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On equilibrium problem for a plate having a crack under the creep condition[1]

by

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Abstract: We consider a boundary problem for equations describing an equilibrium of a plate being under the creep law. The plate is assumed to have a vertical crack. The main peculiarity of the problem is determined by a presence of an inequality imposed on a solution which represents a mutual nonpenetration condition of the crack faces

$$[W]\nu \geq \left|\left[\frac{\partial w}{\partial \nu}\right]\right|,$$

where $W = (w^1, w^2), w$ are horizontal and vertical displacements of mid-surface points of the plate, ν is the normal to the crack shape, $[\cdot]$ is the jump of a function at crack faces. The presence of a crack alone implies a domain wherein the solution is determined to have a nonsmooth boundary, and boundary conditions given at crack faces are of the inequality type.

An existence theorem for the equilibrium problem of the plate is proved. A complete system of equations and inequalities fulfilled at the crack faces is found. Solvability of the optimal control problem with a cost functional characterizing an opening of the crack is established. The solution is shown to belong to the space C^∞ near crack points provided the crack opening is equal to zero.

1. Introduction

Let $\Omega \subset R^2$ be a bounded domain with a smooth boundary $\partial\Omega$, and $y = \psi(x)$ describe a crack shape on the $(x, y)-$plane, $x \in [0, 1]$, $(x, y) \in \Omega$. By Γ_ψ we denote the graph of the function $y = \psi(x), \psi \in H_0^3(0, 1)$. A mid-surface of the plate occupies the domain $\Omega_\psi = \Omega \setminus \Gamma_\psi$. The crack shape as a surface in R^3 can

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be presented in the form $y = \psi(x), -l \leq z \leq l$, where z is the distance from the mid-surface, $2l$ is a plate thickness.

Denote by $W = (w^1, w^2), w$ horizontal and vertical displacements of the mid-surface points, respectively, and write down the formulae for strain and integrated stress tensor components $\varepsilon_{ij}(W)$, $\sigma_{ij}(W)$:

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\right), \quad x_1 = x, x_2 = y,$$

$$\sigma_{11}(W) = \varepsilon_{11}(W) + \kappa\varepsilon_{22}(W),\ \sigma_{22}(W) = \varepsilon_{22}(W) + \kappa\varepsilon_{11}(W),$$

$$\sigma_{12} = (1-\kappa)\varepsilon_{12}(W),\ \kappa = const,\ 0 < \kappa < \frac{1}{2}.$$

Here and everywhere below $i, j = 1, 2$.

Let $\chi = (W, w)$ and

$$\chi^\tau(t, x, y) = \chi(t, x, y) + \int\limits_0^t \chi(\tau, x, y)d\tau\ , \tag{1}$$

$$B(w, \bar{w}) = \int\limits_{\Omega_\psi} (w_{xx}\bar{w}_{xx} + w_{yy}\bar{w}_{yy} + \kappa w_{xx}\bar{w}_{yy} + \kappa w_{yy}\bar{w}_{xx} + 2(1-\kappa)w_{xy}\bar{w}_{xy})d\Omega_\psi.$$

We shall consider an equilibrium problem with a constitutive law corresponding to a creep, in particular, the strain and integrated stress tensor components $\varepsilon_{ij}(W^\tau)$, $\sigma_{ij}(W^\tau)$ will depend on $\chi^\tau = (W^\tau, w^\tau)$, where (W^τ, w^τ) are connected with (W, w) by (1). In this case, the equilibrium equations will be nonlocal with respect to t.

At the external boundary the following boundary conditions are assumed to be satisfied

$$w = \frac{\partial w}{\partial n} = W = 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{2}$$

These conditions correspond to the clamping of the plate at the boundary.

Let Sobolev space $H^{1,0}(\Omega_\psi)$ consist of functions having the first generalized derivatives square integrable in Ω_ψ and which are equal to zero on $\partial\Omega$, the space $H^{2,0}(\Omega_\psi)$ is introduced analogously and consists of functions equal to zero on $\partial\Omega$ with the first derivatives, $H(\Omega_\psi) = H^{1,0}(\Omega_\psi) \times H^{1,0}(\Omega_\psi) \times H^{2,0}(\Omega_\psi)$.

In the domain $Q_\psi = \Omega_\psi \times (0, T)$ we want to find a function (W, w) satisfying the equilibrium equations

$$-\frac{\partial\sigma_{ij}(W^\tau)}{\partial x_j} = u_i, \tag{3}$$

$$\Delta^2 w^\tau = u_3, \tag{4}$$

and boundary condition (2). At the boundary $\Gamma_\psi \times (0,T)$ a system of equations and inequalities is satisfied whose precise form is found in section 3. The function $u = (u_1, u_2, u_3)$ in (3) – (4) is given.

The horizontal displacements of the plate depend on the distance from the mid-surface by the formula from Vol'mir (1972)

$$W(z) = W - z\nabla w, \quad |z| \le l.$$

Hence, the nonpenetration condition of the crack faces can be written as follows

$$[W - z\nabla w]\nu \ge 0 \quad \text{on} \quad \Gamma_\psi \times (0,T), \quad |z| \le l.$$

Here $[U] = U^+ - U^-$ is the jump of a function U on $\Gamma_\psi \times (0,T)$, and $U^\pm$ correspond to the positive and negative directions of the normal ν to the graph Γ_ψ, $\nu = \frac{(-\psi_x, 1)}{\sqrt{1+\psi_x^2}}$, $\nu = (\nu_1, \nu_2)$. The above nonpenetration condition can be rewritten in the equivalent form (Fichera, 1972; Goldshtein, Entov, 1989)

$$[W]\nu \ge l \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \quad \text{on} \quad \Gamma_\psi \times (0,T). \tag{5}$$

For simplicity let $l = 1$.

The structure of the paper is as follows. In Section 2 we prove solvability of the equilibrium problem. This problem is formulated as a variational inequality holding in Q_ψ. The equations (3), (4) are fulfilled in the sense of distributions. On the other hand, if the solution is smooth and satisfies (3), (4) and all the boundary conditions then the above variational inequality holds.

In Section 3 a complete system of equations and inequalities holding on $\Gamma_\psi \times (0,T)$ is found (i.e. boundaty conditions on $\Gamma_\psi \times (0,T)$ are found). Simultaneously, a relationship between two formulations of the problem is established, that is – an equivalence of the variational inequality and the equations (3), (4) with appropriate boundary conditions is proved.

Further, in section 4, an optimal control problem is analysed. The external forces u serve as a control. The solution existence of the optimal control problem with a cost functional describing the crack opening is proved. Finally, in section 5, we prove C^∞-regularity of the solution near crack points having a zero opening.

2. Equilibrium problem. Existence of a solution

In this section we prove an existence theorem of the equilibrium problem for the plate. The problem is formulated as a variational inequality which together with (2), (5) contains a full information about other boundary conditions holding on $\Gamma_\psi \times (0,T)$. An exact form of these conditions is found in the next section.

Let

$$K = \{\chi = (W, w) \in H(\Omega_\psi) |\ [W]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \quad \text{a.e. on} \quad \Gamma_\psi\}.$$

Introduce the set of admissible displacements of the plate,

$$\mathcal{K} = \{\chi \in L^2(0, T; H(\Omega_\psi)) |\ \chi(t) \in K \quad \text{a.e on} \quad (0, T)\},$$

and assume that $u = (u_1, u_2, u_3) \in H^1(0, T; L^2(\Omega_\psi))$. Let the brackets $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2(\Omega_\psi)$.

The following statement provides the solution existence for the equilibrium problem.

THEOREM 2.1 *There exists a unique function χ satisfying the variational inequality*

$$\chi \in \mathcal{K},\ \chi_t \in L^2(0, T; H(\Omega_\psi)),$$

$$\int_0^T B(w^\tau, \bar{w} - w)dt + \int_0^T \langle \sigma_{ij}(W^\tau), \varepsilon_{ij}(\bar{W} - W) \rangle dt \geq \tag{6}$$

$$\geq \int_0^T \langle u, \bar{\chi} - \chi \rangle dt\,, \qquad \forall \quad \bar{\chi} \in \mathcal{K}.$$

Proof. Define the linear and continuous operator $A : L^2(0, T; H(\Omega_\psi)) \to L^2(0, T; H(\Omega_\psi)')$ by the formula

$$A(\chi)(\bar{\chi}) = \int_0^T (B(w^\tau, \bar{w}) + \langle \sigma_{ij}(W^\tau), \varepsilon_{ij}(\bar{W}) \rangle) dt,$$

$$\bar{\chi} = (\bar{W}, \bar{w}) \in L^2(0, T; H(\Omega_\psi)),$$

where (W^τ, w^τ) and (W, w) are connected by the formula (1), and $H(\Omega_\psi)'$ is the space dual to $H(\Omega_\psi)$.

Note that the following inequalities hold in Ω_ψ

$$\langle \sigma_{ij}(W), \varepsilon_{ij}(W) \rangle \geq c\|W\|_1^2, \quad \forall\, W = (w^1, w^2) \in H^{1,0}(\Omega_\psi), \tag{7}$$

$$B(w, w) \geq c\|w\|_2^2, \quad \forall\, w \in H^{2,0}(\Omega_\psi), \tag{8}$$

with the constants c being uniform in W, w, respectively. Hence, owing to the formula

$$A(\chi)(\chi) = \int_0^T \{B(w, w) + \langle \sigma_{ij}(W), \varepsilon_{ij}(W) \rangle\} dt +$$

$$+\frac{1}{2}B(\int_0^T w d\tau, \int_0^T w d\tau) + \frac{1}{2}\langle\sigma_{ij}(\int_0^T W d\tau), \varepsilon_{ij}(\int_0^T W d\tau)\rangle$$

we easily conclude that the operator A is coercive, i.e.

$$\frac{A(\chi)(\chi)}{\|\chi\|_{L^2(0,T;H(\Omega_\psi))}} \to \infty, \quad \|\chi\|_{L^2(0,T;H(\Omega_\psi))} \to \infty.$$

Moreover, the operator A turns out to be pseudomonotone (see definition in Lions, 1969). This implies that the problem

$$A(\chi)(\bar\chi - \chi) \geq \int_0^T \langle u, \bar\chi - \chi\rangle dt, \quad \forall\, \bar\chi \in \mathcal{K}; \quad \chi \in \mathcal{K}, \tag{9}$$

has a solution. In what follows an additional smoothness of the solution χ of (9) with respect to t is proved. To this end, finite differences are used. Let $\varepsilon > 0$ be a parameter and

$$\bar\chi_\varepsilon(\xi) = \begin{cases} \bar\chi, & \xi \in (t-\varepsilon, t+\varepsilon),\ \varepsilon > 0, \\ \chi(\xi), & \text{otherwise} \end{cases}$$

be a test function where $\bar\chi \in K$ is a fixed element. We substitute $\bar\chi_\varepsilon$ in (9) and divide by 2ε the relation obtained. Passing to the limit as $\varepsilon \to 0$ we derive for almost all $t \in (0,T)$

$$B(w^\tau(t), \bar w - w(t)) + \langle\sigma_{ij}(W^\tau(t)), \varepsilon_{ij}(\bar W - W(t))\rangle \geq \tag{10}$$

$$\geq \langle u(t), \bar\chi - \chi(t)\rangle, \quad \forall\, \bar\chi = (\bar W, \bar w) \in K.$$

As seen, the variable t plays a role of a parameter in (10). Let us take $\bar\chi = \chi(t+h)$ as a test function in (10). Then we consider (10) at the point $t+h$ and choose $\bar\chi = \chi(t)$ as a test function. Summing up the obtained inequalities and dividing by h^2 we derive the following relation

$$B(d_h w(t) + d_h^\tau w(t), d_h w(t)) + \langle\sigma_{ij}(d_h W(t) + d_h^\tau W(t)), \varepsilon_{ij}(d_h W(t))\rangle \leq \tag{11}$$

$$\leq \langle d_h u(t), d_h \chi(t)\rangle.$$

Herein the following notations are used

$$d_h v(t) = \frac{v(t+h) - v(t)}{h}, \quad d_h^\tau v(t) = \frac{1}{h}\int_t^{t+h} v(\tau) d\tau.$$

Let, for instance, $h > 0$. The case $h < 0$ can be considered similarly. In view of (7) – (8) we have

$$B(w,w) + \langle\sigma_{ij}(W), \varepsilon_{ij}(W)\rangle \geq c\|\chi\|^2_{H(\Omega_\psi)}, \quad \forall\, \chi = (W,w) \in H(\Omega_\psi).$$

Together with (11) this entails for almost all $t \in (0, T-h)$

$$\|d_h \chi(t)\|^2_{H(\Omega_\psi)} \le c\{\|d_h u(t)\|_0^2 + \|d_h^\tau \chi(t)\|^2_{H(\Omega_\psi)}\} \tag{12}$$

with a constant c uniform in t, h. We next notice that for any smooth function v the following inequalities hold

$$\int_0^{T-h} \|d_h^\tau v(t)\|_0^2 dt \le \int_0^{T-h} d_h\Big(\int_0^t \|v(\tau)\|_0^2 d\tau\Big) dt \le \int_0^T \|v(t)\|_0^2 dt.$$

Hence, the integration of (12) with respect to t from 0 to $T-h$ gives the inequality

$$\int_0^{T-h} \|d_h \chi(t)\|^2_{H(\Omega_\psi)} dt \le c\Big\{\int_0^{T-h} \|d_h u(t)\|_0^2 dt + \int_0^T \|\chi(t)\|^2_{H(\Omega_\psi)} dt\Big\}. \tag{13}$$

The constant c in (13) is uniform in h. Since $u_t \in L^2(Q_\psi)$ we obtain from (13) as $h \to 0$

$$\|\chi_t\|^2_{L^2(0,T;H(\Omega_\psi))} \le c\{\|u_t\|^2_{L^2(Q_\psi)} + \|\chi\|^2_{L^2(0,T;H(\Omega_\psi))}\}.$$

Consequently, the existence of the derivative of the solution to (9) with respect to t is proved. Moreover, by taking $\bar{\chi} = 0$ in (9) we have

$$\|\chi\|^2_{L^2(0,T;H(\Omega_\psi))} \le c\|u\|^2_{L^2(Q_\psi)}.$$

So, the solution of (9) is, in fact, the solution of (6).

The uniqueness of the solution to (6) can be proved by a usual way. As it follows from (6) the difference $\chi = \chi_1 - \chi_2$ of the solutions satisfies the inequality $A(\chi)(\chi) \le 0$. Hence $\chi \equiv 0$. Theorem 2.1 is proved.

Notice that a substitution in (6) of the test functions of the form $\bar{\chi} = \chi + \chi^0$, $\chi^0 \in C_0^\infty(Q_\psi)$, implies that the equations (3), (4) hold in Q_ψ in the sense of distributions. By virtue of the proved inclusion $\chi_t \in L^2(0,T;H(\Omega_\psi))$ the variational inequality (10) is fulfilled for all $t \in (0,T)$.

3. Boundary conditions on $\Gamma_\psi \times (0,T)$

This section is concerned with the search, for boundary conditions holding on $\Gamma_\psi \times (0,T)$, for the solution of (10) or, what is the same, of (9). Our arguments are formal in that the solution is assumed to be smooth enough.

Let $D \subset R^2$ be a bounded domain and let γ be its smooth boundary with the external normal $n = (n_1, n_2)$. We introduce the operators on the boundary γ,

$$M(w) = \kappa\Delta w + (1-\kappa)\frac{\partial^2 w}{\partial n^2}, \quad R(w) = \frac{\partial}{\partial n}\Delta w + (1-\kappa)\frac{\partial^3 w}{\partial n \partial^2 s}, \tag{14}$$

$s = (-n_2, n_1)$.

For any smooth functions $w, v, W, V=(v_1, v_2)$ the following Green formulae hold

$$B_D(w,v)=\langle M(w),\frac{\partial v}{\partial n}\rangle_\gamma-\langle R(w),v\rangle_\gamma+\langle \Delta^2 w,v\rangle_D, \tag{15}$$

$$\langle \sigma_{ij}(W),\varepsilon_{ij}(V)\rangle_D=\langle \sigma_{ij}(W)n_j,v_i\rangle_\gamma-\langle\frac{\partial \sigma_{ij}(W)}{\partial x_j},v_i\rangle_D. \tag{16}$$

The subscripts D,γ denote the integration over the domain D and the boundary γ, respectively. Note that the boundary $\partial\Omega_\psi$ of Ω_ψ is a combination of the sets $\partial\Omega,\Gamma_\psi^+,\Gamma_\psi^-$. The formulae (15), (16) hold true for the domain Ω_ψ despite the absence of regularity of $\partial\Omega_\psi$. To verify this we can extend the graph Γ_ψ so that the domain is divided into two parts. For each of these parts the formulae (15), (16) are valid, hence the statement follows. We should note at this point that the external normals on $\Gamma_\psi^+,\Gamma_\psi^-$ have opposite directions.

To simplify the formulae in this section we shall write w^τ, W^τ, w... instead of $w^\tau(t), W^\tau(t), w(t)$... This means that we fix t and consider the boundary conditions on Γ_ψ for this fixed value t. The same value t is assumed to be chosen in (10).

Introduce the notation $U=(u_1,u_2)$ and take the test functions of the form $(\bar W, w)$ in (10). This implies the variational inequality

$$\langle \sigma_{ij}(W^\tau),\varepsilon_{ij}(\bar W-W)\rangle\ge\langle U,\bar W-W\rangle \tag{17}$$

holding for all functions $\bar W$ such that

$$[\bar W]\nu\ge\left|\left[\frac{\partial w}{\partial\nu}\right]\right|\quad\text{on}\quad\Gamma_\psi,\quad \bar W\in H^{1,0}(\Omega_\psi).$$

On the other hand, we can substitute the test functions of the form $(W,\bar w)$ in (10), which entails the variational inequality

$$B(w^\tau,\bar w-w)\ge\langle u_3,\bar w-w\rangle. \tag{18}$$

The inequality (18) holds for all functions $\bar w$ satisfying the relation

$$[W]\nu\ge\left|\left[\frac{\partial \bar w}{\partial\nu}\right]\right|\quad\text{on}\quad\Gamma_\psi,\quad \bar w\in H^{2,0}(\Omega_\psi).$$

At the boundary Γ_ψ^- we can decompose the vector $\{\sigma_{ij}(W^\tau)\nu_j\}$ into the sum of the normal and tangential components

$$\{\sigma_{ij}(W^\tau)\nu_j\}=\sigma_\nu(W^\tau)\nu+\sigma_s(W^\tau)s,\quad s=(-\nu_2,\nu_1). \tag{19}$$

A similar decomposition takes place on Γ_ψ^+. Let us substitute in (17) the test functions of the form $W+\tilde W$, where smooth functions $\tilde W$ belong to $H^{1,0}(\Omega_\psi)$,

$[\tilde{W}]\nu \geq 0$ on Γ_ψ, and make use of (16). A simple reasoning results in the relations

$$[\sigma_\nu(W^\tau)] = 0, \ \sigma_s(W^\tau) = 0 \quad \text{on} \quad \Gamma_\psi. \tag{20}$$

To proceed, we choose functions of the form $w+\theta$ as test ones in (18), where θ is a smooth function in Ω_ψ having a support in a neighbourhood of a fixed point of Γ_ψ and such that $[\frac{\partial\theta}{\partial\nu}] = 0$. In all, note that $[\theta] \neq 0$. Thanks to (15) this leads to the relations

$$[M(w^\tau)] = 0, \quad R(w^\tau) = 0 \quad \text{on} \quad \Gamma_\psi. \tag{21}$$

We next choose in (10) the test functions of the form $(\bar{W}, \bar{w}) = (0,0)$, $(\bar{W}, \bar{w}) = 2(W, w)$. Using (3), (4) and (15) one easily gets

$$\langle M(w^\tau), \left[\frac{\partial w}{\partial \nu}\right]\rangle_{\Gamma_\psi} + \langle \sigma_\nu(W^\tau), [W]\nu\rangle_{\Gamma_\psi} = 0. \tag{22}$$

On the other hand, a substitution of the test function $(\bar{W}, \bar{w}) = (W, w) + (\tilde{W}, \tilde{w})$ in (10) provides the inequality

$$B(w^\tau, \tilde{w}) + \langle \sigma_{ij}(W^\tau), \varepsilon_{ij}(\tilde{W})\rangle \geq \langle u, \tilde{\chi}\rangle, \tag{23}$$

where $(\tilde{W}, \tilde{w})$ are smooth functions belonging to K. We can integrate here by (3), (4), (15), which gives

$$\langle M(w^\tau), \left[\frac{\partial \tilde{w}}{\partial \nu}\right]\rangle_{\Gamma_\psi} + \langle \sigma_\nu(W^\tau), [\tilde{W}]\nu\rangle_{\Gamma_\psi} \leq 0. \tag{24}$$

Let $(\tilde{W}, \tilde{w})$ be smooth functions having supports in a neighbourhood of a fixed point on Γ_ψ and such that $[\frac{\partial\tilde{w}}{\partial\nu}] = [\tilde{W}]\nu$. We substitute $(\tilde{W}, \tilde{w})$ in (24) and derive

$$M(w^\tau) + \sigma_\nu(W^\tau) \leq 0.$$

Analogously, by choosing $[\frac{\partial\tilde{w}}{\partial\nu}] = -[\tilde{W}]\nu$ one easily gets

$$-M(w^\tau) + \sigma_\nu(W^\tau) \leq 0.$$

Thus, in fact, we have the inequality

$$|M(w^\tau)| \leq -\sigma_\nu(W^\tau) \quad \text{on} \quad \Gamma_\psi. \tag{25}$$

By virtue of (5), (22), (25) we arrive at the conclusion that

$$M(w^\tau)\left[\frac{\partial w}{\partial \nu}\right] + \sigma_\nu(W^\tau)[W]\nu = 0 \quad \text{on} \quad \Gamma_\psi. \tag{26}$$

Hence, the form of the boundary condition on $\Gamma_\psi \times (0,T)$ is completely determined. Together with (5), for all $t \in (0,T)$ the conditions (20) – (21), (25) – (26) hold on Γ_ψ.

Notice that the variational inequality (10) can be derived from (3), (4) and the above boundary conditions. In fact, let us assume that the solution (W, w) is smooth enough and satisfies (3), (4) and the boundary conditions obtained. We choose a smooth function $(\bar{W}, \bar{w}) \in K$ and multiply (3), (4) taken for a fixed $t \in (0,T)$ by $\bar{w}^i - w^i(t), \bar{w} - w(t)$, respectively. We next integrate over Ω_ψ taking into account (5), (20) – (21), (25) – (26). For the value $t \in (0,T)$ chosen above this implies

$$B(w^\tau, \bar{w} - w) + \langle \sigma_{ij}(W^\tau), \varepsilon_{ij}(\bar{W} - W)\rangle - \langle u, \bar{\chi} - \chi\rangle +$$

$$+\langle M(w^\tau), \left[\frac{\partial \bar{w}}{\partial \nu}\right] - \left[\frac{\partial w}{\partial \nu}\right]\rangle_{\Gamma_\psi} + \langle \sigma_\nu(W^\tau), [\bar{W}]\nu - [W]\nu\rangle_{\Gamma_\psi} = 0.$$

According to the boundary conditions the sum of integrals over Γ_ψ is non-positive here, whence (10) follows.

Thus, the boundary problem describing the equilibrium of the plate having a crack can be formulated both in the form (10) (or (6)) and in the form of equations (3), (4) with (5) and conditions (20) – (21), (25) – (26) fulfilled for all $t \in (0,T)$. In this case the latter formulation of the problem is formal in the sense that an additional regularity of the solution is assumed. The solution regularity which follows from (6), in general, does not provide the moments $M(w^\tau)$ and transverse forces $R(w^\tau)$ to be clearly identified at the boundary $\Gamma_\psi \times (0,T)$.

4. Optimal control problem

The goal of this section is to prove an existence theorem for the optimal control problem.

Let $\mathcal{W} \subset H^1(0,T; L^2(\Omega_\psi))$ be a convex, bounded and closed set. For any fixed $u \in \mathcal{W}$ we can find the unique solution $\chi = \chi(u)$ of (6) and define the cost functional characterizing the opening of the crack (see Goldshtein, Entov, 1989)

$$J(u) = \int_0^T \int_{\Gamma_\psi} |[\chi]| d\Gamma_\psi dt.$$

Our aim is to minimize this functional:

$$\inf_{u \in \mathcal{W}} J(u). \tag{27}$$

The result given below provides the solvability of the optimal control problem formulated.

THEOREM 4.1 *There exists a solution of the optimal control problem (27), (6).*

Proof. Let $u^n \in \mathcal{W}$ be a minimizing sequence. By its boundedness in $H^1(0,T;L^2(\Omega_\psi))$ one can assume that as $n \to \infty$

$$u^n \to u \quad \text{weakly in} \quad H^1(0,T;L^2(\Omega_\psi)),\ u \in \mathcal{W}. \tag{28}$$

For every n there exists a unique solution of the variational inequality

$$A(\chi^n)(\bar{\chi} - \chi^n) \geq \int_0^T \langle u^n, \bar{\chi} - \chi^n \rangle dt, \quad \forall\, \bar{\chi} \in \mathcal{K}; \quad \chi^n \in \mathcal{K}. \tag{29}$$

As we are well aware, $\chi_t^n \in L^2(0,T;H(\Omega_\psi))$ and, moreover, it follows from the proof of Theorem 1 that

$$\|\chi^n\|^2_{H^1(0,T;H(\Omega_\psi))} \leq c\|u^n\|^2_{H^1(0,T;L^2(\Omega_\psi))}$$

with a constant c being uniform in n. Without any loss we assume that as $n \to \infty$

$$\chi^n,\ \chi_t^n,\ \int_0^t \chi^n d\tau \to \chi,\ \chi_t,\ \int_0^t \chi d\tau \quad \text{weakly in} \quad L^2(0,T;H(\Omega_\psi)),$$

$$\chi^n \to \chi \quad \text{strongly in} \quad L^2(Q_\psi),$$

$$[\chi^n] \to [\chi] \quad \text{weakly in} \quad L^1(0,T;L^1(\Gamma_\psi)).$$

The last line here is due to the imbedding continuity of $L^2(0,T;H(\Omega_\psi))$ in $L^2(0,T;L^2(\Gamma_\psi))$. The above convergence and (28) allow us to pass to the limit in (29) and to get $\chi_t \in L^2(0,T;H(\Omega_\psi))$,

$$A(\chi)(\bar{\chi} - \chi) \geq \int_0^T \langle u, \bar{\chi} - \chi \rangle dt, \quad \forall \bar{\chi} \in \mathcal{K}; \quad \chi \in \mathcal{K}. \tag{30}$$

The variational inequality (30) precisely means that $\chi = \chi(u)$. On the other hand

$$\inf_{\bar{u} \in \mathcal{W}} J(\bar{u}) = \liminf_{n \to \infty} J(u^n) \geq J(u) \geq \inf_{\bar{u} \in \mathcal{W}} J(\bar{u})$$

so that the constructed function u is a solution of the optimal control problem (27), (6) which completes the proof.

5. Solution regularity near crack points

When $J(u) = 0$ the crack is said to have a zero opening. As it turned out the solution is infinitely differentiable provided that the crack has a zero opening. This assertion, in particular, means that if we have a zero crack opening the presence of the crack has no influence on the displacement field. In this case the plate behaviour precisely coincides with that of the plate without a crack. This property reminds the removable singularity property. We shall prove that C^∞-regularity is a local property. If the crack opening is zero in a vicinity of some fixed point at Γ_ψ for all $t \in (0, t^0)$, then the solution is infinitely smooth near this point for all $t \in (0, t^0)$. Of course, the external force u is assumed to be infinitely smooth in this case. We should also remark that the above regularity property holds provided a zero opening takes place since $t = 0$. In general, if the crack opening is zero for $t \in (t^1, t^0), t^1 > 0, t^0 > t^1$, the solution does not have C^∞-regularity.

Note that asymptotic properties of solutions to the biharmonic equation and the linear elasticity equations near nonsmooth boundaries are analysed in Oleinik, Kondratiev, Kopaček (1981;1982), Morozov (1984), Nicaise (1992). The existence of the so-called extreme crack shapes in plates (i.e. shapes defined through a minimization of cost functionals) has been studied in Khludnev (1992;1994), see also Banichuk (1970).

The arguments given below are concerned with a justification of C^∞-regularity of the solution for the crack of zero opening. We shall prove the solution regularity in a neighbourhood of the line $x^0 \times (0, t^0)$, where $x^0 \equiv (0, 0), t^0 > 0$, i.e. in a vicinity of the crack end. The solution regularity near the line $\bar{x} \times (0, t^0)$, where $\bar{x} \in \Gamma_\psi \setminus \partial\Gamma_\psi$, can be easily proved.

So, let $\mathcal{O}(x^0) \subset R^2$ be a neighbourhood of the point x^0, and let $\mathcal{O} = \mathcal{O}(x^0) \times (0, t^0)$.

Extend the function $\psi(x)$ beyond $x = 0$ assuming that the extension is smooth enough. Denote by $\tilde{\Gamma}_\psi$ the graph of the extended function. Also, let $\mathcal{O}^+(x^0) = \mathcal{O}(x^0) \cap \{y > \psi(x)\}, \mathcal{O}^-(x^0)$ be defined analogously, $\mathcal{O}^\pm = \mathcal{O}^\pm(x^0) \times (0, t^0)$.

As shown, equations (3), (4) hold in $\mathcal{O}^\pm$ in the sense of distributions.

Thanks to the regularity of (W, w) which follows from Theorem 1 we conclude that for all $t \in (0, T)$ in $\mathcal{O}^\pm(x^0)$ the following equations are fulfilled

$$-\frac{\partial \sigma_{ij}(W^\tau(t))}{\partial x_j} = u_i(t), \tag{31}$$

$$\Delta^2 w^\tau(t) = u_3(t) \tag{32}$$

in the sense of (two-dimensional) distributions. As in the above case, let $D \subset R^2$ be a fixed bounded domain with smooth boundary γ. As it turned out, the values $M(w)$ and $R(w)$ can be correctly defined on γ, namely, $M(w) \in H^{-\frac{1}{2}}(\gamma)$,

$R(w) \in H^{-\frac{3}{2}}(\gamma)$ provided that $w \in H^2(D), \Delta^2 w \in L^2(D)$ and, moreover, the following formula holds (see John, Naumann, 1976; Lions, Magenes, 1968)

$$B_D(w,v) = \langle M(w), \frac{\partial v}{\partial n} \rangle_{\frac{1}{2},\gamma} - \langle R(w), v \rangle_{\frac{3}{2},\gamma} + \langle \Delta^2 w, v \rangle_D, \quad \forall\, v \in H^2(D). \tag{33}$$

Here $\langle \cdot, \cdot \rangle_{s,\gamma}$ stands for the duality pairing between $H^{-s}(\gamma)$ and $H^s(\gamma)$.

If $\sigma_{ij}(W) \in L^2(D)$, $\frac{\partial \sigma_{ij}(W)}{\partial x_j} \in L^2(D)$, the values $\sigma_{ij}(W) n_j$ can be correctly defined on γ as elements of $H^{-\frac{1}{2}}(\gamma)$ and (see Temam, 1979)

$$\langle \sigma_{ij}(W), \varepsilon_{ij}(V) \rangle_D = \langle \sigma_{ij}(W) n_j, v_i \rangle_{\frac{1}{2},\gamma} - \langle \frac{\partial \sigma_{ij}(W)}{\partial x_j}, v_i \rangle_D, \tag{34}$$

$$\forall\, V = (v_1, v_2) \in H^1(D).$$

Henceforth the boundaries of $\mathcal{O}^{\pm}(x^0)$ are denoted by $\gamma^{\pm}$, respectively.

Let a function $\varphi \equiv (\varphi_1, \varphi_2)$ belong to the space $C_0^\infty(\mathcal{O}(x^0))$ and be qual to zero beyond $\mathcal{O}(x^0)$. Then $(W(t) + \varphi, w(t)) \in K$. We substitute $(W(t) + \varphi, w(t))$ in (10) as a test function. This implies for all $t \in (0,T)$

$$\langle \sigma_{ij}(W^\tau), \varepsilon_{ij}(\varphi) \rangle_+ + \langle \sigma_{ij}(W^\tau), \varepsilon_{ij}(\varphi) \rangle_- \geq \langle u_i, \varphi_i \rangle.$$

To simplify the formulae here and below we do not show the dependence of the functions on t. Subscripts $+, -$ denote the integration over $\mathcal{O}^{\pm}(x^0)$, respectively. Owing to the formula (34) the last inequality gives for all $t \in (0,T)$:

$$-\langle [\sigma_{ij}(W^\tau) \nu_j], \varphi_i \rangle_{\frac{1}{2},\gamma^-} - \langle \frac{\partial \sigma_{ij}(W^\tau)}{\partial x_j}, \varphi_i \rangle_{\pm} \geq \langle u_i, \varphi_i \rangle. \tag{35}$$

The existence of two angular points on $\gamma^{\pm}$ presents no problems since φ has a compact support. Hence, the inequality (35) with the equations (31) yield the identity

$$\langle [\sigma_{ij}(W^\tau) \nu_j], \varphi_i \rangle_{\frac{1}{2},\gamma^-} = 0, \quad \forall\, \varphi \in C_0^\infty(\mathcal{O}(x^0))$$

and consequently

$$[\sigma_{ij}(W^\tau) \nu_j] = 0 \quad \text{on} \quad \tilde{\Gamma}_\psi \cap \mathcal{O}(x^0). \tag{36}$$

Let $\theta \in C_0^\infty(\mathcal{O}(x^0))$. Beyond $\mathcal{O}(x^0)$ the function θ is assumed to be equal to zero. We substitute $(W(t), \theta + w(t))$ as a test function in (10). As a result the following inequality being valid for all $t \in (0,T)$ follows

$$B_+(w^\tau, \theta) + B_-(w^\tau, \theta) \geq \langle u_3, \theta \rangle. \tag{37}$$

Since the equation (32) holds in $\mathcal{O}^{\pm}(x^0)$ we easily deduce from (37) for all $t \in (0,T)$ that

$$\langle [M(w^\tau)], \frac{\partial \theta}{\partial n} \rangle_{\frac{1}{2},\gamma} = 0, \quad \langle [R(w^\tau)], \theta \rangle_{\frac{3}{2},\gamma} = 0, \quad \forall\, \theta \in C_0^\infty(\mathcal{O}(x^0)).$$

Here γ can coincide with γ^+ or γ^-. Thanks to the arbitrariness of θ the above identities imply for all $t \in (0,T)$

$$[M(w^\tau)] = 0, \quad [R(w^\tau)] = 0 \quad \text{on} \quad \tilde{\Gamma}_\psi \cap \mathcal{O}(x^0). \tag{38}$$

Now we are in a position to prove the result on the regularity of the solution near crack faces.

THEOREM 5.1 *Let $u \in C^\infty(\mathcal{O})$ and*

$$\int_0^{t^0} \int_{\Gamma_\psi \cap \mathcal{O}(x^0)} |[\chi]| d\Gamma_\psi dt = 0.$$

Then

$$\chi \in C^\infty(\mathcal{O}). \tag{39}$$

Proof. The hypotheses of the theorem provide the condition $[\chi] = 0$ on $(\tilde{\Gamma}_\psi \cap \mathcal{O}(x^0)) \times (0, t^0)$, whence

$$[\chi^\tau] = 0 \quad \text{on} \quad (\tilde{\Gamma}_\psi \cap \mathcal{O}(x^0)) \times (0, t^0).$$

Moreover, using (5) we obtain

$$\left[\frac{\partial w^\tau}{\partial \nu}\right] = 0 \quad \text{on} \quad (\tilde{\Gamma}_\psi \cap \mathcal{O}(x^0)) \times (0, t^0).$$

Note that $(W^\tau, w^\tau) \in H^1(0, t^0; H^1(\mathcal{O}^\pm(x^0)) \times H^2(\mathcal{O}^\pm(x^0))$. The above observations concerning the jumps $[\chi^\tau]$, $[\frac{\partial w^\tau}{\partial \nu}]$ imply (see Mikhailov, 1976)

$$(W^\tau, w^\tau) \in H^1(0, t^0; H^1(\mathcal{O}(x^0)) \times H^2(\mathcal{O}(x^0)).$$

Following this inclusion and the conditions (36), (38) we shall prove that the equations (3), (4) hold in $\mathcal{O}$ in the sense of distributions.

Denote by $(\cdot, \varphi)$ the value of a distribution at the point φ. For any $\varphi \in C_0^\infty(\mathcal{O})$ we have

$$-\left(\frac{\partial \sigma_{ij}(W^\tau)}{\partial x_j} + u_i, \varphi\right) = \int_0^{t^0} \langle \sigma_{ij}(W^\tau), \frac{\partial \varphi}{\partial x_j} \rangle_\pm dt -$$

$$-(u_i, \varphi) = -\int_0^{t^0} \langle [\sigma_{ij}(W^\tau)\nu_j], \varphi \rangle_{\frac{1}{2},\gamma^-} dt - \tag{40}$$

$$-\int_0^{t^0} \left\langle \frac{\partial \sigma_{ij}(W^\tau)}{\partial x_j} + u_i, \varphi \right\rangle_\pm dt.$$

Owing to (36), (31) we readily conclude that the right-hand side of (40) is equal to zero, which implies the equations

$$-\frac{\partial \sigma_{ij}(W^\tau)}{\partial x_j} = u_i \quad \text{in} \quad \mathcal{O} \tag{41}$$

holding in the sense of distributions.

Analogously, for any $\varphi \in C_0^\infty(\mathcal{O})$ we derive

$$(\Delta^2 w^\tau - u_3, \varphi) = \int_0^{t^0} B(w^\tau, \varphi)dt - (u_3, \varphi) =$$

$$= \int_0^{t^0} B_\pm(w^\tau, \varphi)dt - (u_3, \varphi) = -\int_0^{t^0} \langle [M(w^\tau)], \frac{\partial \varphi}{\partial \nu}\rangle_{\frac{1}{2},\gamma^-} dt + \tag{42}$$

$$+ \int_0^{t^0} \langle [R(w^\tau)], \varphi\rangle_{\frac{3}{2},\gamma^-} dt + \int_0^{t^0} \langle \Delta^2 w^\tau - u_3, \varphi\rangle_\pm dt.$$

It is evident from (32), (38) that the right-hand side of (42) is equal to zero. Hence, the equation

$$\Delta^2 w^\tau = u_3 \quad \text{in} \quad \mathcal{O} \tag{43}$$

holds in the sense of distributions.

The statement (39) of the theorem clearly follows from (41), (43). In fact, one can locally solve the elliptic equations (41), (43) for each fixed $t \in (0, t^0)$ and get the infinite differentiability with respect to x, y of the functions $\chi^\tau(t) = \chi(t) + \int_0^t \chi(\tau)d\tau$ in any fixed subdomain of $\mathcal{O}(x^0)$ (see Fichera, 1972; Lions, Magenes, 1968). The function $\chi^\tau(t)$ is infinitely differentiable with respect to $t \in (0, t^0)$ and hence $\chi^\tau \in C^\infty(\mathcal{O})$, $\chi \in C^\infty(\mathcal{O})$. The proof is complete. ∎

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