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Shape and material parameters sensitivity analysis in quasi-harmonic problems

by

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Abstract: A general quasi-harmonic boundary-value problem within an anisotropic domain is formulated. Considering an arbitrary behavioural functional, its first-order sensitivities with respect to variation of domain shape and material parameters are derived using the direct and adjoint approaches.

1. Introduction

In the past few years there has been a great deal of interest in methods of calculating the sensitivity of behavioural response with respect to shape and material parameter variations for a wide class of boundary-value problems. The present paper constitutes an extension of previous works in this area done by Dems (1986;1987) and Dems and Mróz (1995), the main interest focusing on the quasi-harmonic boundary-value problems. The variational approach to shape and material parameter sensitivity for an arbitrary behavioural functional associated with boundary-value problem with mixed Dirichlet and Neumann boundary conditions is discussed in details. Céa (1981) classified various domain optimization problems for such type of boundary-value problems. These optimization problems are related, in some sense, to isoparametric problems. Rousselet (1983) considered the shape sensitivity for harmonic problems in structural mechanics, whereas Koda (1984) and Pironneau (1974) discussed these types of shape optimization problems in fluid mechanics. Meric (1988) discussed the shape sensitivity analysis for non-linear anisotropic heat conduction problem and shape optimization by thè BEM. Recently, a systematic approach to shape sensitivity analysis for Laplace problem was presented by Tortorelli & Wang (1993). The first- and second-order sensitivity analysis for domain functional depending on state fields for Dirichlet problem was considered by Fujii (1990) using the Hadamard (1968) method. A similar analysis was performed by Goto et al. (1990) and Simon (1986) for Neumann type problems.

In the present paper much more general form of functional is considered for boundary-value problem with mixed boundary conditions specified within an

anisotropic domain. The material derivative (or rate) concept, widely used in sensitivity analysis cf. for instance Zolesio (1981), Haug et al. (1986), Dems & Haftka (1989) and others, is used in considering the shape modification of problem domain. The derivation of first-order sensitivities is performed by using the direct and adjoint approaches. The respective variation of the considered functional will be thus expressed in terms of the transformation velocity field specifying the shape modification or derivatives of quantities specifying the material properties and solutions of primary and additional direct or adjoint boundary-value problems.

In Section 2, the general expressions for sensitivities and variation of an arbitrary functional with respect to varying domain shape will be derived and the concept of an additional direct and adjoint boundary-value problem will be introduced following the previous works by Dems and Mróz (1984;1995) and Dems (1986;1987). In Section 3, the particular domain transformations will be considered, whereas the transition to the case of energy functionals will be discussed in Section 4. The sensitivity with respect to varying material parameters will be briefly discussed in Section 5. Some simple illustrative examples will be presented in Section 6.

2. Shape sensitivity for an arbitrary functional

Let us consider the following boundary-value problem within an anisotropic domain

$$\left.\begin{array}{l} \mathrm{div}\mathbf{q}(x) - k(x)u(x) = f(x) \\ \mathbf{q}(x) = \mathbf{A}(x) \cdot \nabla u(x) \end{array}\right\} \text{within } \Omega \tag{1}$$

$$\begin{array}{ll} u(\mathbf{x}) = u_0(\mathbf{x}) & \text{on } \Gamma_u \\ q_n(\mathbf{x}) = \mathbf{n} \cdot \mathbf{q}(\mathbf{x}) = q_{n0}(\mathbf{x}) & \text{on } \Gamma_q \end{array} \tag{2}$$

where u and $\mathbf{q}$ denote the scalar state function and its flux vector within the problem domain Ω bounded by boundary $\Gamma = \Gamma_u \cup \Gamma_q$, Fig. 1a. $A(\mathbf{x})$ denotes the symmetric matrix of anisotropy coefficients, $\mathbf{n}$ denotes the unit normal vector directed outward on Γ, and ∇ is the gradient operator. On the boundary portion Γ_u the state field u is prescribed (Dirichlet condition) whereas the flux q_n proportional to its normal derivative (Neumann condition) is specified on the remaining boundary portion Γ_q. Thus, we assume that the functions $u_0(x)$ on Γ_u and $q_{n0}(\mathbf{x})$ on Γ_q are known in advance and they are specified on respective portions of Γ or are given functions of three-dimensional Euclidean space. Furthermore, we assume that the non-negative function $k(\mathbf{x})$ and source term $f(\mathbf{x})$ are also known. It is well known fact that Eq. (1) together with boundary conditions (2) can describe a wide class of physical problems, depending on a proper specification of state function $u(\mathbf{x})$, source term $f(\mathbf{x})$, variable coefficient $k(\mathbf{x})$ and boundary term $q_{n0}(\mathbf{x})$. Such physical problems come, for instance, from

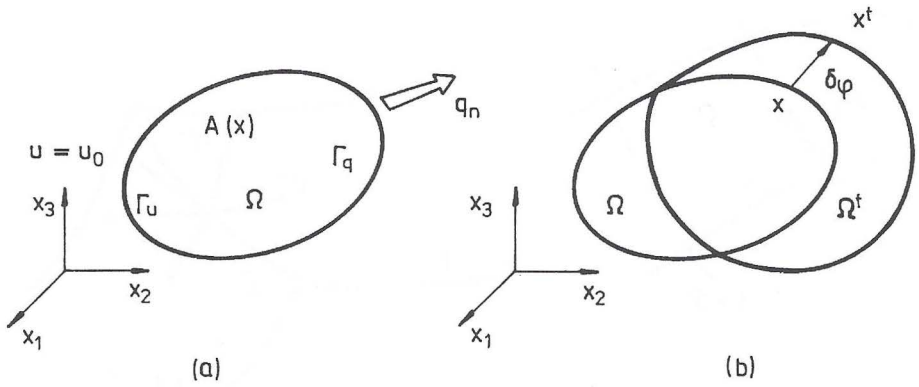

Figure 1. Domain Ω with boundary (a) and its modification (b)

structural mechanics (e.g. membrane, torsion), fluid mechanics, steady heat transfer, electric or magnetic field theory, and so on.

The equivalent weak variational form of boundary-value problem (1)-(2) can be written as the variational equation

$$B(z,u) - l(z,q_n) = 0 \tag{3}$$

where

$$\begin{aligned} B(z,u) &= \int (\nabla z \cdot \mathbf{A} \cdot \nabla u + zku) d\Omega \\ l(z,q_n) &= -\int zf d\Omega + \int zq_0 d\Gamma_q + \int z_0 q_n d\Gamma_u \end{aligned} \tag{4}$$

and $z(\mathbf{x})$ is an arbitrary admissible test function satisfying the condition $z(\mathbf{x}) = z_0(x)$ on Γ_u. This weak variational form will be used later on in deriving the first-order sensitivities of an arbitrary functional.

Let us now consider the variation of domain Ω together with its boundary Γ, Fig. 1b, due to an infinitesimal transformation process defined as

$$\Omega \to \Omega^t : \; x^t = x + \delta\phi(x,\mathbf{b}) = x + v^p(x,\mathbf{b})\delta b_p \tag{5}$$

where the transformation field $\phi(x,\mathbf{b})$ is a given function of space and depends on a set $\mathbf{b}$ of independent shape design parameters b_p, $p = 1,2,\ldots,P$, and $v^p = \partial\phi/\partial b_p$ denotes a transformation velocity field associated with shape parameter b_p treated as time-like parameter.

To make our subsequent analysis more clear and easy to understand, let us introduce some useful notations. First of all, let us introduce an orthogonal curvilinear coordinate system (t,s) on the surface Γ, Fig. 2a, coinciding with the principal curvature lines on Γ and let, for simplicity, t and s denote the arc parameters of these lines. The unit vectors along these lines are denoted by $\mathbf{t}$, $\mathbf{s}$, so that vectors $\mathbf{t}$, $\mathbf{s}$, $\mathbf{n}$ constitute a local orthogonal coordinate system on Γ.

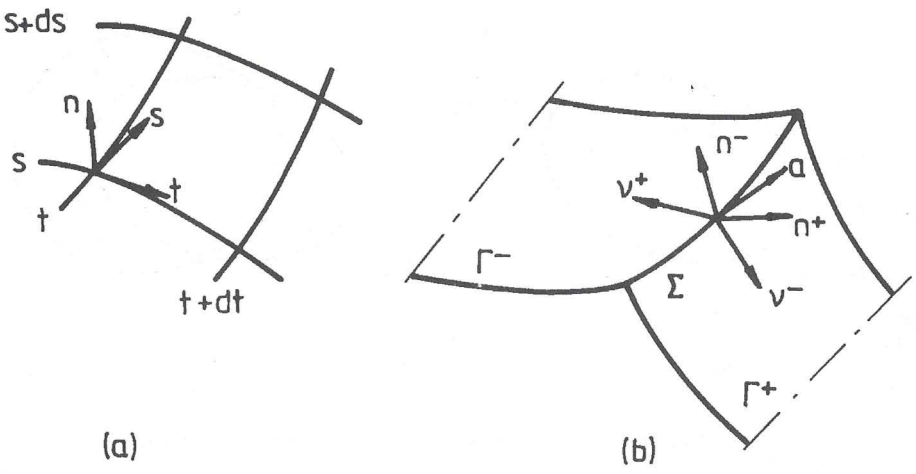

Figure 2. Local coordinate system on Γ (a), and boundary intersection curve Σ (b)

If the surface Γ is piecewise smooth, then the intersection curve between two adjacent parts of Γ is denoted by Σ with the unit tangent vector $\mathbf{a}$, as shown in Fig. 2b, and two unit vectors ν^+ and ν^- normal to Σ and tangential to parts of Γ, respectively, are defined by

$$\nu^- = \mathbf{a} \times \mathbf{n}^-, \; \nu^+ = \mathbf{a} \times \mathbf{n}^+ \tag{6}$$

The principal curvatures of Γ will be denoted by K_t, K_s and its mean curvature is H.

For any smooth scalar field $g(\mathbf{x})$ defined on Γ, its gradient in a plane tangential to Γ is given by

$$\nabla_\Gamma g(x) = \mathbf{t}(x) g_{/t}(x) + \mathbf{s}(x) g_{/s}(x) \; x \in \Gamma \tag{7}$$

where comma preceding an index denotes partial differentiation of the respective quantity with respect to surface or space variable. Obviously, the vector $\nabla_\Gamma g$ lies in a plane tangential to Γ at point $\mathbf{x}$. If $g(\mathbf{x})$ is defined not only on Γ but also in its neighborhood, then the gradient of g in a three-dimensional space is expressed as

$$\nabla g(x) = \nabla_\Gamma g(x) + \mathbf{n} g_{/n}(x) \; x \in \Gamma \tag{8}$$

Similarly, let $\mathbf{h}(x) = \mathbf{t}h_t(x) + \mathbf{s}h_s(x)$ be a continuous and differentiable vector field defined on Γ. Then, the divergence of $\mathbf{h}(x)$ on Γ is defined by

$$\mathrm{div}_\Gamma \mathbf{h}(x) = h_{t/t}(x) + h_{s/s}(x) \tag{9}$$

If $\mathbf{h}(x)$ is defined in a neighborhood of Γ, then its divergence is expressed as

$$\begin{aligned} \mathrm{div}\mathbf{h}(x) &= \mathrm{div}_\Gamma \mathbf{h}(x) + h_{n/n}(x) - 2H(x)h_n(x) \\ &= \mathrm{div}_\Gamma \mathbf{h} + \mathbf{n}\cdot D\mathbf{h}\cdot\mathbf{n}\; x\in\Gamma \end{aligned} \tag{10}$$

where $D\mathbf{h} = [h_{i/j}]$ denotes the gradient matrix of $\mathbf{h}$ and $\mathbf{n}\cdot D\mathbf{h}\cdot\mathbf{n} = n_i h_{i/j} n_j$.

The Gauss divergence and Stokes theorem for the domain Ω bounded by a closed surface Γ can be now written in the form

$$\begin{aligned} \int \mathrm{div}\mathbf{h}(x)d\Omega &= \int \mathbf{h}(x)\cdot\mathbf{n}(x)d\Gamma \\ \int \mathrm{div}_\Gamma \mathbf{h}(x)d\Gamma &= \int \langle \mathbf{h}(x)\cdot\nu(x)\rangle d\Sigma \end{aligned} \tag{11}$$

where the integral on the right-hand side of second equality of (11) is expanded over all intersection curves of partly smooth surface Γ and $\langle\cdot\rangle$ denotes the jump of enclosed quantities calculated as the difference on adjacent parts of Γ along Σ. Note that for a smooth and closed surface Γ this integral vanishes.

Finally, we should note that during transformation process of the domain Ω with the boundary Γ defined by (5), the following relations hold for any continuous function $g(x,\mathbf{b})$ (cf. Haug et al., 1986)

$$\begin{aligned} g_p &= g_{/p} + \nabla g\cdot\mathbf{v}^p \\ (\nabla g)_p &= \nabla g_{/p} + D(\nabla g)\cdot\mathbf{v}^p = \nabla g_p - \nabla g\cdot D\mathbf{v}^p \end{aligned} \tag{12}$$

where $g_p = \frac{dg}{db_p}$ denotes the total (or material) derivative of g with respect to design parameter b_p and the local (or domain) derivative of g with respect to this parameter for fixed domain Ω is $g_{/p} = \frac{\partial g}{\partial b_p}$. The total derivatives of domain and boundary elements as well as of the unit normal vector to the boundary with respect to design parameter are expressed by (cf. Dems & Mróz, 1984, Dems & Haftka, 1989)

$$\begin{aligned} (d\Omega)_p &= \mathrm{div}\mathbf{v}^p d\Omega \\ (d\Gamma)_p &= (\mathrm{div}\mathbf{v}^p - \mathbf{n}\cdot D\mathbf{v}^p\cdot\mathbf{n})d\Gamma = \mathrm{div}_\Gamma \mathbf{V}^p d\Gamma \end{aligned} \tag{13}$$

and

$$\mathbf{n}_p = \mathbf{n}(\mathbf{n}\cdot D\mathbf{v}^p\cdot\mathbf{n}) - \mathbf{n}\cdot D\mathbf{v}^p \tag{14}$$

After these preliminaries we can pass to studying the variation of an arbitrary functional due to variation of the domain of specified boundary-value problem. Consider then an arbitrary behavioural functional given in the form

$$J = \int_{\Omega(b)} \Psi(u,\nabla u)d\Omega + \int_{\Gamma(b)} \Phi(u,q_n)d\Gamma \tag{15}$$

where Ψ and Φ are continuous and differentiable functions of their arguments. In what follows it is assumed that the particular forms of functional J are admissible from the point of view of solution of well posed boundary value

problem (1 - 2), so that J may exist in Ω and on Γ. The first variation of functional (15) with respect to variation of the domain can be written as

$$\delta J = J_p \delta b_p \tag{16}$$

where $J_p = \frac{dJ}{db_p}$ denotes the sensitivity, or in other words the total derivative, of J with respect to shape design parameter b_p. The sensitivity of J is now expressed in the form

$$\begin{aligned} J_p = &\int [\Psi_p d\Omega + \Psi(d\Omega)_p] + \int [\Phi_p d\Gamma + \Psi(d\Gamma)_p] = \\ &\int [\Psi_{\prime u} u_p + \nabla_u \Psi \cdot (\nabla u)_p + \Psi \mathrm{div} \mathbf{v}^p] d\Omega + \int [\Phi_{\prime u} u_p + \Phi_{\prime q_n} (q_n)_p + \\ &\Phi \mathrm{div}_\Gamma \mathbf{v}^p] d\Gamma \end{aligned} \tag{17}$$

where the use of (13) was made and $\nabla_u \Psi$ denotes the gradient of Ψ with respect to gradient components of u, namely

$$\nabla_u \Psi = \left(\frac{\partial \Psi}{\partial u_{\prime 1}}, \frac{\partial \Psi}{\partial u_{\prime 2}}, \frac{\partial \Psi}{\partial u_{\prime 3}} \right) \tag{18}$$

Note now that in view of (12) and (14) we can write

$$\begin{aligned} (q_n)_p &= (\mathbf{n} \cdot \mathbf{q})_p = \mathbf{n}_p \cdot \mathbf{q} + \mathbf{n} \cdot \mathbf{q}_p \\ &= \mathbf{n} \cdot [(\mathbf{n} \cdot D\mathbf{v}^p \mathbf{n}) - D\mathbf{v}^p] \cdot \mathbf{q} + \mathbf{n} \cdot (\mathbf{q}_{\prime p} + D\mathbf{q}\mathbf{v}^p) = \\ &\quad q_{n\prime p} + \nabla q_n \cdot \mathbf{v}^p - \mathbf{q} \cdot \nabla_\Gamma \nu_n^p \end{aligned} \tag{19}$$

where ν_n^p denotes the normal component of $\mathbf{v}^p$ on Γ. Making use of Gauss divergence theorem (11), (12) and (19) in (17), the sensitivity J_p can be rewritten as

$$\begin{aligned} J_p = &\int (\Psi_{\prime u} u_{\prime p} + \nabla_u \Psi \cdot \nabla u_{\prime p}) d\Omega + \\ &\int \Psi \mathbf{v}^p \cdot \mathbf{n} d\Gamma + \int (\Phi_{\prime u} u_{\prime p} + \Phi_{\prime q_n} q_{n\prime p}) d\Gamma + \\ &\int (\nabla \Phi \cdot \mathbf{v}^p - \Phi_{\prime q_n} \mathbf{q} \cdot \nabla_\Gamma \nu_n^p + \Phi \mathrm{div}_\Gamma \mathbf{v}^p) d\Gamma \end{aligned} \tag{20}$$

In view of (8) and (11), the last integral on the right-hand side of (20) can be transformed as follows

$$\begin{aligned} &\int (\nabla \Phi \cdot \mathbf{v}^p - \Phi_{\prime q_n} \mathbf{q} \cdot \nabla_\Gamma \nu_n^p + \Phi \mathrm{div}_\Gamma \mathbf{v}^p) d\Gamma = \\ &\int (\Phi_{\prime n} \mathbf{v}^p \cdot \mathbf{n} - \Phi_{\prime q_n} \mathbf{q} \cdot \nabla_\Gamma \nu_n^p) d\Gamma + \int \langle \Phi \mathbf{v}^p \cdot \nu \rangle d\Sigma \end{aligned} \tag{21}$$

Integrating now by parts the second term in the first integral on the right-hand side of (20) and using (21), J_p can be written in the form

$$J_p = \int [\Psi_{\prime u} - \mathrm{div}(\nabla_u \Psi)] u_{\prime p} d\Omega + \int [(\Phi_{\prime u} + \nabla_u \Psi \cdot \mathbf{n}) u_{\prime p} + \Phi_{\prime q_n} q_{n\prime p} + (\Psi + \Phi_{\prime m}) \mathbf{v}^p \cdot \mathbf{n} - \Phi_{\prime q_n} \mathbf{q} \cdot \nabla_\Gamma \nu_n^p] d\Gamma + \int \langle \Phi \mathbf{v}^p \cdot \nu \rangle d\Sigma \tag{22}$$

Since now, in view of (12) and (19), we have

$$u_{\prime p} = u_p - \nabla u \cdot \mathbf{v}^p; \; q_{n\prime p} = (q_n)_p - \nabla q_n \cdot \mathbf{v}^p + \mathbf{q} \cdot \nabla_\Gamma \nu_n^p \text{ on } \Gamma \tag{23}$$

then (22) can be presented in the form

$$\begin{aligned} J_p = & \int [\Psi_{\prime u} - \mathrm{div}(\nabla_u \Psi)] u_{\prime p} d\Omega + \int (\Phi_{\prime u} + \nabla_u \Psi \cdot \mathbf{n}) u_{\prime p} d\Gamma_q + \\ & \int \Phi_{\prime q_n} q_{n\prime p} d\Gamma_u + \int \Phi_{\prime q_n} (q_{n0})_p d\Gamma_q + \int (\Phi_{\prime u} + \nabla_u \Psi \cdot \mathbf{n}) u_{0p} d\Gamma_u + \\ & \int (\Phi_{\prime u} + \nabla_u \Psi \cdot \mathbf{n})(\nabla u \cdot \mathbf{v}^p) d\Gamma_q + \\ & \int \Phi_{\prime q_n} (\nabla q_n \cdot \mathbf{v}^p - \mathbf{q} \cdot \nabla_\Gamma \nu_n^p) d\Gamma_u + \\ & \int [\Psi \mathbf{n} - (\nabla_u \Psi \cdot \mathbf{n}) \nabla u - \nabla_\Gamma] \cdot \mathbf{v}^p d\Gamma + \\ & \int \langle \Phi \mathbf{v}^p \cdot \nu \rangle d\Sigma \end{aligned} \tag{24}$$

We should note that u_{0p} on Γ_u and $(q_{n0})_p$ on Γ_q are known in advance due to the specified form of boundary conditions (2) of the primary boundary-value problem. On the other hand, $u_{\prime p}$ in Ω and on Γ_q and $q_{n\prime p}$ on Γ_u are still unknown and should be specified. The calculation of these derivatives can be performed, for instance, by using a so called *direct approach* as it was presented by Dems (1986) for the case of heat transfer problem. In fact, the fields $u_{\prime p}$ and $q_{n\prime p}$ can be obtained as the solution of an additional direct boundary-value problem, which follows from differentiating with respect to b_p the equations (1)-(2). Thus, this problem is specified by the following set of equations

$$\begin{aligned} \mathrm{div}\mathbf{q}_{\prime p} - k u_{\prime p} = 0, \; \mathbf{q}_{\prime} p = A \cdot \nabla u_{\prime p} \quad & \text{within} \quad \Omega \\ u_{\prime p} = u_{0p} - \nabla u \cdot \mathbf{v}^p \quad & \text{on} \quad \Gamma_u \\ q_{n\prime p} = (q_{n0})_p - \nabla q_n \cdot \mathbf{v}^p + \mathbf{q} \cdot \nabla_\Gamma \nu_n^p \quad & \text{on} \quad \Gamma_q \end{aligned} \tag{25}$$

In writing (25) we assumed that both coefficient k and source term f are design independent. The boundary-value problem (25) should be solved with respect to all design parameters b_p in order to calculate all desired sensitivities $u_{\prime p}$, and then (24) can be used to evaluate all components J_p of first-order sensitivity

vector of functional (15). It is thus seen that for n shape design parameters, evaluation of the first-order sensitivity vector of J by the *direct method* would require solution of primary problem (1-2) and n additional direct problems (25).

An alternative method for calculating first-order sensitivities of J is an *adjoint approach*, where in order to eliminate $u_{\prime p}$ in Ω and on Γ_q and $q_{n\prime p}$ on Γ_u, an adjoint state field u^a is used. To do this, let us introduce an adjoint boundary-value problem of the same form as the primary one, specified within domain Ω and along Γ by the variational equation

$$B(u_{\prime p}, u^a) - l(u_{\prime p}, q_n^a) = 0 \tag{26}$$

where $B(\cdot,\cdot)$ and $l(\cdot,\cdot)$ are defined by (4). Thus, (26) can be written explicitly as

$$\int (\nabla u_{\prime p} \cdot \mathbf{A} \cdot \nabla u^a + u_{\prime p} k u^a + u_{\prime p} f^a) d\Omega - \int u_{\prime p} q_0^a d\Gamma_q - \int u_{0\prime p} q_n^a d\Gamma_u = 0 \tag{27}$$

Integrating by parts the first integral on the left-hand side of (27) and using the reciprocity theorem for the fields $u_{\prime p}$ and u^a, after simple transformations, the variational equation (26) is written in the form

$$\int u_{\prime p} f^a d\Omega + \int u^a q_{n\prime p} d\Gamma - \int u_{\prime p} q_0^a d\Gamma_q - \int u_{0\prime p} q_n^a d\Gamma_u = 0 \tag{28}$$

In view of (23), Eq. (28) can be further transformed as follows

$$\begin{aligned} &\int u_{\prime p} f^a d\Omega - \int u_{\prime p} q_{n0}^a d\Gamma_q + \int q_{n\prime p} u^a d\Gamma_u = + \int u_{0p} q_n^a d\Gamma_u - \\ &\int (q_{n0})_p u^a d\Gamma_q + \int (u^a \mathbf{q} \cdot \nabla_\Gamma \nu_n^p - u^a \nabla q_n \cdot \mathbf{v}^p) d\Gamma_u + \int q_n^a \nabla u \cdot \mathbf{v}^p d\Gamma_q - \\ &\int [u^a \mathbf{q} \cdot \nabla_\Gamma \nu_n^p - (u^a \nabla q_n - q_n^a \nabla u) \cdot \mathbf{v}^p] d\Gamma \end{aligned} \tag{29}$$

Comparing (24) and (29) we observe that the source term f^a within Ω and boundary conditions u^a on Γ_u and q_n^a on Γ_q for adjoint problem should be specified in the form

$$f^a = \Psi_{\prime u} - \mathrm{div}(\nabla_u \Psi) \text{ within } \Omega \tag{30}$$

and

$$u^a = u_0^a = \Phi_{\prime q_n} \text{ on } \Gamma_u,\ q_n^a = q_{n0}^a = -(\Phi_{\prime u} + \nabla_u \Psi \cdot \mathbf{n}) \text{ on } \Gamma_q \tag{31}$$

Thus the adjoint boundary value-problem introduced by variational equation (26) can be also specified in the strong form by the differential equation

$$\mathrm{div}\mathbf{q}^a - k u^a = f^a,\ \mathbf{q}^a = \mathbf{A} \cdot \nabla u^a \text{ in } \Omega \tag{32}$$

with the mixed boundary conditions (31) of Dirichlet and Neumann type and source term specified by (30). Both the boundary conditions and the source

term depend on the form of integrands appearing in the considered functional (15).

Using now (29) in (24) and noting (30)-(31), the first order sensitivity of J with respect to design parameter is written as

$$J_p = \int (\Phi_{/u} + \nabla_u \Psi \cdot \mathbf{n} + q_n^a) u_{0p} d\Gamma_u + \int (\Phi_{/q_n} - u^a)(q_{n0})_p d\Gamma_q + \int \{[\Psi \mathbf{n} - (\nabla_u \Psi \cdot \mathbf{n}) \nabla u - \nabla_\Gamma \{\Phi + u^a \nabla q_n - q_n^a \nabla u] \cdot \mathbf{v}^p - u^a \mathbf{q} \cdot \nabla_\Gamma \nu_n^p\} d\Gamma + \int \langle \Phi \mathbf{v}^p \cdot \nu \rangle d\Sigma \tag{33}$$

Now, we can finally express (33) in two equivalent alternate forms. The first form expresses J_p entirely in the local coordinate system (t, s, n) on Γ and is useful when the both primary and adjoint state fields and their gradients are expressed in terms of this coordinate system on Γ. To obtain this form, we can decompose the transformation velocity vector $\mathbf{v}^p$ on Γ in the form

$$\mathbf{v}^p = \mathbf{v}_\Gamma^p + \mathbf{n} \nu_n^p \tag{34}$$

where $\mathbf{v}_\Gamma^p$ denotes the transformation velocity vector in a plane tangential to Γ, expressed as

$$\mathbf{v}_\Gamma^p = \mathbf{t} v_t^p + \mathbf{s} \nu_s^p \tag{35}$$

Furthermore, the following identity can be written for any scalar and vector functions $g(\mathbf{x})$ and $\mathbf{h}(x)$ on Γ

$$\nabla q \cdot \mathbf{h}_\Gamma = \nabla_\Gamma g \cdot \mathbf{h} = \nabla_\Gamma g \cdot \mathbf{h}_\Gamma \tag{36}$$

where $\mathbf{h}_\Gamma$ is defined similarly to (35).

Using now (36) and (8) in (33), the first form of sensitivity J_p can be written as

$$\begin{aligned} J_p &= \int (\Phi_{/u} + \nabla_u \Psi \cdot \mathbf{n} + q_n^a)(u_0)_p d\Gamma_u + \int (\Phi_{/q_n} - u^a)(q_{n0})_p d\Gamma_q \\ &+ \int \{[\Psi - (\nabla_u \Psi \cdot \mathbf{n}) u_{/n} + u^a q_{n/n} - q_n^a u_{/n}] v_n^p \\ &+ (u^a \nabla_\Gamma q_n - q_n^a \nabla_\Gamma u - \nabla_\Gamma \Phi) \cdot \mathbf{v}_\Gamma^p - \\ &\quad (\nabla_u \Psi \cdot \mathbf{n})(\nabla_\Gamma u \cdot \mathbf{v}_\Gamma^p) - u^a \mathbf{q}_\Gamma \cdot \nabla_\Gamma v_n^p\} d\Gamma + \int \langle \Phi \mathbf{v}^p \cdot \nu \rangle d\Sigma \end{aligned} \tag{37}$$

On the other hand when all primary and adjoint quantities are expressed in a fixed rectangular coordinate system (x_1, x_2, x_3) it is more convenient to have the expression for J_p written in terms of these coordinates. Using (8) to transform the gradient in a plane tangential to Γ to the gradient in three-dimensional

space, we obtain from (33) the following expression for the first-order sensitivity of functional (15)

$$
\begin{aligned}
J_p \;=\; & \int (\Phi_{/u} + \nabla_u \Psi \cdot \mathbf{n} + \mathbf{n} \cdot q^a)(u_0)_p d\Gamma_u + \int (\Phi_{/q_n} - u^a)(q_{n0})_p s\Gamma_q + \\
& \int \{(\Psi + u^a f + u^a k u + \nabla u^a \cdot \mathbf{q})\mathbf{n} \cdot \mathbf{v}^p - \\
& (\mathbf{q} \cdot \mathbf{n})(\nabla u^a \cdot \mathbf{v}^p) - (\mathbf{q}^a \cdot \mathbf{n})(\nabla u \cdot \mathbf{v}^p) + \\
& (\nabla_u \Psi \cdot \mathbf{n})(\nabla u \cdot \mathbf{v}^p) + (u^a \mathbf{q} \cdot \mathbf{n} - \Phi)(\mathbf{n} \cdot D\mathbf{v}^p \mathbf{n} - \mathrm{div}\mathbf{v}^p)\} d\Gamma \qquad (38)
\end{aligned}
$$

As one can easily observe, the sensitivity expressions (37) or (38) are written in terms of integrands of functional (15), state fields of primary and adjoint problems and the transformation velocity field associated with a particular design parameter. Whereas the primary problem is defined by strong differential equation (1) with proper set of boundary conditions (2), the adjoint problem is described by equation (32) with boundary conditions (31) and source term specified by (30). It is easy to notice that the calculation of this adjoint source term requires the knowledge of the second derivatives of primary state field with respect to space variables. When the solution of a boundary-value problem is performed analytically and the exact analytical form of $u(\mathbf{x})$ has been found, there is no trouble to calculate the accurate values of second derivatives of u at any point of the problem domain. On the other hand, when the numerical method of solution is used, due to complexity of the problem, the calculated second derivatives of approximation of $u(\mathbf{x})$ can suffer some inaccuracy which will affect, via source term (29), the accuracy of the solution for the adjoint problem. To avoid this inaccuracy concerned with calculation of the adjoint source term in the form (30), we can formulate, following Tortorelli & Haber (1989), the other equivalent adjoint problem with a fictitious initial flux $\mathbf{q}^{ai}$ within the domain Ω and the same state field $u^a(\mathbf{x})$ as for the adjoint problem (30)-(32). This new adjoint problem is formulated as follows

$$
\mathrm{div}\mathbf{q}^a - k u^a = \hat{f}^a \text{ in } \Omega \qquad (39)
$$

with

$$
\mathbf{q}^a = \mathbf{A} \cdot \nabla u^a - \mathbf{q}^{ai} \text{ in } \Omega \qquad (40)
$$

and the boundary conditions of the form

$$
u^a = u_0^a \text{ on } \Gamma_u, \; \mathbf{q}^a \cdot \mathbf{n} = \hat{q}_{n0}^a \text{ on } \Gamma_q \qquad (41)
$$

Substituting (40) into (39) and (41) we get

$$
\mathrm{div}(\mathbf{A} \cdot \nabla u^a) - k u^a = \hat{f}^a + \mathrm{div}\mathbf{q}^{ai} \text{ in } \Omega \qquad (42)
$$

and

$$
u^a = u_0^a \text{ on } \Gamma_u, \; (\mathbf{A} \cdot \nabla u^a) = \hat{q}_{n0}^a + \mathbf{q}^{ai} \cdot \mathbf{n} \text{ on } \Gamma_q \qquad (43)
$$

Comparing now (42)-(43) with (30)-(32), we can observe that

$$\hat{f}^a = -\Phi_{,u},\ \mathbf{q}^{ai} = -\nabla_u \Phi \text{ in } \Omega$$
$$u_0^a = \Psi_{,q_n} \text{ on } \Gamma_u,\ \hat{q}_{n0}^a = -\Psi_{,u} \text{ on } \Gamma_q \tag{44}$$

Thus, instead of solution of adjoint problem (30)-(32), we can solve the equivalent adjoint problem (39)-(41) with source term, initial flux and boundary conditions specified by (44). Note that calculation of (44) does not require the knowledge of second derivatives of primary state field and then it is more useful in numerical solution methods. Obviously, the solutions $u^a(\mathbf{x})$ for both adjoint problems are exactly the same.

To facilitate analysis of the next Section, let us finally write (38) in the following form

$$J_p = \int [\Psi_{,u} + (\Psi_{,u_{,i}} + q_i^a n_i)](u_0)_p d\Gamma_u + \int (\Psi_{,q_n} - u^a)(q_{n0})_p d\Gamma_q +$$
$$\int \{[(\Phi + u^a f + u^a k u + u_{,i}^a q_i) n_k - (q_l u_{,k}^a + q_l^a u_{,k}) n_l - \Phi_{,u_{,i}} u_{,k} n_i] v_k^p +$$
$$(u^a q_l n_l - \Psi)(n_k n_l - \delta_{kl}) v_{k,l}^p \} d\Gamma \tag{45}$$

which constitutes the foundation for deriving the sensitivity expressions of (15) for some particular transformations of problem domain Ω together with its boundary Γ.

3. Sensitivity of an arbitrary functional with respect to translation, rotation and scale change of problem domain

Consider now three most fundamental modifications of problem domain, namely its translation, rotation and scale change and derive the variation of an arbitrary functional associated with these modifications. When the functional is expressed in the form (15), then its first variation takes the form (16), where the sensitivities with respect to an arbitrary domain variation are expressed either using direct approach by (24) or by (37), (38) or (45) when the adjoint method is applied. For the purpose of our subsequent analysis we shall select the expression (45) as the desired sensitivity expression for J and next particularize it for particular class of domain variation.

3.1. Translation of problem domain

In the case of translation of problem domain, the set of design parameters is reduced to the components of a constant translation vector $\mathbf{a}$, so that domain transformation process (5) can be written as

$$\Omega \Rightarrow \Omega^t : \mathbf{x}^t = \mathbf{x} + \delta\mathbf{a} \tag{46}$$

and transformation velocity fields $\mathbf{v}^p$, $p = 1, 2, 3$, associated with components a_p of translation vector $\mathbf{a}$ are reduced to the constant vector fields $v_k^p = \delta_{pk}$ over domain Ω and its boundary Γ. The boundary conditions (2) of the primary problem are also translated correspondingly, and then their variations are

$$\delta u_0 = (u_0)_p \delta q_p = 0 \text{ on } \Gamma_u, \ \delta q_{n0} = (q_{n0})_p \delta a_p = 0 \text{ on } \Gamma_q \tag{47}$$

It follows therefore from (47) that during translation of boundary conditions there is $(u_0)_p = 0$ and $(q_{n0})_p = 0$. The first variation of J, expressed generally by (16) can now be written in the form

$$\delta J = (J_p^T)_\Gamma \delta a_p \ p = 1, 2, 3 \tag{48}$$

where $(J_p^T)_\Gamma$ follows from (45), and in view of (46) and (47) takes the form

$$(J_p^T)_\Gamma = \int [(\Psi + u^a f + u^a k u + u_{,l}^a q_l)\delta_{ip} - (q_l^a u_{,p} + q_l u_{,p}^a)\delta_{il} - \Psi_{,u_{,i}} u_{,p}] n_i d\Gamma \tag{49}$$

Assume now that the domain of the problem is homogeneous and the source term f is constant within the whole domain Ω. In that case the integral $(J_p^T)_\Gamma$ vanishes for any closed surface within the problem domain, yielding

$$(J_p^T)_\Gamma = 0 \ p = 1, 2, 3, \ \delta J = 0 \tag{50}$$

To prove this, let us transform (49) into a domain integral and use (1)-(2) and (30)-(32), obtaining

$$\begin{aligned}(J_p^T)_\Gamma &= \int [(\Psi + u^a f + u^a k u + u_{,l}^a q_l)_{,p} - (q_l^a u_{,p} + q_l u_{,p}^a)_{,l} - (\Psi_{,u_{,i}} u_{,p})_{,i}] d\Omega \\ &= \int [\Psi_{,u} u_p + u_{,p}^a f + u_{,p}^a k u + u^a k u_{,p} - q_{i,i}^a u_{,p} - q_{i,i} u_{,p}^a - (\Psi_{,u_{,i}})_{,i} u_{,p}] d\Omega = \\ &\quad - \int [u_{,p}^a (q_{l,l} - k u - f) + u_{,p} (q_{l,l}^a - k u^a - f^a)] d\Omega = 0 \end{aligned} \tag{51}$$

since for a homogeneous domain $\mathbf{A}(\mathbf{x})$ and $k(\mathbf{x})$ are constant. Note that the isotropy of $\mathbf{A}(\mathbf{x})$ is not required.

On the other hand, for a non-homogeneous domain, the integral (49), according to (48), represents the variation of the functional J due to infinitesimal translation of the boundary with respect to inhomogeneity. Alternatively, we can consider the translation of inhomogeneity or internal void with external boundary fixed, see Fig. 3. In Fig. 3a, the external boundary does not vary and the void of surface Γ_0 translates through the distance $\delta\mathbf{a}$ within the homogeneous domain. The variation of J can now be calculated from (48) by considering the integral (49) along the void surface Γ_0. For the free surface Γ_0 on which $q_n^a = q_n = 0$, the expression (49) is simplified, namely

$$(J_p^T)_{\Gamma_0} = \int (\Psi + u^a f + u^a k u + u_{,l}^a q_l) n_p d\Gamma_0 \tag{52}$$

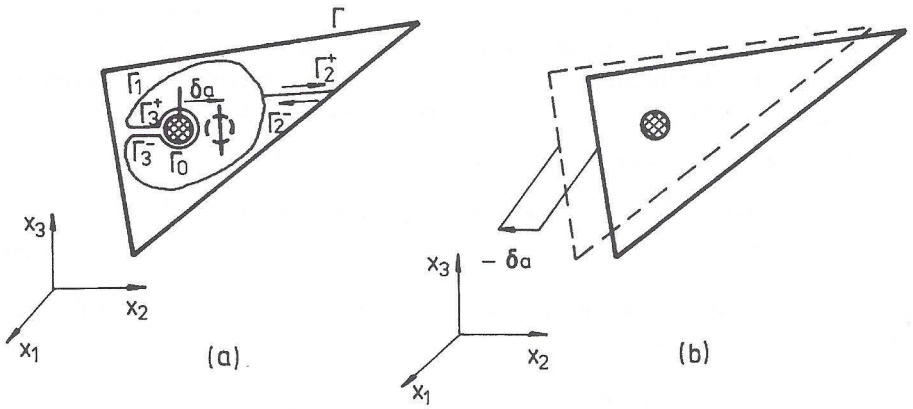

Figure 3. Translation of inclusion (a) and translation of domain with respect to fixed inclusion (b)

Let be given now an arbitrary closed surface Γ_1 enclosing the cavity. We can connect this surface to the cavity surface Γ_0 by the cuts Γ_3^+ and Γ_3^-. Since the integral J_p^T taken along the surface $\Gamma_1 \cup \Gamma_3^- \cup \Gamma_0 \cup \Gamma_3^+$ vanishes and the integrals along Γ_3^+ and Γ_3^- cancel, we obtain

$$(J_p^T)_{\Gamma_0} + (J_p^T)_{\Gamma_1} = 0, \ (J_p^T)_{\Gamma_0} = -(J_p^T)_{\Gamma_1} = -(J_p^T)_{\Gamma} \tag{53}$$

The transition from Γ_1 to Γ can be performed similarly by cuts Γ_2^+ and Γ_2^-. An alternate way to calculate the variation of J is to consider the translation of the domain through the vector $-\delta\mathbf{a}$ with the cavity fixed in space, Fig. 3b. The transition from the boundary surface Γ to an arbitrary closed surface Γ_1 enclosing the cavity or to the cavity surface Γ_0 is also obtained by considering the cuts between these surfaces.

Thus, we showed that the integral (49) associated with variation of an arbitrary functional J can be taken along any arbitrary closed and piecewise smooth surface within the problem domain providing always the same value. In this sense, the integral J_p^T is *path-independent.*

3.2. Rotation of problem domain

Consider now the case when the domain Ω is rotated with respect to its primary configuration, and denote the infinitesimal rotation vector by $\delta\omega$. Therefore the infinitesimal domain transformation process (5) can now be described as

$$\Omega \Rightarrow \Omega^t : \mathbf{x}^t = \mathbf{x} + \delta\phi = \mathbf{x} + \delta\omega \times \mathbf{x} \tag{54}$$

and

$$\delta\phi_k = e_{kpl} x_l \delta\omega_p, \ v_k^p = e_{kpl} x_l \tag{55}$$

where e_{kpl} is the permutation tensor and v_k^p denotes the components of transformation velocity field $\mathbf{v}^p$ associated with p-th component of infinitesimal rotation vector. Assuming furthermore that the boundary conditions of primary problem u_0 on Γ_u and q_{n0} on Γ_q are also rotated correspondingly, we obtain

$$\delta u_0 = 0 \; (u_0)_p = 0 \text{ on } \Gamma_u, \; \delta q_{n0} = 0 \; (q_{n0})_p = 0 \text{ on } \Gamma_q \tag{56}$$

Thus, the variation of functional J can now be presented in the form

$$\delta J = (J_p^R)_\Gamma \delta\omega_p \tag{57}$$

where $(J_p^R)_\Gamma$ follows from general expression (45) and, by virtue of (55) and (56), can be written as

$$\begin{aligned}(J_p^R)_\Gamma \;=\; & e_{kpl} \int [(\Psi + u^a f + u^a k u + u^a_{\prime j} q_j) x_l \delta_{ki} - \\ & (q_j^a u_{\prime k} + q_j u^a_{\prime k}) x_l \delta_{ji} - \Psi_{\prime u_{\prime i}} u_{\prime k} x_l] n_i d\Gamma\end{aligned} \tag{58}$$

Assume now that the source term f of primary problem is constant within the homogeneous and isotropic domain Ω, i.e. $k(\mathbf{x}) = \text{const}$. For constant source term f and $\mathbf{A}(\mathbf{x}) = \lambda\delta$ where λ denotes the constant coefficient of isotropy, the expression (57) can further be transformed into the domain integral

$$\begin{aligned}&(J_p^R)_\Gamma = \\ &e_{kpl} \int \{[(\Psi + u^a f + u^a k u + u^a_{\prime j} q_j) x_l]_{\prime k} - [(q_j^a u_{\prime k} + q_j u^a_{\prime k}) x_l]_{\prime j} - \\ &(\Psi_{\prime u_{\prime i}} u_{\prime k} x_l)_{\prime i}\} d\Omega = e_{kpl} \int \Psi_{\prime u} u_{\prime k} + u^a_{\prime k} f + u^a_{\prime k} k u + u^a k u_{\prime k} - q^a_{j\prime j} u_{\prime k} - \\ &q_{j\prime j} u^a_{\prime k} - (\Psi_{\prime u_{\prime i}}) u_{\prime k}] x_l - (q_l^a u_{\prime k} + q_l u^a_{\prime k}) - \Psi_{\prime u_{\prime l}} u_{\prime k}\} d\Omega = \\ &-e_{kpl} \int [u^a_{\prime k} (q_{j\prime j} - k u - f) x_l + u_{\prime k} (q^a_{j\prime j} - k u^a - f^a) x_l + \\ &\lambda (u^a_{\prime l} u_{\prime k} + u_{\prime l} u^a_{\prime k}) + \Psi_{\prime u_{\prime l}} u_{\prime k}] d\Omega = \\ &-e_{kpl} \int [\lambda (u^a_{\prime l} u_{\prime k} + u_{\prime l} u^a_{\prime k}) + \Psi_{\prime u_{\prime l}} u_{\prime k}] d\Omega\end{aligned} \tag{59}$$

In writing the final form of (59) the use of (1) and (32) was made. Using now the equality

$$e_{kpl} u^a_{\prime l} u_{\prime k} = -e_{kpi} u^a_{\prime k} u_{\prime l} \tag{60}$$

the expression (59) for $(J_p^R)_\Gamma$ can be reduced to the form

$$(J_p^R)_\Gamma = -e_{kpl} \int \Psi_{\prime u_{\prime l}} u_{\prime k} d\Omega \tag{61}$$

Assume now that $\Psi = \Psi(u, \nabla u)$ is an isotropic function of its arguments, thus

$$\Psi = \Psi(u, I) \tag{62}$$

where

$$I = \nabla u \cdot \nabla u \tag{63}$$

is the invariant of ∇u. Under this assumption, we have

$$\Psi_{/u_{,l}} = \frac{\partial \Psi}{\partial I}\frac{\partial I}{\partial u_{,l}} = 2\frac{\partial \Psi}{\partial I}u_{,l} \tag{64}$$

Using now (64) in (61), we obtain

$$(J_p^R)_\Gamma = -2\int e_{kpl}\frac{\partial \Psi}{\partial I}u_{,l}u_{,k}d\Omega = 0 \tag{65}$$

Thus, it has been shown that for an isotropic and homogeneous domain Ω, with constant source term f, the surface integral $(J_p^R)_\Gamma$ defined by (58) vanishes for any closed surface within the problem domain, provided the adjoint problem satisfies (30) and (32) and the function $\Psi = \Psi(u, \nabla u)$ is the isotropic function of its arguments. Note furthermore that the function $h = h(u, q_n)$ appearing in the adjoint boundary conditions (30) is also isotropic from definition.

The transition of the integral (58) from external surface boundary Γ to an arbitrary closed surface within the domain Ω can be performed similarly as for the case of translation and then the integral $(J_p^R)_\Gamma$ can be considered as *path-independent.*

3.3. Expansion or contraction of problem domain

Considering the scale change of problem domain Ω, we shall confine ourselves to the particular cases of the general functional (15). The transformation process (5) can now be written in the form

$$\Omega \Rightarrow \Omega^t : x^t = x + x\delta p = (1 + \delta p)x, \ \mathbf{v}^p = x \tag{66}$$

where p denotes the scale change parameter standing for only one component of design parameter set $\mathbf{b}$ and $\mathbf{v}^p = \mathbf{x}$ is the associated transformation velocity field.

Following Dems and Mróz (1986) let us assume that the state field in a transformed domain can now be written in the form

$$u^t = (1 + \xi\delta p)u \tag{67}$$

where ξ is a constant to be determined later on. The variation of state field and its sensitivity with respect to p are expressed as

$$\delta u = u^t - u = \xi\delta p u, \ u_p = \xi u \tag{68}$$

The sensitivity of a gradient of state field, in view of the second equation of (12) and (66), is now

$$(\nabla u)_p = \nabla u_p - \nabla u \cdot D\mathbf{v}^p = \xi\nabla u - \nabla u \cdot D\mathbf{x} = (\xi - 1)\nabla u \tag{69}$$

Note furthermore that during transformation process (66) the unit normal vector to Γ does not change as it can be deduced from (14), and then $\mathbf{n}_p = 0$.

Assuming the homogeneous domain Ω, (constant $\mathbf{A}(\mathbf{x})$ and $k(\mathbf{x})$) during transformation (66), the sensitivities of boundary conditions (2), in view of (68) and (69), are

$$\begin{aligned} (u_0)_p &= \xi u_0 \text{ on } \Gamma_u \\ (q_{n0})_p &= \mathbf{q}_p \cdot \mathbf{n} + \mathbf{q} \cdot \mathbf{n}_p = (\xi - 1_{\mathbf{q}} \cdot \mathbf{n} = (\xi - 1) q_i n_i = (\xi - 1) q_{n0} \\ &\quad \text{on } \Gamma_q \end{aligned} \tag{70}$$

Let now the source term f and coefficient k vanish within the domain of primary problem, that is $f = 0$ and $k = 0$ in Ω. The first particular form of functional (15) can be assumed as

$$J_{1\Psi} = \int \Psi_1(u) d\Omega \tag{71}$$

Comparing (71) with (15) it is easy to see that now $\Psi(u, \nabla u) = \Psi_1(u)$ and $h(u, q_n) = 0$. If $\Psi_1(u)$ is a homogeneous function of u of order η, then

$$\Psi_1(tu) = t^{\eta} \Psi_1(u), \ \Psi_{1\prime u} u = \eta \Psi_1(u) \tag{72}$$

The constant ξ is now determined by requiring the invariance of $\Psi_1 d\Omega$ under the transformation (66), that is

$$(\Psi_1)_p = \Psi_{1p} d\Omega + \Psi_1 (d\Omega)_p = 0 \tag{73}$$

In view of (13), (66), (68) and (72), we can write

$$(d\Omega)_p = \mathrm{div}\mathbf{x} d\Omega = \delta_{kk} d\Omega, \ \Psi_{1p} = \Psi_{1\prime u} u_p = \Psi_{1\prime u} u \xi = \xi \eta \Psi_1 \tag{74}$$

where δ denotes the Kronecker's symbol and $\delta_{kk} = 3$ for three-dimensional scale change and $\delta_{kk} = 2$ for planar case. Using now (74) in (73), we can determine the value of constant ξ, namely

$$(\xi \eta + \delta_{kk}) \Psi_1 d\Omega = 0, \ \xi = -\frac{\delta_{kk}}{\eta} \tag{75}$$

The variation of functional $J_{1\Psi}$ can now be expressed as follows

$$\delta J_{1\Psi} = (J^E_{1\Psi})_{\Gamma} \delta p \tag{76}$$

where $(J^E_{1\Psi})_{\Gamma}$ following from (45), in view of (67), takes the form

$$\begin{aligned} (J^E_{1\Psi})_{\Gamma} &= \int \xi u q^a_i n_i d\Gamma_u - \int (\xi - 1) u^a q_i n_i d\Gamma_q + \\ &\int [(\Psi_1 + u^a_{\prime i} q_i) n_k x_k - (q^a_i u_{\prime k} + q_i u^a_{\prime k}) n_i x_k + u^a q_i n_i (1 - \delta kk)] d\Gamma \end{aligned} \tag{77}$$

Noting that, in view of (30), there is $u^a = 0$ on Γ_u and $q_n^a = q_i^a n_i = 0$ on Γ_q, and taking into account (75), expression (77) is transformed as follows

$$(J_{1\Psi}^E)_\Gamma = \int \{(\Psi_1 + u_{\prime k}^a q_k)x_i - (q_i^a u_{\prime k} + q_i u_{\prime k}^a)x_k + \xi u q_i^a + [2 + \xi(\eta - 1)]u^a q_i\} n_i d\Gamma \tag{78}$$

Let us show now that the integral (78) vanishes for any closed surface Γ within a homogeneous domain Ω. Transforming (78) into domain integral, we obtain

$$(J_{1\Psi}^E)_\Gamma = \int \{(\Psi_1 + u_{\prime k}^a q_k)x_i - (q_i^a u_{\prime k} + q_i u_{\prime k}^a)x_k + \xi u q_i^a + [2 + \xi(\eta - 1)]u^a q_i\}_{\prime i} d\Omega = \int \{(\Psi_1 + u_{\prime k}^a q_k)\delta_{ii} - [u_{\prime k}(q_{i\prime i}^a - \Psi_{1\prime u}) + u_{\prime k}^a q_{i\prime i}]x_k + \xi u q_{i\prime i}^a + \xi\eta u_{\prime i}^a q_i + [2 + \xi(\eta - 1)]u^a q_{i\prime i}\} d\Omega \tag{79}$$

Keeping in mind that $f = 0$ and $k = 0$, and making use of (1), (30), (32) and (75), expression (78) can be written in the form

$$(J_{1\Psi}^E)_\Gamma = \xi \int (-\eta\Psi_1 + \Psi_{1\prime u} u) d\Omega = 0 \tag{80}$$

assuming the second equality of (72) holds.

Consider now the second form of functional (15), namely

$$J_{2\Psi} = \int \Psi_2(\nabla u) d\Omega \tag{81}$$

where $\Psi_2(\nabla u)$ is assumed to be a homogeneous function of order η. The first variation of (81) equals

$$\delta J_{2\Psi} = (J_{2\Psi}^E)_\Gamma \delta p \tag{82}$$

The constant ξ is now determined from the condition of invariance of $\Psi_2 d\Omega$ under transformation (66). In view of (13), (66) and (69) this condition is written in the form

$$[\eta(\xi - 1) + \delta_{kk}]\Psi_2 d\Omega = 0 \tag{83}$$

yielding

$$\xi = 1 - \frac{\delta_{kk}}{\eta} \tag{84}$$

The integral $(J_{2\Psi}^E)_\Gamma$ follows once more from the general expression (45) and now takes the form

$$(J_{2\Psi}^E)_\Gamma = \int \{(\Psi_2 + u_{\prime k}^a q_k)x_i - (q_i^a u_{\prime k} + q_i u_{\prime k}^a)x_k - \Psi_{2\prime u_{\prime i}} u_{\prime k} x_k + \xi(\Psi_{2\prime u_{\prime i}} + q_i^a)u + [1 + (1 - \xi)(1 - \eta)]u^a q_i\} n_i d\Gamma \tag{85}$$

The proof that integral (85) vanishes for any closed surface Γ follows the similar steps as previously. Converting (85) into domain integral and using the state equations of primary and adjoint problems, we obtain

$$(J^E_{2\Psi})_\Gamma = (1-\xi)\int(\eta\Psi_2 - \Psi_{2\prime u_{\prime i}} u_{\prime i})d\Omega = 0 \tag{86}$$

due to the assumption of homogeneity of Ψ_2 with respect to ∇u.

The third form of the functional J is assumed as

$$J_{1\Phi} = \int \Phi_1(u)d\Gamma \tag{87}$$

where $\Phi_1(u)$ is a homogeneous function of order η. From the condition of invariance of $\Phi_1 d\Gamma$ under transformation (66) we obtain the following value of constant ξ

$$\xi = \frac{1-\delta_{kk}}{\eta} \tag{88}$$

The variation of functional (87) is written now in the form

$$\delta J_{1\Phi} = (J^E_{1\Phi})_\Gamma \delta p \tag{89}$$

where

$$(J^E_{1\Phi})_\Gamma = \int\{u^a_{\prime k} q_k x_i - (q^a_i u_{\prime k} + q_i u^a_{\prime k})x_k + \xi q^a_i u + [1+\xi(\eta-1)]u^a q_i\}n_i d\Gamma \tag{90}$$

The proof that $(J^E_{1\Phi})_\Gamma = 0$ for any closed surface Γ is straightforward. We convert (90) into the domain integral, differentiating each term and making use of definitions of primary and adjoint problems.

Finally consider the functional

$$J_{2\Phi} = \int \Phi_2(q_n)d\Gamma \tag{91}$$

where, similarly as previously, $\Phi_{2\Phi}(q_n)$ is a homogeneous function of order η. The value of constant ξ, following from the invariance of $\Phi_2 d\Gamma$ under the transformation (66), now is

$$\xi = 1 + \frac{1-\delta_{kk}}{\eta} \tag{92}$$

The variation of (91) is expressed by

$$\delta J_{2\Phi} = (J^E_{2\Phi})_\Gamma \delta p \tag{93}$$

where

$$(J^E_{2\Phi})_\Gamma = \int\{u^a_{\prime k} q_k x_i - (q^a_i u_{\prime k} + q_i u^a_{\prime k})x_k + \xi q^a_i u + (\xi-1)(\eta-1)u^a q_i\}n_i d\Gamma \tag{94}$$

The proof that $(J^E_{2\Phi})_\Gamma$ vanishes for any closed surface within a homogeneous domain follows the similar steps as before and is not presented here.

Thus, it is clear from the above analysis that for a homogeneous domain Ω the surface integrals (78), (85), (90) and (94) vanish provided the integrands of proper functionals are the homogeneous functions of their arguments and there is also $k = 0$ and $f = 0$ for the primary problem. The transition from the external boundary Γ to an arbitrary closed surface within the domain Ω can be performed similarly as before and then these integrals can be considered also as *path-independent.*

The additional two remarks should be stated at this point of analysis. Namely, the combined transformation of problem domain consisting of simultaneous translation and rotation or translation and expansion can be considered as a pure rotation or expansion with respect to properly selected center of rotation or expansion. It should be also noted that in the case of several inclusions or cavities in non-homogeneous domain the path-independent integrals can also be applied in sensitivity analysis with respect to translation, rotation or scale change of each particular inclusion. In this case each inclusion is surrounded independently by a closed surface not penetrating the other inclusions or cavities and the path-independent integral associated with respective domain variation can be calculated along this surface.

4. Sensitivity expressions for energy functional

The transition to the case when the functional (15) coincides with the energy functional associated with weak variational form (3) can be obtained by specifying the adjoint problem and using the general expressions for J_p.

The energy functional associated with the problem (1)-(2) can be assumed in the form

$$\Pi_u = \int \left[\frac{1}{2}(\nabla u \cdot \mathbf{A} \cdot \nabla u + ku^2) + fu\right] d\Omega - \int q_{n0} u d\Gamma_q \tag{95}$$

Comparing (95) with (15), we can observe that

$$\begin{aligned} \Psi &= \frac{1}{2}(\nabla u \cdot \mathbf{A} \cdot \nabla u + ku^2) + fu \text{ in } \Omega \\ \Phi &= 0 \text{ on } \Gamma_u,\ \Phi = -q_{n0} u \text{ on } \Gamma_q \end{aligned} \tag{96}$$

and then, in view of (30)-(31), the adjoint system is now specified as follows

$$\begin{aligned} f^a &= ku + f - \text{div}\mathbf{q} = 0 \text{ in } \Omega \\ u^a &= 0 \text{ on } \Gamma_u,\ q^a_n = 0 \text{ on } \Gamma_q \end{aligned} \tag{97}$$

The adjoint solution satisfying conditions (97) is $u^a = 0$. Thus, the general expressions for J_p given by (37) takes the form

$$J_p = (\Pi_u)_p = \int q_n (u_0)_p d\Gamma_u - \int u (q_{n0})_p d\Gamma_q - \int q_n (\nabla_\Gamma u \cdot \mathbf{v}^p_\Gamma) d\Gamma_u +$$

$$\int u(\nabla_\Gamma q_n \cdot \mathbf{v}_\Gamma^p) d\Gamma_q + \frac{1}{2}\int (\mathbf{q}_\Gamma \cdot \nabla_\Gamma u - q_n u_{,m}) v_n^p d\Gamma_u \tag{98}$$

In particular cases of translation and rotations, discussed in the previous section, the sensitivities J_p^T and J_p^R given by (49) and (58) take the following forms

$$(J_p^T)_\Gamma = [(\Pi_u)_p^T]_\Gamma =$$
$$\int \left\{ \left[\frac{1}{2}(\nabla u \cdot \mathbf{A} \cdot \nabla u + k u^2) + f u \right] \delta_{ip} - q_i u_{,p} \right\} n_i d\Gamma_u$$
$$(J_p^R)_\Gamma = [(\Pi_u)_p^R]_\Gamma =$$
$$e_{kpl} \int \left\{ \left[\frac{1}{2}(\nabla u \cdot \mathbf{A} \cdot \nabla u + k u^2) + f u \right] \delta_{ki} - q_i u_{,p} \right\} x_i n_i d\Gamma_u \tag{99}$$

In the case of scale change note that there has to be $f = 0$ and $k = 0$, and the general expression (85) for $(J_{2\Psi}^E)_\Gamma^D$ is simplified to the form

$$(J_{2\Psi}^E)_\Gamma =$$
$$[(\Pi_u)^E]_\Gamma = \int \left[\frac{1}{2} \nabla u \cdot \mathbf{A} \cdot \nabla u x_i + q_i u_{,k} x_k + \left(1 - \frac{1}{2}\delta_{kk}\right) q_i u \right] n_i d\Gamma_u \tag{100}$$

5. Sensitivity with respect to material parameters

In this section we consider the boundary-value problem (1)-(2) specified within fixed domain Ω bounded by external boundary Γ. We assume now that the anisotropy matrix $\mathbf{A}$, function k and source term f can depend on a set of material design parameters $\mathbf{a}$, $\mathbf{A} = \mathbf{A}(\mathbf{x}, \mathbf{a})$, $k = k(\mathbf{x}, \mathbf{a})$, $f = f(\mathbf{x}, \mathbf{a})$, and derive the sensitivity of functional (15) with respect to these parameters. The functional (15) can now be written as

$$J = \int_\Omega \Psi(u, \nabla u, \mathbf{a}) d\Omega + \int_\Gamma \Phi(u, q_n, \mathbf{a}) d\Gamma \tag{101}$$

and its first variation with respect to $\mathbf{a}$ is

$$\delta J = J_r \delta a_r \tag{102}$$

where $J_r = \frac{dJ}{da_r}$ denotes now the sensitivity of J with respect to material design parameter a_r. The sensitivity J_r can be expressed in the form

$$J_r = \int \Psi_r d\Omega + \int \Phi_r d\Gamma = \int [\Psi_{,u} u_{,r} + \nabla_u \Psi \cdot \nabla u_{,r} + \Psi_{,r}] d\Omega +$$
$$\int \Phi_{,u} u_{,r} d\Gamma_q + \int \Phi_{,q_n} q_{n,r} d\Gamma_u + \int \Phi_{,r} d\Gamma \tag{103}$$

assuming the boundary conditions as design independent, that is $u_{,r} = 0$ on Γ_u and $q_{n,r} = 0$ on Γ_q. Expression (103) constitutes the base for calculating the first-order sensitivities of J using either direct or adjoint approaches.

Let us discuss first the direct method requiring the knowledge of state function sensitivity $u_{,r}$. To formulate the equations for calculating the desired sensitivity we differentiate the primary state equations (1) - (2) with respect to design parameter a_r, obtaining

$$\begin{aligned} &\mathrm{div}\mathbf{q}_{,r} - ku_{,r} = f_{,r} + k_{,r}u,\ \mathbf{q}_{,r} = \mathbf{A}\cdot\nabla u_{,r} + \mathbf{A}_{,r}\cdot\nabla u \text{ within } \Omega \\ &u_{,r} = 0 \text{ on } \Gamma_u,\ q_{n,r} = \mathbf{n}\cdot\mathbf{q}_{,r} = 0 \text{ on } \Gamma_q \end{aligned} \tag{104}$$

Denoting now $u_{,r} = \bar{u}$, Eqs. (104) can be rewritten in the form

$$\begin{aligned} &\mathrm{div}\bar{\mathbf{q}} - k\bar{u} = \bar{f},\ \bar{f} = f_{,r} + k_{,r}u - \mathrm{div}(\mathbf{A}_{,r}\cdot\nabla u)\ \bar{\mathbf{q}} = \mathbf{A}\cdot\nabla\bar{u} \text{ within } \Omega \\ &\bar{u} = 0 \text{ on } \Gamma_u,\ \bar{q}_n = \mathbf{n}\cdot\bar{\mathbf{q}} = -\mathbf{n}\cdot(\mathbf{A}_{,r}\cdot\nabla u) \text{ on } \Gamma_q \end{aligned} \tag{105}$$

constituting the state equations of additional direct boundary-value problem associated with design parameter r. The sensitivities of primary fields are related to state fields of this problem as follows

$$\begin{aligned} &u_{,r} = \bar{u},\ \mathbf{q}_{,r} = \bar{\mathbf{q}} + \mathbf{A}_{,r}\cdot\nabla u \text{ within } \Omega \\ &q_{n,r} = \bar{q}_n + \mathbf{n}(\mathbf{A}_{,r}\cdot\nabla u) \text{ on } \Gamma \end{aligned} \tag{106}$$

The above procedure has to be repeated to evaluate the sensitivity derivatives with respect to all design parameters a_r, $r = 1, 2, \ldots, m$, and then Eq.(103) can be used to evaluate the sensitivity vector of functional (101).

An alternative method for deriving the sensitivities of (101) would require, similarly as for the case of shape sensitivity, the adjoint solution u^a of adjoint boundary-value problem (30)-(32). We start with integrating by parts the second term in domain integral of (103) and make use of equations (30)-(31) defining the source term and boundary conditions of adjoint problem. Thus, we can write (103) in the form

$$\begin{aligned} J_r \;=\; &\int\{[\Psi_{,u} - \mathrm{div}(\nabla_u\Psi)]u_{,r} + \Psi_{,r}\}d\Omega + \int(\Phi_{,u} + \nabla_u\Psi\cdot\mathbf{n})u_{,r}d\Gamma_q + \\ &\int\Phi_{,q_n}q_{n,r}d\Gamma_u + \int\Phi_{,r}d\Gamma = \int(f^a u_{,r} + \Psi_{,r})d\Omega - \\ &\int q^a_{n0}u_{,r}d\Gamma_q + \int u^a q_{n,r}d\Gamma_u + \int\Phi_{,r}d\Gamma \end{aligned} \tag{107}$$

The adjoint problem is specified by the variational equation

$$B(u_{,r}, u^a) - l(u_{,r}, q^a_n) = 0 \tag{108}$$

where $B(\cdot,\cdot)$ and $l(\cdot,\cdot)$ are defined by (4). Integrating by parts the first term of B in (108), we get

$$\begin{aligned} &\int(f^a u_{,r} - \nabla u^a\cdot\mathbf{A}_{,r}\cdot\nabla u - f_{,r}u^a - u^a k_{,r}u)d\Omega - \int u_{,r}q^a_{n0}d\Gamma_q + \\ &\int u^a q_{n,r}d\Gamma_u = 0 \end{aligned} \tag{109}$$

Substituting now (109) into (107), we finally obtain the expression for sensitivities of (101) written in terms of primary and adjoint state fields, namely

$$J_r = \int (\nabla u^a \cdot \mathbf{A}_{lr} \cdot \nabla u + u^a k_{lr} u + u^a f_{lr} + \Psi_{lr}) d\Omega + \int \Phi_{lr} d\Gamma \tag{110}$$

Thus, similarly as for shape sensitivity, the sensitivities of J with respect to material design parameters are obtained as the result of only primary and adjoint solutions.

Let us note that the adjoint problem used here is exactly the same as for shape sensitivity. Thus, using the adjoint approach to sensitivity analysis, we need only the same two solutions for both shape and material variations. However, when there are n design parameters and m functionals, the direct approach would still require $n+1$ solutions independently of the number of functionals, whereas the adjoint approach would need $m+1$ solutions to generate the sensitivities of m functionals. Thus, the choice between the two methods depends on the ratio of m to n as well as relative difficulty of obtaining adjoint solutions versus direct solutions.

6. Illustrative applications

In this section we will illustrate the applicability of derived sensitivity expressions in optimal shape and material design by considering some simple analytical and numerical examples.

EXAMPLE 6.1 *Consider a prismatic bar with the elliptical cross-section of prescribed area D subjected to torsion, Fig. 4, and derive the optimal values of semi-axes b_1 and b_2 for which the bar attains its maximal torsional rigidity. The bar is made from stratified orthotropic material with orthotropy coefficients $A_{11} = 1/G_1$, $A_{12} = A_{21} = 0$ and $A_{22} = 1/G_2$, where G_1 and G_2 are the shear module in a plane of the strata and direction perpendicular to the strata, respectively. Introducing the Prandtl's stress function u, the torsion problem is defined as the boundary-value problem (1) - (2), particularized to the form*

$$\begin{aligned} &\frac{1}{G_1} u_{l11} + \frac{1}{G_2} u_{l22} = -2\Theta \text{ within } \Omega \\ &u = 0 \text{ on } \Gamma_u \end{aligned} \tag{111}$$

where Θ denotes the angle of twist per unit length of the bar. The solution of the problem (111) can be written as

$$u = -\frac{b_1^2 b_2^2 G_1 G_2 \Theta}{b_1^2 G_1 + b_2^2 G_2} \left(\frac{x_1^2}{b_1^2} + \frac{x_2^2}{b_2^2} - 1 \right) \tag{112}$$

The transformation of cross-sectional domain associated with the change of the length of semi-axes b_1 and b_2 is described by transformation velocity fields as-

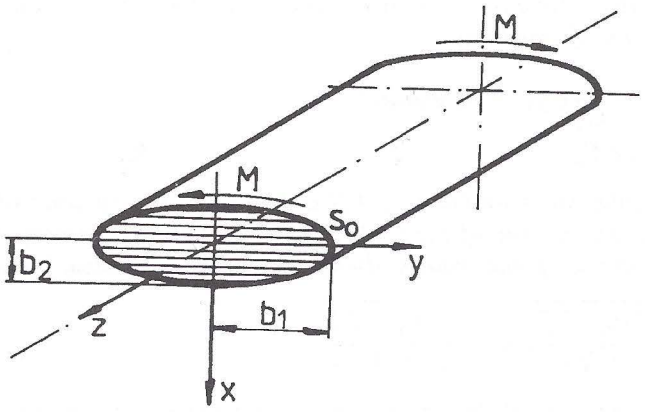

Figure 4. Bar with elliptical cross-section subjected to torsion

sumed in the form

$$\begin{aligned} v_1^{b_1} &= \frac{x_1}{b_1},\ v_2^{b_1} = 0 \\ v_1^{b_2} &= 0,\ v_2^{b_2} = \frac{x_2}{b_2} \end{aligned} \tag{113}$$

Denoting the torsional rigidity of a bar as

$$K = 2\int u d\Omega \Rightarrow \max. \tag{114}$$

the optimization problem can now be formulated as follows

$$\textit{Minimize } J = \int (-2u) d\Omega \tag{115}$$

$$\textit{subject to} D - D_0 = 0 \tag{116}$$

where $D = \int d\Omega$ *is the area of the cross-section of the bar and* D_0 *is prescribed quantity. Comparing (115) with (15) one can observe that* $\Psi = -2u$ *and* $\Phi = 0$. *The optimality conditions for the problem (115) follow from the stationarity of Lagrange functional* $J' = J - \lambda(D - D_0)$ *and takes the form*

$$\begin{aligned} J_{b_1} &= \lambda D_{b_1} \\ J_{b_2} &= \lambda D_{b_2} \\ \pi b_1 b_2 &= D_0 \end{aligned} \tag{117}$$

where J_{b_1} *and* J_{b_2} *are the sensitivities of objective functional (115), expressed using the adjoint approach by (38), and* D_{b_1} *and* D_{b_2} *denote the sensitivities of*

constraint (116). The adjoint problem, in view of (30)-(32), is now specified by the following set of equations

$$\frac{1}{G_1}u^a_{,11} + \frac{1}{G_2}u^a_{,22} = -2 \text{ within } \Omega$$
$$u^a = 0 \text{ on } \Gamma_u \tag{118}$$

and then its solution is similar to (112) with $\Theta = 1$. *The solution of optimality conditions (117), in view of (38), (113) and solutions of primary and adjoint problems, yield the optimal values of cross-sectional semi-axes, namely*

$$b_1 = \sqrt{\frac{D_0}{\pi}\sqrt{\frac{G_2}{G_1}}}, \; b_2 = \sqrt{\frac{D_0}{\pi}\sqrt{\frac{G_1}{G_2}}} \tag{119}$$

and then the optimal external boundary of bar cross-section is described by the equation

$$G_1 x_1^2 + G_2 x_2^2 - \frac{D_0}{\pi}\sqrt{G_1 G_2} = 0 \tag{120}$$

which is equivalent to that obtained earlier by Banichuk (1975). The relative torsional rigidity, defined as the ratio of rigidity of optimal bar to the rigidity of a bar with circular cross-section of the same area, is expressed as

$$K_r = \frac{K_{optimal}}{K_{circular}} = \frac{G_1 + G_2}{2\sqrt{G_1 G_2}} \tag{121}$$

It follows from (121) that the effectiveness of optimization is increasing for both $G_1/G_2 \to 0$ *and* $G_1/G_2 \to \infty$. *For* $G_1 = G_2$ *the relative rigidity* $K_r = 1$, *what is equivalent to the well known fact that in the case of isotropy the circle is the optimal cross-section of a bar in torsion.*

The next two examples will illustrate the applicability of path-independent integrals derived in Section 3 in sensitivity calculations associated with basic modifications of problem domain. These integrals can be applied in sensitivity analysis of an arbitrary functional with respect to translation, rotation or expansion of internal defects such as cracks, cavities or inclusions. In fact, the functional (15) can be given different interpretation depending on the type of problem considered. It can represent, for instance, an energy functional associated with physical problem to be considered, as well as a local or averaged measure of state field or even a distance norm between measured and theoretical values of state field within a body and/or on its boundary. Thus, this functional can be used in optimization or identification problems associated with position and sizes of internal cavities or inclusions.

EXAMPLE 6.2 *Consider a circular isotropic domain of external radius b with an internal hole of radius a and isotropy coefficient* $\lambda = 1$, *Fig. 5. The values of state function u are prescribed at the internal and external perimeters, namely*

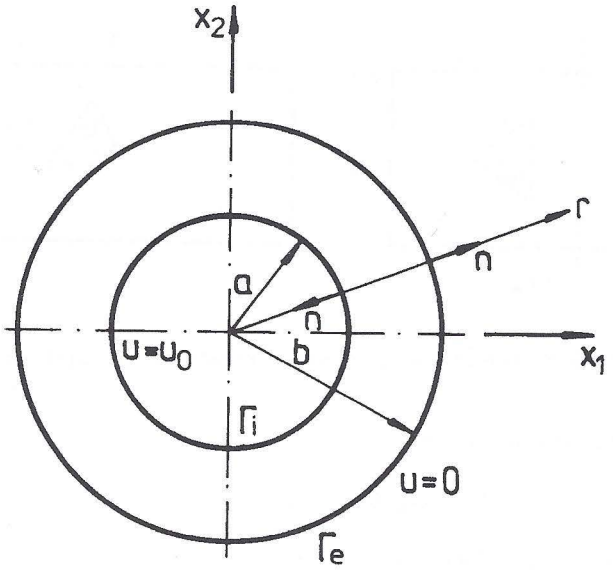

Figure 5. Circular disk with a hole

$$u = u_0 \text{ for } r = a,\ u = 0 \text{ for } r = b \tag{122}$$

The state field distribution referred to the polar coordinate system, due to the axial symmetry of the problem, is as follows

$$u = u_0 \frac{\ln \frac{r}{b}}{\ln \frac{a}{b}} \tag{123}$$

Consider now an averaged measure of flux intensity over external boundary Γ_b, given in the form

$$J + \frac{1}{2\pi b} \int q_n d\Gamma_b = u_{,n} \mid_{r=b} \tag{124}$$

and determine its variation associated with an expansion of the internal hole. According to (91), (93) and (94), this variation can be calculated by means of a path-independent integral over the external perimeter. Since the degree of homogeneity of functional (124) is $\eta = 1$, then, in view of (92), $\xi = 0$ and

$$\delta J = (J^E_{2\Phi})_{\Gamma_b} \delta p,\ (J^E_{2\Phi})_{\Gamma_b} = \int u_{,r} u^a_{,r} r d\Gamma_b = 2\pi b^2 (u_{,r}, u^a_{,r}) \mid_{r=b} \tag{125}$$

The adjoint boundary-value problem associated with functional (124), in view of (30)-(31), is specified by the equation (32) with $\mathbf{A} = \mathbf{I}$, $k = 0$, $f^a = 0$ and

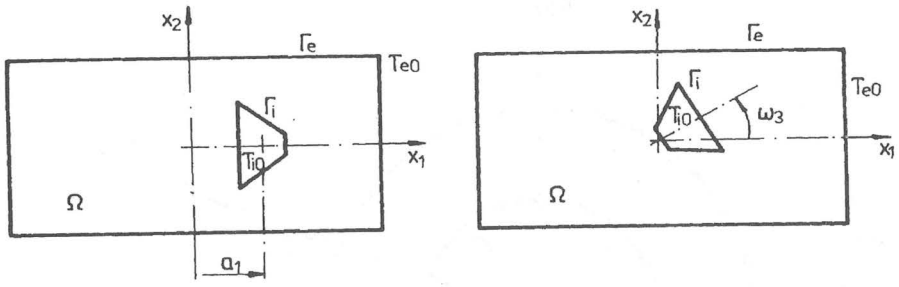

Figure 6. Disk with a hole subjected to translation (a) and rotation (b)

boundary conditions of the form

$$u^a = 0 \text{ for } r = a,\ u^a = \frac{1}{2\pi b} \text{ for } r = b \tag{126}$$

The solution of this problem has the form

$$u^a = -\frac{1}{2\pi b}\frac{\ln\frac{r}{a}}{\ln\frac{a}{b}} \tag{127}$$

and then, in view of (126), we have

$$(J^E_{2\Phi})_{\Gamma_b} = -\frac{u_0}{b\left(\ln\frac{a}{b}\right)^2},\ \delta J = -\frac{u_0}{b\left(\ln\frac{a}{b}\right)^2}\delta p = -\frac{u_0}{ab\left(\ln\frac{a}{b}\right)^2}\delta a \tag{128}$$

since, in view of (66), $\delta a = a\delta p > 0$ corresponds to the expansion of the hole. The result (128) can easily be verified by direct differentiation of (123) with respect to a.

EXAMPLE 6.3 *This example is related to the steady heat transfer problem in rectangular isotropic disk with trapezoidal hole, Fig. 6. The prescribed temperature is specified along internal and external boundaries Γ_i and Γ_e, respectively,*

$$T = T_{i0} \text{ on } \Gamma_i,\ T = T_{e0} \text{ on } \Gamma_e \tag{129}$$

where T denotes the temperature field within disk domain.

Consider now the functional representing the amount of heat transfer through the external boundary from the disk domain to the surrounding environment

$$J = \int q_n d\Gamma_e \tag{130}$$

and determine its sensitivities with respect to translation of the hole along x_1-axis, Fig. 6a, and rotation about x_3-axis, Fig. 6b. The adjoint problem

associated with the functional (130) is specified by equation (32) with boundary conditions following from (31) in the form

$$T^a_{i0} = 0 \text{ on } \Gamma_i,\ T^a_{e0} = 1 \text{ on } \Gamma_e \tag{131}$$

The temperature within the domain of adjoint disk is denoted by T^a. The particular forms of path-independent integrals associated with infinitesimal translation δa_1 and rotation $\delta\omega_3$ of the hole follow from general expressions (49) and (58), respectively. Taking these integrals along external boundary of a disk, we obtain in the case of translation

$$\frac{dJ}{da_1} = (J_1^T)_{\Gamma_e} = \lambda \int [(T_{,1}T^a_{,1} - T_{,2}T^a_{,2})n_1 + (T_{,2}T^a_{,1} - T_{,1}T^a_{,2})n_2]d\Gamma_e \tag{132}$$

whereas for the case of hole rotation we have

$$\begin{aligned}\frac{dJ}{d\omega_3} &= (J_1^R)_{\Gamma_e} = \\ &\lambda \int [(T_{,2}T^a_{,2} - T_{,1}T^a_{,1})(x_2n_1 + x_1n_2) \\ &+(T_{,1}T^a_{,2} + T_{,2}T^a_{,1})(x_1n_1 - x_2n_2)]d\Gamma_e\end{aligned} \tag{133}$$

where λ denotes the conductivity coefficient of an isotropic disk.

The analytic steps for the primary and adjoint problems for different locations of the hole were performed using the finite element method and the desired sensitivities were next calculated from the expressions (132) and (133). In the case of translation, 504 elements with 1932 degrees of freedom were used, whereas for rotation case the disk was discretized using 592 elements with 2266 degrees of freedom.

The results of calculation for the case of hole translation are shown in Fig. 7, where the plots of functional value and its sensitivity following from (132) are depicted. It is easy to observe that the minimal value of functional J corresponds to zero value of its sensitivity. To verify the values of sensitivities calculated from (132), the similar calculations using the central finite differences were also performed for some selected hole locations, as it is shown in Fig. 7b. The similar results obtained for the case of hole rotation are depicted in Fig. 8, where the plots of functional values and its sensitivities with respect to rotation angle are presented. It follows from Fig. 8b that the functional (130) attains in this case three local extrema corresponding to zero values of its sensitivity.

The last example will illustrate the applicability of the results obtained in section 5 for sensitivities with respect to variation of material parameters.

EXAMPLE 6.4 *Consider a heat transfer problem in an isotropic infinite cylinder of internal radius a and external radius b, as shown in Fig. 5. Assume the specified temperature distribution on both sides of the cylinder, namely*

$$u(a) = u_i,\ u(b) = u_e \tag{134}$$

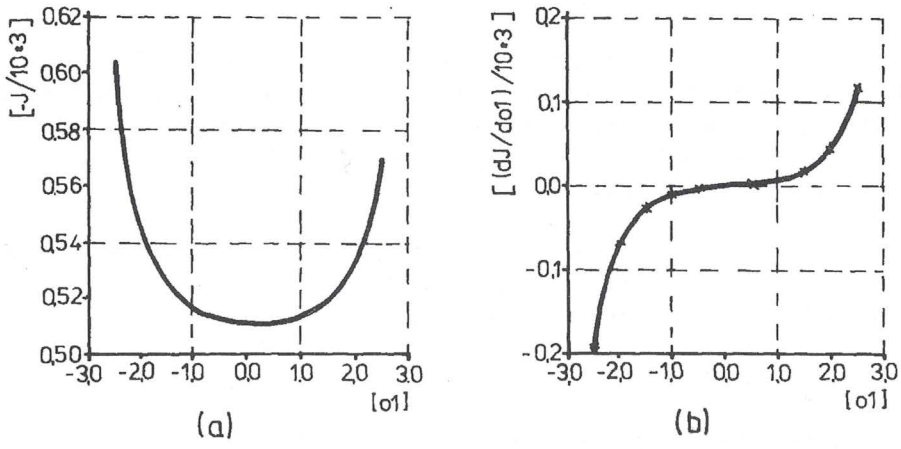

Figure 7. Functional value (a) and its sensitivity (b) versus hole translation

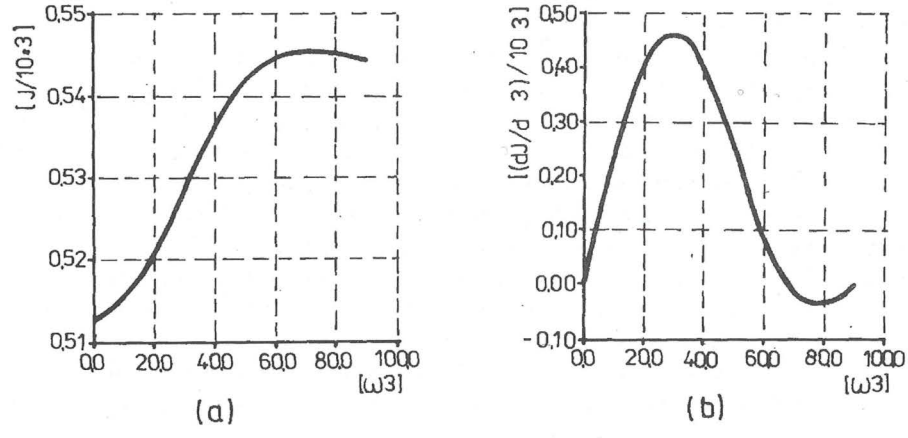

Figure 8. Functional value (a) and its sensitivity (b) versus hole rotation

and the rate of uniform heat generation within the cylinder domain equal to f. *The mathematical formulation of the problem follows immediately from (1). The differential equation in cylindrical coordinates is*

$$\lambda\left(u_{,rr}+\frac{1}{r}u_{,r}\right)=f \tag{135}$$

with boundary conditions (134). Thus, the temperature distribution within the cylinder can be expressed as

$$u=u_i+(u_e-u_i)\frac{\ln(r/a)}{\ln(b/a)}+\frac{f}{4\lambda}\left[r^2-a^2-\frac{(b^2-a^2)\ln(r/a)}{\ln(b/a)}\right] \tag{136}$$

Now consider the surface integral of the form

$$J=\int q_n d\Gamma_e \tag{137}$$

and assume thermal conductivity λ as the only material design parameter a_1. Comparing (137) with (101), we note that $\Psi=0$ within Ω, $\Phi=0$ on Γ_i and $\Phi=q_n$ on Γ_e. We shall derive the sensitivity of J by using both the direct and adjoint approaches.

Using direct approach, we have to calculate the sensitivity of the temperature field with respect to λ. Thus, we have to solve the boundary-value problem (105), which is now particularized as follows

$$\begin{aligned}&\lambda\left(\bar{u}_{,rr}+\frac{1}{r}\bar{u}_{,r}\right)=-\frac{f}{\lambda}\\&\bar{u}(a)=0,\ \bar{u}(b)=0\end{aligned} \tag{138}$$

The solution of (138) takes the form

$$\bar{u}=-\frac{f}{4\lambda}\left[r^2-a^2-\frac{(b^2-a^2)\ln(r/a)}{\ln(b/a)}\right] \tag{139}$$

and then, in view of (103) and (106), the sensitivity of (137) is

$$J_\lambda=\int\left(\bar{q}_n+\frac{1}{\lambda}q_n\right)d\Gamma_e=\frac{2\pi(u_i-u_e)}{\ln(b/a)} \tag{140}$$

To derive the first-order sensitivity of J using the adjoint approach, we solve the adjoint heat transfer problem described by (30)-(32), which are now simplified to the following set of equations

$$\begin{aligned}&u^a_{,rr}+\frac{1}{r}u^a_{,r}=0\\&u^a(a)=0,\ u^a(b)=1\end{aligned} \tag{141}$$

Thus the adjoint temperature field is expressed in the form

$$u^a = \frac{\ln(r/a)}{\ln(b/a)} \tag{142}$$

and, in view of (110), the sensitivity J_λ is

$$J_\lambda = \int u^a_{,r} u_{,r} d\Omega = 2\pi \int_a^b u^a_{,r} u_{,r} r dr = \frac{2\pi(u_i - u_e)}{\ln(b/a)} \tag{143}$$

It is easy to verify (140) or (143) by integrating (137) and then explicitly differentiating the result with respect to λ.

7. Concluding remarks

In the present paper a shape and material parameter sensitivity analysis for quasi-harmonic problem with mixed boundary conditions was discussed using the direct and adjoint approaches. In carrying out the shape sensitivity analysis the material derivative concept was also introduced. The paper provides two methods widely used in structural sensitivity analysis, for generating the first-order sensitivities expressions for an arbitrary response functional. Such functionals can arise in studying anisotropic quasi-harmonic problems describing a wide class of physical and technical problems. As an additional effect of shape sensitivity analysis, a class of path-independent integrals associated with the three most fundamental domain transformations was also presented. This class is associated with variation of an arbitrary behavioural functional due to infinitesimal translation, rotation or scale change of an inhomogeneity within the physically linear domain. The derived path-independent integrals can be applied in studying variation of any functional with respect to size, orientation and growth of single or multiple inclusions, cavities or voids.

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