Control and Cybernetics

vol. **25** (1996) No. 5

How to prove symmetry in Shape Optimization Problems?

by

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Abstract: In this paper we give some different methods for proving the symmetry of the solution of some shape optimization problems. We begin with the Steiner symmetrization, then we show how to use the optimality conditions together with maximum principle, and finally we present another method also based on the optimality conditions.

1. Introduction

In this paper, we want to give different methods for proving that the solution of some shape optimization problem has some symmetry. Of course, it is necessary that the data of the problem themselves possess some symmetry. Our aim is here to give three different kind of methods. The two first methods are directly inspired by the standard situation of calculus of variations. Indeed let us consider a function u being the solution of some problem of minimization:

$$J(u) = \min_{v \in V} J(v).$$

In this situation, if we want to prove that the function u has some symmetry, we can follow two plans:

- we introduce a symmetrization or rearrangement $u^\star$ of u (which has the symmetry we have in mind) and we prove that $J(u^\star) \leq J(u)$.
- we express the optimality conditions (or Euler equations) and we work directly on the differential equations, often together with maximum principle, to prove that the solution u has some symmetry.

In shape optimization problems, these two differents ideas can also work. In the second section, we are working with the Steiner symmetrization. Indeed, in shape optimization problems, this kind of symmetrization can be more convenient than the other ones: it is the only symmetrization which allows to prove symmetry with respect to one hyperplane. The ideas presented in this section are essentially due to Polya and Szegö (1952).

In the third section, after making explicit the optimality conditions thanks to the derivative with respect to the domain, we obtain an overdetermined system and we prove symmetry of the domain using the so-called moving plane method due to Alexandroff and popularized by Gidas, Ni, Niremberg (1979), and, for this kind of situation, Serrin (1971).

In the last section, we present an original method. It consists in introducing a new shape optimization problem whose minima are the solution of the overdetermined optimality system obtained in the previous section. Then, making explicit once again the optimality condition of this new problem, we obtain some new information on the minimal domain allowing us to conclude. This work is taken from a recent paper of the author Choulli and Henrot (to appear).

For the sake of simplicity, we are going to deal in this paper with the following shape optimization problem governed by a linear elliptic equation. Given is Ω an open bounded connected regular subset in $\mathbb{R}^N$, $N \geq 2$ which corresponds to an extremum (minimum or maximum, it will depend) of the following functional

$$J(\omega) = \int_\omega |\nabla u_\omega(x)|^2 \, dx \tag{1}$$

with u_ω the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_\omega = f & \text{in } \omega \\ u_\omega = 0 & \text{on } \partial\omega \end{cases} \tag{2}$$

where f is a given positive function in $L^2_{\text{loc}}(\mathbb{R}^N)$. A natural question which arises in this context is: if we assume some symmetry for the data f, are we able to prove the same kind of symmetry for the minimal domain Ω? More precisely, we are going to give here three different methods to prove the following kind of result (we will precise below):

THEOREM 1.1 *Assume that f is symmetric with respect to the hyperplane $\{x_N = 0\}$, that is to say $f(x', -x_N) = f(x', x_N) \quad \forall x' \in \mathbb{R}^{N-1}, x_N \in \mathbb{R}$, then the domain Ω extremum of the functional J defined in (1), (2) is symmetric with respect to the hyperplane $\{x_N = 0\}$.*

Remark: As a corollary, we obtain that, in the case $f = 1$, the domain Ω must be a ball, giving an answer to a conjecture of Saint-Venant (looking at a maximum of the torsional rigidity of a cross-section of a beam, see Polya, 1948). In the last section, we will furthermore restrict ourselves to this case ($f = 1$) to prove the theorem.

2. Using Steiner symmetrization

Let us first recall the definition of the Steiner symmetrization for sets and functions, we refer to Hardy, Littlewood and Polya (1952), Polya and Szegö (1952) or Kawohl (1985) for more details and proofs. In the sequel, we will denote by $|M|$ the Lebesgue measure of any (measurable) set M. If there is no possible misunderstanding, we will not precise the dimension of the sets considered (sometimes we will be concerned by one-dimensional sets, sometimes by N-dimensional sets).

Let Ω be any open subset in $\mathbb{R}^N$, $N \geq 2$. We set

$$\Omega' := \{x' \in \mathbb{R}^{N-1} \text{ such that } (x', x_N) \in \Omega\}, \text{ (the projection of } \Omega \text{ on } \mathbb{R}^{N-1})$$

and

$$\Omega(x') := \{x_N \in \mathbb{R} \text{ such that } (x', x_N) \in \Omega \ , \ x' \in \Omega'\}$$

(the intersection of Ω with $(x', \mathbb{R})$.
Note that the sets $\Omega(x')$ are open for any $x' \in \mathbb{R}^{N-1}$.

DEFINITION 2.1 *Let Ω be any open subset in $\mathbb{R}^N$. Then the set*

$$\Omega^\star := \left\{x = (x', x_N) \text{ such that } -\frac{1}{2}|\Omega(x')| < x_N < \frac{1}{2}|\Omega(x')|, \ x' \in \Omega'\right\}$$

is called the Steiner-symmetrization of Ω with respect to $x_N = 0$.

Let us notice that, even if Ω is symmetric with respect to $x_N = 0$, it may not coincide with its Steiner-symmetrization. Indeed, this one must be, by construction, convex in the x_N direction, so we have

$$\Omega = \Omega^\star \iff \begin{cases} \Omega \text{ is symmetric with respect to } x_N = 0 \\ \Omega \text{ is convex in the } x_N \text{ direction} \end{cases}$$

Let us now consider a positive function u defined on Ω which has the property that for every $c > 0$

$$\text{the level sets } \{x_N \in \mathbb{R}, \ u(x', x_N) > c\} \text{ have finite Lebesgue measure.} \qquad (3)$$

We can define, for almost every $x' \in \mathbb{R}^{N-1}$, the distribution function of u by

$$m_u(x', c) := |\{x_N \in \mathbb{R}; \ u(x', x_N) > c\}|, \qquad c > 0 \qquad (4)$$

DEFINITION 2.2 *Let u satisfy (3), and $m_u(x', c)$ be defined by (4). We consider the function $y = Y(x', c) := \frac{1}{2}m_u(x', c)$. Its inverse function, denoted by $u^\star$, satisfies*

$$c = u^\star(x', y) = u^\star(x', -y)$$

and is called the Steiner symmetrization of u with respect to $x_N = 0$.

We notice, as for Ω, that the level sets of $u^\star$ are convex in the x_N direction and symmetric in x_N. So, we have the same property: if f is a positive function defined on $\mathbb{R}^N$ and satisfying (3), then

$$f = f^\star \iff \begin{cases} \forall x' \in \mathbb{R}^{N-1}, \forall x_N \in \mathbb{R} \;\; f(x', x_N) = f(x', -x_N) \\ \text{The level sets of } f \text{ are convex in the } x_N \text{ direction} \end{cases} \tag{5}$$

Let us gather in one single theorem the main classical results and properties that are satisfied by the Steiner-symmetrization.

THEOREM 2.1 *Let Ω be an open subset in $\mathbb{R}^N$, u, v two positive functions defined on Ω and satisfying (3), let $\Omega^\star$, $u^\star$ and $v^\star$ be their Steiner-symmetrizations, respectively, then*

(i) $|\Omega| = |\Omega^\star|$

(ii) *If F is continuous from $\mathbb{R}_+^\star$ into $\mathbb{R}$, then*

$$\int_\Omega F(u)(x)\,dx = \int_{\Omega^\star} F(u^\star)(x)\,dx$$

(iii)

$$\int_\Omega uv(x)\,dx \le \int_{\Omega^\star} u^\star v^\star(x)\,dx$$

(iv) *If u belongs to the Sobolev space $W_0^{1,p}(\Omega)$, with $p > 1$, then $u^\star \in W_0^{1,p}(\Omega^\star)$ and*

$$\int_\Omega |\nabla u(x)|^p\,dx \ge \int_{\Omega^\star} |\nabla u^\star(x)|^p\,dx$$

We come back now to the problem. We are now in position to prove the following theorem which is inspired by the paper of Polya (1948).

THEOREM 2.2 *Let $f \in L^2_{\text{loc}}(\mathbb{R}^N)$ be given satisfying (3), assume moreover that f satisfies the following symmetry assumption*

$$f = f^\star \quad \textit{i.e. } f \textit{ satisfies } (5),$$

then if the functional J defined in (1), (2) has a maximum (with or without volume constraint), there exists at least one maximum which is symmetric with respect to the hyperplane $\{x_N = 0\}$.

Proof: Let us denote by Ω a maximum of the functional J. From the variational formulation, it is well known that the solution u_Ω of the problem (2) can be characterized as the minimum on the Sobolev space $H_0^1(\Omega)$ of the functional

$$j_\Omega(v) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx - \int_\Omega fv(x)\,dx.$$

Moreover, also thanks to the variational formulation, we have

$$\int_\Omega |\nabla u_\Omega(x)|^2\,dx = \int_\Omega fu_\Omega(x)\,dx$$

and then

$$j_\Omega(u_\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega(x)|^2\,dx = -\frac{1}{2}J(\Omega).$$

Let us introduce $\Omega^\star$ the Steiner symmetrization of Ω with respect to $\{x_N = 0\}$ (notice that since $|\Omega| = |\Omega^\star|$, if there is a volume constraint, it will also be satisfied by $\Omega^\star$). Now, let us prove that $J(\Omega^\star) \geq J(\Omega)$. As above, $u_{\Omega^\star}$ is the minimum on the Sobolev space $H_0^1(\Omega^\star)$ of the functional

$$j_{\Omega^\star}(v) := \frac{1}{2}\int_{\Omega^\star} |\nabla v(x)|^2\,dx - \int_{\Omega^\star} fv(x)\,dx$$

and

$$j_{\Omega^\star}(u_{\Omega^\star}) = -\frac{1}{2}\int_{\Omega^\star} |\nabla u_{\Omega^\star}(x)|^2\,dx = -\frac{1}{2}J(\Omega^\star)$$

Let us denote by $u^\star$ the (Steiner) symmetrization function of u_Ω. By definition of the Steiner symmetrization, $u^\star$ belongs to the Sobolev space $H_0^1(\Omega^\star)$. So, thanks to the variational formulation, we have

$$j_{\Omega^\star}(u_{\Omega^\star}) \leq j_{\Omega^\star}(u^\star)$$

that is to say

$$\frac{1}{2}\int_{\Omega^\star} |\nabla u_{\Omega^\star}(x)|^2\,dx - \int_{\Omega^\star} fu_{\Omega^\star}(x)\,dx \leq \frac{1}{2}\int_{\Omega^\star} |\nabla u^\star(x)|^2\,dx - \int_{\Omega^\star} fu^\star(x)\,dx.$$

Now, according to theorem 2 , we have

$$\int_{\Omega^\star} |\nabla u^\star(x)|^2\,dx \leq \int_\Omega |\nabla u(x)|^2\,dx \quad \text{and} \quad \int_{\Omega^\star} f^\star u^\star(x)\,dx \geq \int_\Omega fu(x)\,dx$$

therefore

$$-\frac{1}{2}J(\Omega^\star) = j_{\Omega^\star}(u_{\Omega^\star}) = \frac{1}{2}\int_{\Omega^\star} |\nabla u_{\Omega^\star}(x)|^2\,dx - \int_{\Omega^\star} fu_{\Omega^\star}(x)\,dx \leq$$
$$\leq \tfrac{1}{2}\textstyle\int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega fu(x)\,dx = -\tfrac{1}{2}J(\Omega)$$

what proves the theorem. ■

Remark: The assumption on the function f can appear as purely technical, but it is absolutely essential. Indeed, the result is wrong if the function f doesnot satisfy (5) as proved by the following one-dimensional example. Let f be the symmetric function defined by:

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in]-\infty, -4] \cup [-2, 2] \cup [4, +\infty[\\ -\frac{1}{2}x^2 - 3x - 3 & \text{if} \quad x \in]-4, -2[\\ -\frac{1}{2}x^2 + 3x - 3 & \text{if} \quad x \in]2, 4[\end{cases}$$

then the maximum, amongst the domains of length 1, of the functional $J(\omega) := \int_\omega u'^2(x)\,dx$ where u is the solution of $-u" = f$ in ω, with $u = 0$ on $\partial\omega$ is achieved for the interval $]5/2, 7/2[$ (or the symmetric one $]-7/2, -5/2[$) and not by $]-1/2, 1/2[$.

3. Using optimality condition with maximum principle

Let us assume in this section that Ω is a regular (at least C^2) open set which is a maximum or a minimum of the functional J amongst all the domains with given volume. Then, we are going to use the standard tool of the domain derivative to write down the optimality condition (we refer to Simon, 1980, and Sokołowski and Zolesio, 1992, in order to have more details on this topic). Let us consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$, then the classical Hadamard formulae yield for the derivative of J with respect to the deformation V (or in the direction V):

$$dJ(\Omega; V) = \int_\Omega \nabla u_\Omega . \nabla u'(x)\, dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u_\Omega(x)|^2 V.n\, dx \tag{6}$$

where n is the exterior normal vector to $\partial\Omega$ and u' the derivative of u_Ω which can be defined as the solution of the following p.d.e.:

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega \\ u' = -\frac{\partial u_\Omega}{\partial n} V.n & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Now using the Green formula we can evaluate $dJ(\Omega; V)$ also by

$$dJ(\Omega; V) = \int_{\partial\Omega} u_\Omega \frac{\partial u'}{\partial n}\, dx - \int_\Omega u_\Omega \Delta u'(x)\, dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u_\Omega(x)|^2 V.n\, dx$$

that is to say, thanks to (6) and (7)

$$dJ(\Omega; V) = \frac{1}{2}\int_{\partial\Omega} |\nabla u_\Omega(x)|^2 V.n\, dx$$

Now, the derivative of the volume with respect to the deformation V is given by

$$d\mathrm{Vol}(\Omega; V) = \int_{\partial\Omega} V.n\, dx. \tag{8}$$

So, for every domain Ω which is a maximum or a minimum (or more generally a critical point) of the functional J with a volume constraint, we have the existence of a Lagrange multiplier, say C such that, for every displacement field V, we have

$$dJ(\Omega; V) = C\, d\mathrm{Vol}(\Omega; V)$$

that is to say

$$\forall V \in C^2(\mathbb{R}^N; \mathbb{R}^N), \quad \frac{1}{2}\int_{\partial\Omega} |\nabla u_\Omega(x)|^2 V.n\, dx = C \int_{\partial\Omega} V.n\, dx.$$

Therefore, we have proved

PROPOSITION 3.1 *Let Ω be a regular critical point of the functional J defined in (1), (2) with a volume constraint. Then, there exists a constant c such that $|\nabla u_\Omega(x)| = c$ on $\partial\Omega$.*

We have thus obtained, as it is classical in shape optimization, that the extrema of the functional J are also characterized as solutions of an overdetermined problem. To prove the symmetry of the domain Ω, we can now use this information. We are going to use the classical method of moving plane, introduced by Alexandroff and popularized by J. Serrin (1971) and also by Gidas, Ni, Niremberg (1979). First of all, let us give the precise assumptions needed to prove the symmetry result which is essentially due to J. Serrin.

THEOREM 3.1 *Let f be a positive continuous function defined on $I\!R^N$ and satisfying*

- $\forall x_1 \in I\!R\ ,\ \forall x' \in I\!R^{N-1} \quad f(-x_1, x') = f(x_1, x')$
- $x_1 \mapsto f(x_1, x')$ *is a nonincreasing function on* $I\!R_+$.

Let Ω be a C^2 domain which maximizes or minimizes the functional J defined by (1), (2) with a volume constraint, then there exists a real λ such that Ω is symmetric with respect to $x_1 = \lambda$.

Remark: Without more precise assumptions on f, we cannot ensure Ω to be symmetric with respect to $x_1 = 0$. Indeed, in the one-dimensional example where

$$f(x) = \begin{cases} 1 & if \quad x \in [-1,1] \\ 1/|x| & if \quad x \notin [-1,1] \end{cases}$$

with a volume constraint equal to 1, it is easy to verify that the solutions are given by any interval $]a, a+1[$ contained in $[-1,1]$ (see also the counterexample at the end of section 2).

Proof: Preliminary remark: Let us denote by $\tilde{\Omega}$ the symmetric of Ω with respect to $x_1 = 0$. Then, thanks to the assumption on f, it is clear that u_Ω and $u_{\tilde{\Omega}}$ are equal up to a symmetry. So $J(\Omega) = J(\tilde{\Omega})$ and then $\tilde{\Omega}$ is also a maximum (or a minimum) (of course, it is not a proof of the theorem, since we do not know anything about the uniqueness of the solution of our problem).

Let us denote by T_λ the hyperplane $x_1 = \lambda$. When λ is large enough, T_λ does not intersect Ω. We decrease λ, at some moment T_λ begins to intersect Ω, and from that moment T_λ cut off from Ω a cap $\Sigma(T_\lambda)$. We denote by $\Sigma'(T_\lambda)$, the reflexion of $\Sigma(T_\lambda)$ with respect to T_λ. At the beginning of the process, $\Sigma'(T_\lambda)$ stays entirely inside Ω, until one of the following two events occurs:

1. $\Sigma'(T_\lambda)$ becomes internally tangent to the boundary of Ω at some point P not on T_λ
2. T_λ reaches a position where it is orthogonal to the boundary of Ω at some point Q.

We still denote by T_λ the hyperplane when it reaches either one of these positions and by $\Sigma \equiv \Sigma(T_\lambda)$ and $\Sigma' \equiv \Sigma'(T_\lambda)$. We define now a new function v on Σ' by reflexion:

$$v(x_1, x') := u_\Omega(2\lambda - x_1, x').$$

By construction, v satisfies

$$\begin{cases} -\Delta v(x_1, x') = f(2\lambda - x_1, x') & \text{in } \Sigma' \\ v = u & \text{on } \partial\Sigma' \cap T_\lambda \\ v = 0, \quad \frac{\partial v}{\partial n} = \mathrm{c} & \text{on } \partial\Sigma' \cap T_\lambda^c. \end{cases} \tag{9}$$

Since Σ' is contained in Ω by construction, we can consider the function $w = u - v$ in Σ'. It satisfies

$$-\Delta w(x_1, x') = f(x_1, x') - f(2\lambda - x_1, x') \tag{10}$$

Let us prove first that

$$\forall (x_1, x') \in \Sigma' \quad f(x_1, x') - f(2\lambda - x_1, x') \geq 0 \tag{11}$$

First of all, according to the preliminary remark, we can always assume that $\lambda \geq 0$. Indeed, if it were not the case, we could work with the symmetric domain $\tilde{\Omega}$ for which we would evidently have $\lambda \geq 0$. If $\lambda = 0$ the result is obvious since f is symmetric w.r. to $x_1 = 0$. So assume $\lambda > 0$: on Σ' we have $x_1 < \lambda$, so $x_1 < 2\lambda - x_1$, and then if $\lambda > x_1 \geq 0$, the fact that $f(x_1, x') - f(2\lambda - x_1, x') \geq 0$ is a consequence of the assumption on f (monotonicity). Now, if we assume $-(2k+2)\lambda \leq x_1 \leq -2k\lambda \leq 0$, with $k \geq 0$, we have $0 \leq (2k+2)\lambda \leq 2\lambda - x_1 \leq (2k+4)\lambda$, and then, according to the monotonicity of f on $\mathbb{R}_+$ w.r. to the first variable, we have

$$f(2\lambda - x_1, x') \leq f((2k+2)\lambda, x') = f(-(2k+2)\lambda, x') \leq f(x_1, x')$$

what proves (11). Then, it follows from (10) that w is superharmonic on Σ' and therefore has its minimum on the boundary of Σ'. Now, by construction $w \geq 0$ on $\partial\Sigma'$, then it follows, by the strong maximum principle, that either

$$u - v > 0 \quad \text{at all interior points of } \Sigma' \tag{12}$$

or else $u \equiv v$ in Σ'. In the latter case it is clear that the reflected cap Σ' must coincide with the part of Ω which is on the left of T_λ, that is, Ω must be symmetric about T_λ what will prove the theorem.

So, it remains to prove that (12) cannot happen. Suppose first that we are in case 1., that is Σ' is internally tangent to the boundary of Ω at some point P. Then $u - v = 0$ at P, consequently, thanks to (12) and the Hopf boundary point lemma, we can conclude that

$$\frac{\partial u - v}{\partial n} > 0 \quad \text{at } P.$$

This however contradicts the fact that $\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = c$ at P, hence (12) is impossible in the case 1.

In case 2, the situation is more complicated since we cannot apply the boundary point lemma (the point Q is a right angle corner of Σ'). Consequently we must proceed in an alternate way. We can first prove that $u - v$ has a zero of second order at Q and then a contradiction will be obtained from a more delicate version of the boundary point maximum principle due to J. Serrin (we refer to Serrin, 1971, for more details):

LEMMA 3.1 (SERRIN) *Let $D^\star$ be a domain with C^2 boundary and let T be an hyperplane containing the normal to $\partial D^\star$ at some point Q. Let D then denote the portion of $D^\star$ lying on some particular side of T.*
Assume that w is a superharmonic function of class C^2 in the closure of D, while also $w \geq 0$ in D and $w = 0$ at Q. Let s be any direction at Q which enters D nontangentially. Then either

$$\frac{\partial w}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2} > 0 \quad \text{at } Q$$

unless $w \equiv 0$.

To finish the proof of the theorem, we apply this lemma to our situation. Since $w = u - v > 0$ in Σ' and $w = 0$ at Q, this yields

$$\frac{\partial u - v}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2 u - v}{\partial s^2} > 0 \quad \text{at } Q$$

contradicting the fact that both u and v have the same first and second partial derivative at Q. This completes the proof of the theorem.

4. Using the optimality condition (bis)

In this section, we are going to restrict ourselves to the case $f = 1$, that is to say the problem of maximizing or minimizing the torsional rigidity, with a volume constraint. We have seen in section 3 that the optimal domain was such that there exists a function u solution of the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = \text{constant} & \text{on } \partial\Omega. \end{cases} \tag{13}$$

The new idea that we are going to develop here is the following: we want to construct a domain functional $J_1 = J_1(\omega)$ defined for each bounded domain in $\mathbb{R}^N$ and prove that the solutions of problem (13) are exactly the minima of the functional J_1. Then, expressing the optimality condition by means of the domain derivative, we are able to show that the mean curvature of the boundary of a minimizer must be constant. Then, in some sense we are going to use a new shape optimization problem to prove the symmetry result we have in mind (the original work where this method has been first developed is to appear in *Math. Nach.*, see Choulli and Henrot, to appear).

As in the previous sections, we are going to prove:

THEOREM 4.1 (SERRIN) *Let Ω be a bounded open connected domain of class C^2. Then the equation (13) has a solution u belonging to the Sobolev space $H^2(\Omega)$ if and only if Ω is a ball.*

We denote by $\mathcal{O}$ the set of all bounded open connected domains of class C^2 in $\mathbb{R}^N$. To each $\omega \in \mathcal{O}$ we associate u_ω, the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_\omega = 1 \text{ in } \omega \\ u_\omega \in H_0^1(\omega). \end{cases} \tag{14}$$

Since ω is of class C^2, by classical regularity results, u_ω belongs to the space $C^1(\overline{\omega})$ (see for instance Dautray and Lions, 1984). The new functional J_1 that we want to minimize is

$$J_1(\omega) = N \int_{\partial\omega} |\nabla u_\omega|^3 d\sigma - (N+2) \int_\omega |\nabla u_\omega|^2 dx, \quad \omega \in \mathcal{O}.$$

LEMMA 4.1 *$J_1(\omega) \geq 0$ for any ω in $\mathcal{O}$ and $J_1(\omega) = 0$ if ω is a solution of the problem (13).*

Proof: We begin by a simple inequality, already used by Weinberger in his paper Weinberger (1971). For every $\omega \in \mathcal{O}$, we have

$$1 = (\Delta u_\omega)^2 \leq N \sum_{i=1}^N \left(\frac{\partial^2 u_\omega}{\partial x_i^2}\right)^2 \leq N \sum_{i,j=1}^N \left(\frac{\partial^2 u_\omega}{\partial x_i \partial x_j}\right)^2. \tag{15}$$

But

$$2 \sum_{i,j=1}^N \left(\frac{\partial^2 u_\omega}{\partial x_i \partial x_j}\right)^2 = \Delta(|\nabla u_\omega|^2) - 2\nabla(\Delta u_\omega).\nabla u_\omega = \Delta(|\nabla u_\omega|^2). \tag{16}$$

Multiplying (15) by u_ω (which is positive) and integrating over ω, we obtain the inequality

$$\int_\omega u_\omega dx \leq \frac{N}{2} \int_\omega u_\omega \Delta(|\nabla u_\omega|^2) dx. \tag{17}$$

From Green formula and the fact that u_ω vanishes on $\partial\omega$ (the normal derivative of u_ω on the boundary is then given by $\dfrac{\partial u_\omega}{\partial n} = -|\nabla u_\omega|$) it follows that

$$\int_\omega u_\omega dx \leq \frac{N}{2}\left[\int_{\partial\omega} |\nabla u_\omega|^3 d\sigma + \int_\omega |\nabla u_\omega|^2 \Delta u_\omega dx\right]; \tag{18}$$

and then, using equation (14) in its direct form ($\Delta u_\omega = -1$ in ω) and in its variational form

$$\int_\omega |\nabla u_\omega|^2 dx = \int_\omega u_\omega dx$$

yields the desired inequality:

$$0 \leq \frac{N}{2} \int_{\partial\omega} |\nabla u_\omega|^3 d\sigma - (\frac{N}{2} + 1) \int_\omega |\nabla u_\omega|^2 dx. \tag{19}$$

Next, we have to prove that (19) becomes an equality if $|\nabla u_\omega| = -\frac{\partial u_\omega}{\partial n}$ is constant on the boundary of ω. For this, we use the well known formula of Rellich (see for instance Rellich, 1940), valid for any $v \in C^1(\overline{\omega}) \cap H^2(\omega)$,

$$\begin{aligned} &2\int_{\partial\omega} (x.\nabla v)\frac{\partial v}{\partial n} d\sigma - \int_{\partial\omega} (x.n)|\nabla v|^2 d\sigma \\ &= 2\int_\omega (x.\nabla v)\Delta v dx + (2-N)\int_\omega |\nabla v|^2 dx. \end{aligned} \tag{20}$$

Replacing in (20) v by u_ω and using again $\nabla u_\omega = -|\nabla u_\omega| n$ on the boundary, we find

$$\int_{\partial\omega} (x.n)|\nabla u_\omega|^2 d\sigma = -2\int_\omega (x.\nabla u_\omega) dx + (2-N)\int_\omega |\nabla u_\omega|^2 dx. \tag{21}$$

Again Green formula gives

$$\int_\omega x.\nabla u_\omega dx = -N\int_\omega u_\omega dx = -N\int_\omega |\nabla u_\omega|^2 dx.$$

We obtain then the identity

$$\int_{\partial\omega} (x.n)|\nabla u_\omega|^2 = (2+N)\int_\omega |\nabla u_\omega|^2 dx. \tag{22}$$

Assume that $|\nabla u_\omega| = \text{constant} = c$ on $\partial\omega$. Integrating (14) on ω yields

$$V(\omega) = \int_\omega dx = -\int_\omega \Delta u_\omega dx = -\int_{\partial\omega} \frac{\partial u_\omega}{\partial n} d\sigma = c\int_{\partial\omega} d\sigma = cP(\omega); \tag{23}$$

where $V(\omega)$ and $P(\omega)$ denote respectively the volume and the perimeter of ω; while replacing $|\nabla u_\omega| = c$ in (22) gives

$$(2+N)\int_\omega |\nabla u_\omega|^2 dx = c^2 \int_{\partial\omega} x.n d\sigma = Nc^2 V(\omega). \tag{24}$$

Then $J_1(\omega) = Nc^3 P(\omega) - Nc^2 V(\omega) = 0$ at ω a solution of the problem (13), according to (24) what finishes the proof of Lemma 2.

Now, we want to use classical differentiation with respect to the domain to find information on a minimizer of the shape functional J_1 defined in Lemma 2.

Lemma 4.2 *The derivative of the functional J_1 at ω in the direction θ is given by*

$$dJ_1(\omega,\theta) = \int_{\partial\omega} ([(2N-2)|\nabla u_\omega|^2 - 2N(N-1)H|\nabla u_\omega|^3]\theta.n - 3N|\nabla u_\omega|^2 \frac{\partial u'_\omega}{\partial n}) d\sigma \quad (25)$$

Here H is the mean curvature of $\partial\omega$ and u'_ω is defined by (7).

The proof is very classical for those who are familiar with domain derivative. We refer to Choulli and Henrot (to appear) for the details.

Now, let Ω be a solution of the problem (13). According to Lemma 2, Ω is therefore a minimizer of the functional J_1. Then for every vector field θ in $C^2(\mathbb{R}^N, \mathbb{R}^N)$ we must have $dJ_1(\Omega,\theta) = 0$. The computation of the derivative of J_1 at Ω must be completed since $|\nabla u_\Omega| = c = \dfrac{V(\Omega)}{P(\Omega)}$ on the boundary of Ω. Replacing in (25) yields

$$dJ_1(\Omega,\theta) = 2c^2(N-1)\int_{\partial\Omega} [1 - NHc]\theta.n d\sigma - 3Nc^2 \int_{\partial\Omega} \frac{\partial u'_\Omega}{\partial n} d\sigma.$$

But $\displaystyle\int_{\partial\Omega} \frac{\partial u'_\Omega}{\partial n} d\sigma = \int_\Omega \Delta u'_\Omega dx = 0$, and then

$$dJ_1(\Omega,\theta) = 2c^2(N-1)\int_{\partial\Omega} [1 - NHc]\theta.n d\sigma = 0 \text{ for every } \theta \in C^2(\mathbb{R}^N, \mathbb{R}^N).$$

So the mean curvature of $\partial\Omega$ is constant (and it is equal to $\dfrac{P(\Omega)}{NV(\Omega)}$). The theorem follows using the classical result of Alexandroff.

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