Z |p(x)|^m dZ \right)^{\frac{1}{m}} \qquad 2 \leq m < +\infty$$

It is important to notice that the considered observations *do not depend* on Ω.

Due to the Sobolev embedding theorems, the space $H^1(\Omega)$ can be continuously embedded in $L^m(\Omega)$ for $2 \leq m < +\infty$. Hence, the observations $J_m(p)$ are well defined for any p, element of $H^1(\Omega)$.

The *cost function*, given by $j_m(\Omega) = J_m(p_\Omega)$, is differentiable from $D(\hat{\Omega})$ onto $I\!R$ w.r.t. the domain variable Ω (in the sense of *definition 4*), as a composition of the differentiable mappings $p \longmapsto J_m(p)$ and $\Omega \longmapsto p_\Omega$.

4.3. The domain derivative

In order to compute the domain derivative of the cost j_m, we introduce an adjoint state variable $p^*_\Omega \in H^1(\Omega)$, which is the unique solution to the *adjoint state equation*:

$$\exists! p^*_\Omega \in H^1(\Omega), \quad \forall q \in H^1(\Omega), \quad a(\Omega; p^*_\Omega, q) = \frac{\partial J_m}{\partial v}(\Omega; p_\Omega).q \tag{13}$$

where

$$\frac{\partial J_m}{\partial v}(\Omega;\, p_\Omega).q \;=\; \left(\int_Z |p(x)|^m dZ\right)^{\frac{1}{m}-1} . \int_Z |p(x)|^{m-1}.q\, dZ$$

The derivative of $j_m(\Omega)$ in a given Ω, along some direction $V \in W^{1,\infty}(\Omega;\, I\!R^2)$ is then obtained by the formula (7). Using the results of section 3.3. and the same notations, one has:

$$\begin{aligned}\frac{d}{d\Omega} j_m(\Omega).V \;&=\; -\int_{\Omega/Z} \nabla p_\Omega.\nabla p_\Omega^*\, div(V) d\Omega + \\ &+ \int_{\Omega/Z} < [DV]\nabla p_\Omega,\, \nabla p_\Omega^* > d\Omega \\ &+ \int_{\Omega/Z} S.p_\Omega^*\, div(V) d\Omega \qquad (14)\end{aligned}$$

Remark 4.1 *Remember that the target zone must remain stable when the initial domain Ω is perturbed by a field V, giving a domain $\Omega + V(\Omega)$. In order to assure this, one can impose the condition $V = 0$ over the region Z.*

That is the reason why the above integrals are set over Ω/Z, and not over the whole Ω.

5. Numerical implementation

There are mainly three kinds of approximation that intervene in shape optimal design:

- *The domain approximation*, which consists in discretizing the infinite dimensional space of controls, using geometrical approximation tools, e.g. *cubic splines*, to approximate the boundary of the domain.
- *The direct and adjoint state approximation*
- *The gradient approximation*

In the following, we give more details about each of the above approximations.

5.1. The domain approximation

In a first step, we choose to approximate the general shape of the target element by *cubic splines*, as shown Fig.2, using a finite number N of interpolation nodes, which are called *master nodes*. The optimization variables become now *the master nodes coordinates.*

In order to get a finite dimensional space W_N, which approximates the space $W^{1,\infty}(\Omega;\, I\!R^2)$, necessary for the gradient approximation, we first compute the N basic elements $V_i,\ i = 1, ..., N$ such that:

$$\left\{\begin{array}{rcl} -\Delta V_i &=& 0 \ \ over\ \Omega/Z \\ V_i &=& 0 \ \ over\ \Gamma_2 \cup \partial Z \\ V_i &=& s_i \ \ over\ \Gamma_1 \end{array}\right. \qquad (15)$$

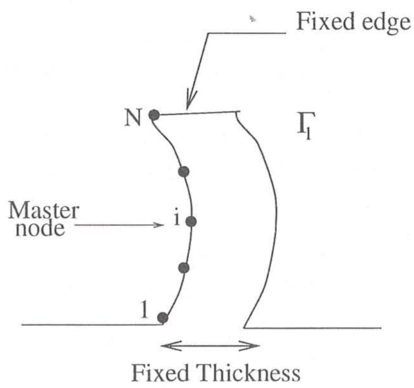

Figure 2. Approximation of the target element

where the functions s_i are *the N basic cubic S-splines*[3] depending on the N master nodes.

The discrete space W_N is then the one *spanned* by the V_i's, i.e. the successive domains will be of the form:

$$\Omega^{k+1} = \Omega^k + \sum_{i=1}^{i=N} \alpha_i V_i(\Omega^k)$$

5.2. The state variables approximation

The method used to compute approximations of the (V_i) defined above, of the acoustical pressure and of the adjoint state variable is the *Finite Element Method (FEM)*.

In the present paper, we used piecewise linear triangular finite element approximation, which is of first order precision. The choice of this low order method is guided by the fact that the optimization process needs many FE computations, and one has to make a compromise between cheap computations and precise results. The method used turned out to be optimal from this point of view.

5.3. The gradient approximation

The exact derivative of j_m w.r.t. the domain variable is given by the formula (14). This derivative is called *a continuous derivative*, in the sense that it is *directly derived from the continuous model.* We define the *discretized continuous*

[3]S-spline basis is **not** B-spline one. In particular, one has $s_i(x_j) = \delta_{ij}$ where x_j is the jth master node

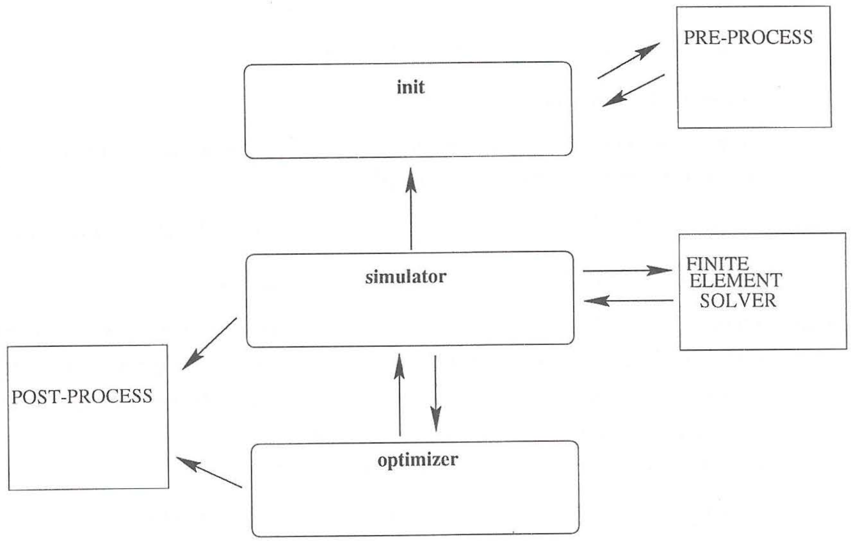

Figure 3. The general software structure

gradient DCG as a numerical approximation to the quantity:

$$G_i = \frac{d}{d\Omega} j_m(\Omega).V_i \tag{16}$$

As the domain Ω is triangulated, and approximate numerical values of the V_i, p_Ω *and* p^*_Ω are provided by the FE solver, it is possible to compute a numerical approximation to the G_i's, by means of any *quadrature formula* that approximates accurately volume or surface integrals.

Remark 5.1 *It is shown, in Masmoudi (1987), that the order of precision of the approximate value of G_i is the same as that of the finite element method used to compute the direct and adjoint states.*

5.4. The software design

The structure of the implemented software is the one recommended by the MODULOPT project of INRIA, France, as shown in Fig. 3. The keywords are *simulator* and *optimizer*.

- **init :** read initial data; select master nodes; compute the V_i; ...etc
- **simulator :**
  - given a new geometry of the target element $\Gamma_1^{(k)}$, update the mesh, and save the new domain $\Omega^{(k)}$,
  - test the mesh quality : re-mesh and go to **init** step if necessary,

- compute the direct state approximation $p^{(k)}$,
- evaluate the cost $j_m^{(k)}$,
- update the second member of the adjoint state equation, and compute the adjoint state approximation $p^{*(k)}$,
- given $\Omega^{(k)}$, the V_i's, $p^{(k)}$ and $p^{*(k)}$, compute the $DCG^{(k)}$ components,
- output $\Gamma_1^{(k)}$, $j_m^{(k)}$ and $DCG^{(k)}$ to the optimizer

- **optimizer :** It is the actual *controller* of the whole optimization process. It runs the optimization algorithm, asks the simulator to output $j_m^{(k)}$ and $DCG^{(k)}$ for an input $\Gamma_1^{(k)}$, and then, performs a new improved geometry $\Gamma_1^{(k+1)}$, which is a new input to the simulator...etc.
The process stops when one of the following tests is true:
  - maximum of simulation/optimization iterations reached,
  - $\|\Gamma_1^{(k+1)} - \Gamma_1^{(k)}\| <$ some tolerance parameter,
  - $\|j_m^{(k+1)} - j_m^{(k)}\| <$ some tolerance parameter,
  - $\|DCG^{(k+1)} - DCG^{(k)}\| <$ some tolerance parameter

6. Numerical optimization results

6.1. The optimization model

- The domain Ω is the square unit, with hard-sound boundaries.
- The acoustical source is a Dirac point-wise distribution, with origin located at (0.9,0.2) and magnitude 1.
- The sensitive area Z is a rectangular zone, as represented in Fig. 4.
- The target element, i.e. the design variable, is the left side of the square, approximated by cubic splines. The master nodes are allowed to move horizontally, with a *box constraint* i.e. all the abscissae x_i of the nodes fulfill the condition $-0.05 \leq x_i \leq 0.05$.

This model configuration may e.g. correspond to the situation where a machine user is in front of a central processor unit box, which is made of sound-hard material (seek of comfort for the user). The ventilator fixed at the back side, generates harmonic acoustical waves that may deteriorate the computer performances. We then seek for the front side the geometries that minimize the effects of the ventilator on the electronic chips (this is a simplified approach; in fact, a pertinent model must take into account at least the coupled vibro-acoustic behavior.)

The optimization computations are done for:

- different wave numbers : $k_1 = 1,\ k_2 = 4,\ k_3 = 10$

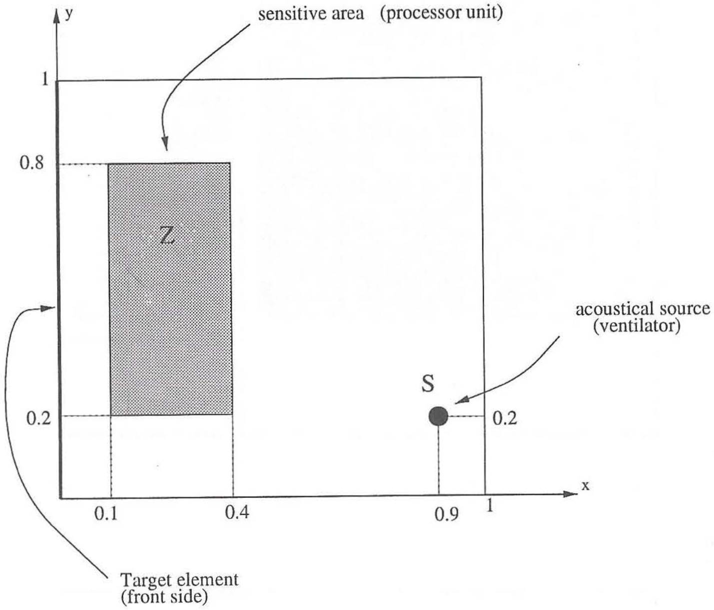

Figure 4. Description of the model configuration.

- different cost functions :

$$\begin{aligned} j_2(\Omega) &= \int_Z |p_\Omega(x)|^2 dZ \\ j_{10}(\Omega) &= \left(\int_Z |p_\Omega(x)|^{10} dZ\right)^{1/10} \\ j_\infty(\Omega) &= \max_{x\in\overline{Z}} |p_\Omega(x)|^2 \end{aligned}$$

6.2. Optimization results

The MODULEF finite element library from INRIA was used in order to compute the direct acoustical pressure and its adjoint.

The optimization algorithms used are a Quasi-Newton program from MODULOPT project (INRIA, France) for the minimization of the differentiable costs j_2 and j_{10}, and the bundle algorithm M2FC1 (from MODULOPT) for the minimization of the nonsmooth cost j_∞.

We present some results in Figs. 5-9, obtained for different wave numbers: $k = 1$, $k = 4$ and $k = 10$. The figures show the optimal geometry and pressure distribution, starting from the above initial configuration.

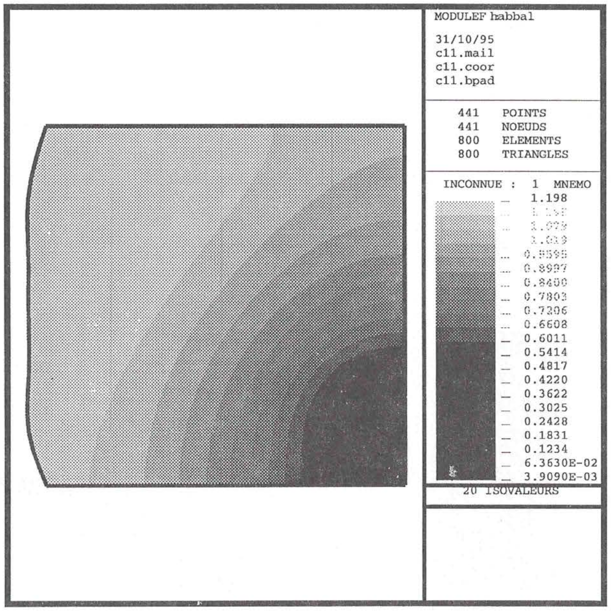

Figure 5. Optimal geometry and pressure distribution for the cost j_2. $k = 1$

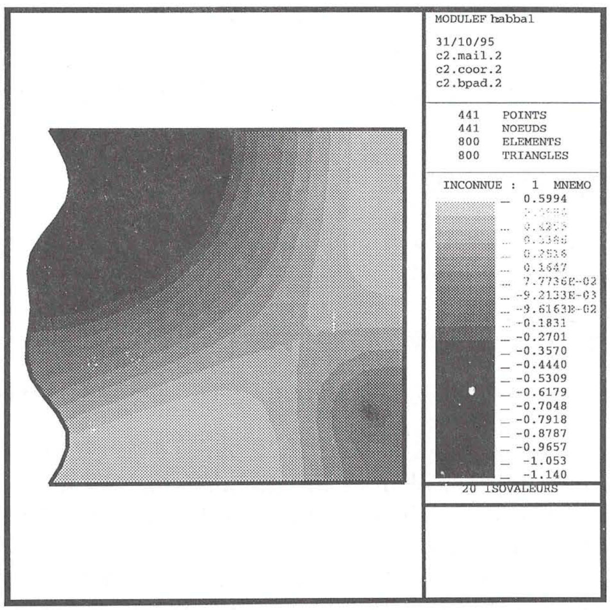

Figure 6. Optimal geometry and pressure distribution for the cost j_2. $k = 4$

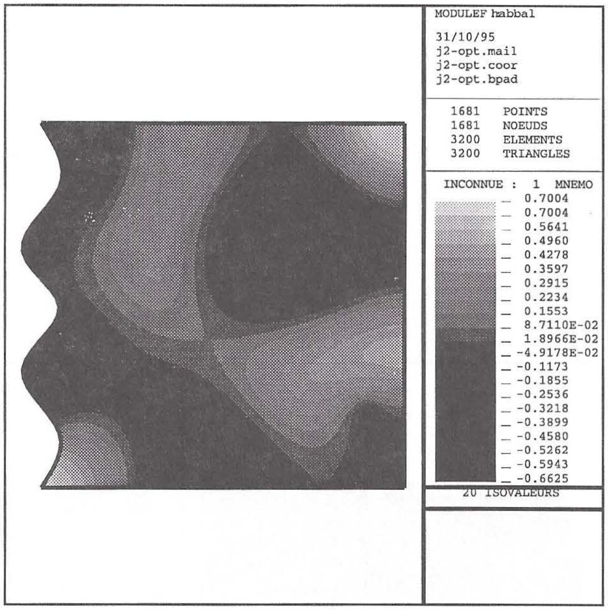

Figure 7. Optimal geometry and pressure distribution for the cost j_2. $k = 10$

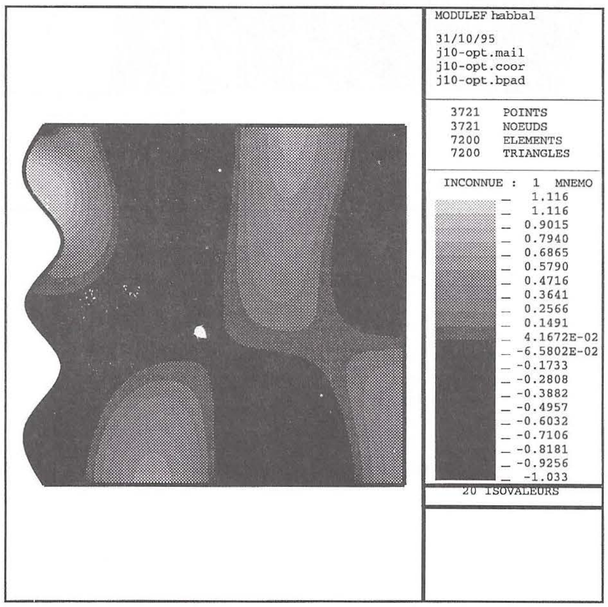

Figure 8. Optimal geometry and pressure distribution for the cost j_{10}. $k = 10$

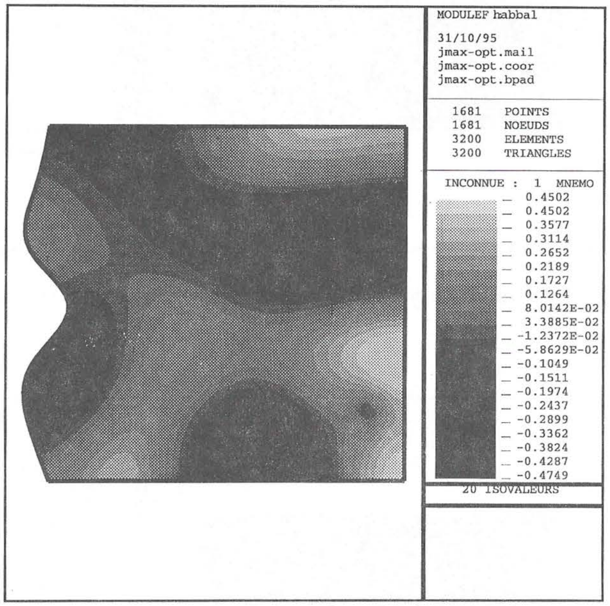

Figure 9. Optimal geometry and pressure distribution for the cost j_{∞}. $k = 10$

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