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Anti-optimization technique to include uncertainties in structural optimization problems

by

C. Cinquini & M. Lombardi
Department of Structural Mechanics, University of Pavia,
Via Ferrata 1, I-27100 Pavia - Italy

Abstract: This paper deals with the problem of dependence that solution of structural optimization problems may have on the uncertainties of some problem parameters, as for instance value of material moduli, geometry and dimensions of the structure, loads magnitude and/or unprescribed design load combinations. The problem is approached with a non-probabilistic method called anti-optimization, first proposed by Elishakoff and subsequently used by several researchers for different structural problems.

The optimization of the uncertain structure is carried out by alternating the optimization with the search of the value of the uncertainties that pictures the worst scenario for the considered problem which is identified with the anti-optimization. The method is applied to solve simple linear and non-linear optimization problems.

1. Introduction

In structural analysis and design the problem of uncertainties has been studied for quite a long time, and the use of probability theory is nowadays common practice whenever the variation of data can have non negligible effect on the behavior of the structure. Although uncertainty has always been associated with probabilistic models, probability is not its only natural solution. The probabilistic approach does not always lead to easy formulation of the problems, and sometimes the probability functions used to describe the uncertain variables do not represent their real distributions, leading to largely unprecise results. What else then? According to Elishakoff (1990), if a good knowledge of the uncertainty should lead one to use probabilistic methods, a non complete knowledge of data should be handled by choosing one of the methods at the other corners of the uncertainty triangle, Fig.1. The choice between fuzzy methods, and convex methods depends on the level of available information.

In fact when the amount of data is limited one may not be able to define a precise probability distribution, but may at least define bounds for the uncertain variable, and try to obtain bounds on the solution of the problem. The

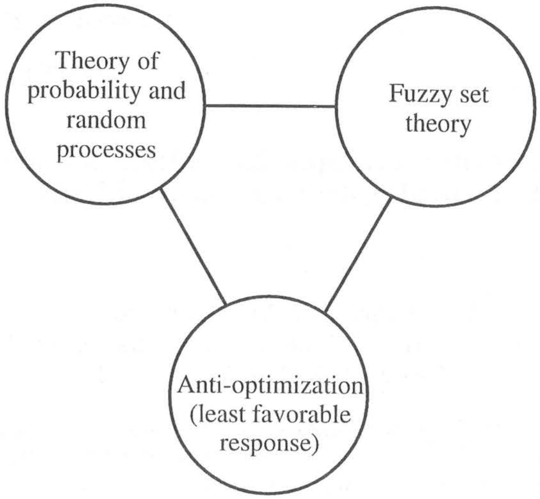

Figure 1. Elishakoff's Uncertainty Triangle

designer will generally seek the least favorable solution for the structure by finding extrema of a given function, within the domain defined by the bounds, which generally results to be convex. This search for the worst condition is called anti-optimization.

The problem of data uncertainty may be even more crucial in optimization, where structure resources are exploited at a maximum level, and safety factors may be very low, increasing the structure sensibility to uncertainties. It is then very important to find the worst value of the uncertain variables, such that, once the optimal design is found, possible variation of the variables does not lead to violation of the constraints. If a probabilistic description of the uncertainties is used, a certain probability of occurrence for the worst scenario will be prescribed. In the convex model the worst case will be searched in the domain defined by the bounds defined for the uncertain variables. The latter description will be used here.

2. Previous works

The framework of the original formulation of the of anti-optimization based on the convex models is due to Ben-Haim and Elishakoff (1990), and was first used as an alternative to probabilistic approaches to define bounds for the problem response, thus its maximum and minimum value, from given bounds for the uncertain variables and under certain initial conditions Ben-Haim (1994), Elishakoff, Cai and Starnes Jr. (1994), and Elishakoff, Li and Starnes Jr. (1994). In a recent work by Adali et al. (1995), this formulation was used for the optimization of composite plates for minimum weight, with buckling and displacement constraints, subject to uncertain transverse and in-plane loads. Optimum design is sought using worst load combination, found with anti-optimization.

The name anti-optimization was also used in an extended sense, to describe the search of the worst scenario for a given structural problem. The convex formulation is no longer used in these works, but only the idea of the method is retained. Gangadharan, Nikolaidis, Lee and Haftka (1993) used anti-optimization to evaluate the difference between two different finite element models of a car welded joint by comparing the value of their strain energy. In another work Lee, Haftka, Griffin Jr., Watson and Sensmeier (1994) proposed a method based on anti-optimization for detecting delamination fronts in a composite laminate beam. In Van Wamelen, Johnson and Haftka (1993) anti-optimization is used to find the loads and the stacking sequence that maximize the difference in prediction between two first-ply-failure models for laminated composites. In the following an optimization scheme that accounts for problem uncertainties is described. The algorithm, referred to as Two Step Method (TSM), presents two separate steps which are alternatively repeated until convergence occurs, Lombardi (1995).

3. Convex modeling

The basic theory of the convex set approach to uncertainty, as described in the monograph by Ben-Haim and Elishakoff (1990), is presented here. Basic definitions of convexity and convex spaces will be given, to conclude with the theorem by Kelly and Weiss (1979) which represents the base of the convex theory.

DEFINITION 3.1 *If S is a set of point in the space E^N then S is convex if given any two points $\mathbf{p}, \mathbf{q} \in S$ there is*

$$\mathbf{r} = \alpha\mathbf{p} + (1-\alpha)\mathbf{q} \in S \quad 0 \le \alpha \le 1 \tag{1}$$

If S is convex then given $\mathbf{N}$ points of S

$$\mathbf{r} = \sum_{i=1}^{N} \alpha_i \mathbf{p}_i \quad \text{where} \quad 0 \le \alpha_i \quad \text{and} \quad \sum_{i=1}^{N} \alpha_i = 1 \tag{2}$$

is the convex combination of the points and it can be shown that $\mathbf{r}$ belongs to S. Going from points to functions, one has

DEFINITION 3.2 *A scalar function $f(\mathbf{x}) : E^N \rightarrow E^1$ is convex if*

$$f\left(\sum_{i=1}^{N} \alpha_i \mathbf{p}_i\right) \le \sum_{i=1}^{N} \alpha_i f(\mathbf{p}_i) \quad \text{where } \mathbf{p}_i \in S,\ 0 \le \alpha_i \ \text{ and } \ \sum_{i=1}^{N} \alpha_i = 1 \tag{3}$$

The relation between set convexity and function convexity is established using the definition of epigraph of a function $EG(f)$.

DEFINITION 3.3 *Given $D \subset E^N$ and a scalar function $f(\mathbf{x}) : D \rightarrow E^1$, the epigraph of f, $EG(f)$, is defined as*

$$EG(f) = \{(\mathbf{x}, y) : \mathbf{x} \in D,\ y \in E^1,\ y \geq f(\mathbf{x})\} \tag{4}$$

It can be easily seen that $EG(f)$ is a set in E^{N+1}, and that $f(\mathbf{x})$ is convex if and only if $EG(f)$ is a convex set.

DEFINITION 3.4 *Given a set of points A of E^N, the convex hull of the set A, $Ch(A)$, is defined as the intersection of all possible convex sets including A, i.e. $Ch(A)$ is the smallest convex set including all points of A.*

THEOREM 3.1 (BALAKRISHNAN, 1981) *A close and bounded convex set S of E^N can be seen as the convex hull of the set of its extreme points.*

In general this is true if for extreme points we mean the boundary of the set S, but if S is bounded by linear functions then the set of extreme points can be limited to the set of corners of S.

DEFINITION 3.5 *The convex span of a set of points B is the set of all possible convex combinations of the points of B.*

THEOREM 3.2 (KELLY AND WEISS, 1979) *The convex hull of a set A of E^N can be seen as the set of convex span of all possible subset of A.*

If Theorems 1 and 2 are now considered, a convex set can be seen as the convex hull of its extreme points, which in turn can be seen as intersection of all convex spans of subsets of extreme points, i.e. each element of the set can be represented as a convex combination of a subset of the extreme points. The usefulness of this assumption will become more clear after introducing the last theorem.

DEFINITION 3.6 *A vector function $\mathbf{f}(\mathbf{x}) : E^N \rightarrow E^M$ is said to be affine if it can be expressed as:*

$$\mathbf{f}(\mathbf{x}) = [A]\mathbf{x} + \mathbf{p} \tag{5}$$

where $[A]$ is an $M \times N$ matrix and $\mathbf{p}$ is an M-vector.

THEOREM 3.3 (KELLY AND WEISS, 1979) *If $\mathbf{f}(\mathbf{x})$ is an affine function and S is a compact set, then $\mathbf{f}(\mathbf{x})$ assumes the same minimum and maximum values on S and $Ch(S)$.*

The extremes of an affine function over a compact set can then be sought on its convex hull, or alternatively on its boundary region. If the region is bounded by linear functions the extremes of $\mathbf{f}(\mathbf{x})$ can be looked for on the corners of the convex hull, thus in a limited number of points. For instance, let the following optimization problem be considered

$$\begin{aligned} &\min_{\mathbf{x}} \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g_i(\mathbf{x}) \leq 0 \quad j = 1, \dots, N_g \end{aligned} \tag{6}$$

The set S of points satisfying problem constraints is defined as.

$$S = \{\mathbf{x} : \mathbf{x} \in R^N, \; g_i \leq 0, \; i = 1, \dots, N_g\} \tag{7}$$

i.e. S represents the feasible domain. The problem can then be written as

$$\begin{aligned} &\min_{\mathbf{x}} \quad f(\mathbf{x}) \\ &\text{s.t.} \quad \mathbf{x} \in S \end{aligned} \tag{8}$$

If for the sake of simplicity S is assumed to be convex then the solution of the problem can be found by solving

$$\begin{aligned} &\min_{\mathbf{x}} \quad f(\mathbf{x}) \\ &\text{s.t.} \quad \mathbf{x} \in \partial S \end{aligned} \tag{9}$$

that is searching the solution on the boundary of the set S. Assume as a simple example the following optimization problem, which is also taken from Ben-Haim and Elishakoff (1990)

$$\begin{aligned} &\min \quad \mathbf{a}^T\mathbf{x} \\ &\text{s.t.} \quad \mathbf{x}^T\mathbf{x} \leq 1 \end{aligned} \quad \rightarrow \quad \begin{aligned} &\min \quad \mathbf{a}^T\mathbf{x} \\ &\text{s.t.} \quad \mathbf{x}^T\mathbf{x} = 1 \end{aligned} \tag{10}$$

The two problems share the same solution, and thus the problem on the right can be solved instead. This gives the following Lagrange functional

$$L = \mathbf{a}^T\mathbf{x} + \lambda\left(\mathbf{x}^T\mathbf{x} - 1\right) \tag{11}$$

At the optimum one has

$$\begin{aligned} &\frac{\partial L}{\partial \mathbf{x}} = \mathbf{a} + 2\lambda\mathbf{x} = 0 \;\rightarrow\; \mathbf{x} = -\frac{1}{2\lambda}\mathbf{a} \\ &\mathbf{x}^T\mathbf{x} = 1 \;\rightarrow\; \frac{1}{4\lambda^2}\mathbf{a}^T\mathbf{a} = 1 \\ &\lambda = \pm\frac{1}{2}\sqrt{\mathbf{a}^T\mathbf{a}} \end{aligned} \tag{12}$$

Thus, by restricting the search to the boundary of the admissible domain it has been possible to find the value of the Lagrange multiplier and of the optimal point. Example of use of this strategy can be found in Adali, Richter and Verijenko (1995).

4. Two step algorithm

As it was mentioned in the introductory section, the feasibility of the solution of an optimization problem may be greatly affected by the variation of uncertain initial conditions. Uncertainty may lie in the value of material moduli, geometry, dimensions or in the fact that once the set of loads acting on the structure is given, the relevant combination to be used is not known. It is then important to find the worst scenario for the structure before proceeding to its optimization. In this way the optimal structure is optimal and feasible for any possible value of the problem parameters. Anti-optimization is one of the possible tools one can use to find the worst initial conditions for the structure.

The TSM finds the optimal design with an iterative scheme that alternates between two different phases:

- solve main optimization problem to find the design variables within the feasible region,
- use anti-optimization to find the value of the uncertain parameters that extremize the value of the constraint functions

Suppose that the solution of the following optimization problem is sought:

$$\begin{aligned} &\min_{\mathbf{x}} \quad f(\mathbf{x}, \mathbf{c}) \\ &\text{s.t.} \quad g_j(\mathbf{x}, \mathbf{c}) \leq 0 \quad j = 1, \ldots, N_g \end{aligned} \tag{13}$$

where $\mathbf{c}$ represents the vector of uncertain or undetermined parameters. The dependence of the objective function and of the constraints from $\mathbf{c}$ could be eliminated by finding the values of $\mathbf{c}$ that extremize the g_j's, so that at the optimum of f the feasibility of the solution would no longer depend on $\mathbf{c}$. For each constraint the following anti-optimization problem could then be solved

$$\begin{aligned} &\max_{\mathbf{c}} \quad g_j(\mathbf{x}, \mathbf{c}) \\ &\text{s.t.} \quad h_l(\mathbf{c}) \leq 0 \quad l = 1, \ldots, N_h \end{aligned} \tag{14}$$

which will seek the value of $\mathbf{c}$ that extremizes that particular constraint function. The h_l's represent linear constraints on the values of the uncertain parameters, e.g. the maximum variation from the nominal value, or bounds on the value of a load multiplier or again constraints on the combinations of these parameters. The optimal value of the parameter vector will then be $\mathbf{c}^{(j)}$, to indicate that this is the solution of the j^{th} anti-optimization problem. Once a solution of all the anti-optimization problems, i.e. one for each constraint, is found, problem (13) is solved again, verifying each constraint with respect to the corresponding extremizing vector of uncertain parameter $\mathbf{c}^{(j)}$, i.e.

$$\begin{aligned} &\max_{\mathbf{x}} \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g_j(\mathbf{x}, \mathbf{c}^{(i)}) \leq 0 \quad j = 1, \ldots, N_g \end{aligned} \tag{15}$$

The combination of problem (15) and problem(s) (14) represents one cycle of the TSM algorithm, and is iterated until convergence occurs. This is done because in general the solution of the optimization problem (15) may represent a substantially changed structure, for which a different worst condition, i.e. the solution of the anti-optimization problem(s) (14) can be found. In turn this different worst condition may lead to a different optimal solution and so on. Experience shows that if the constraints of the optimization problems are linear, then the worst conditions for the structure are found at the first cycle, and further iterations lead to no more changes in the uncertain parameter vectors $\mathbf{c}^{(j)}$. This can be explained by the fact that the uncertainty space will be linearly bounded, and the solution will be in one of the corners of this space. Thus, unless the changes in the structure given by problem (15) are drastic, the *slope* of the objective function of the anti-optimization problems will not change by a large amount, and the solution will always be in the same corner. A different behavior is shown by problems with non-linear objective functions and/or non-linear constraints. Here iteration will lead to different solutions for both problems at each cycle. Different convergence criteria can be adopted, the easiest one being based on the relative changes of the objective function and of the value of $\mathbf{c}^{(j)}$'s.

As the first illustrative example, the method is applied to the optimization of a composite laminate plate for minimum weight, Lombardi, Cinquini, Contro and Haftka (1995), subject to multiple loads of unknown combinations and magnitudes. Suitable upper bounds on the displacements of some given points are prescribed as behavioral constraints. A general purpose finite element code is used for the analysis and the optimization of the structure, in particular the MSC/NASTRAN program is adopted here, while *dsplp*, a FORTRAN routine for LP problems written by Hanson and Hiebert from the SLATEC Common Mathematical Library, Fong, Jefferson, Suyehiro and Walton (1992), is used to solve the anti-optimization problems. Use of the method is next shown for the optimization of a clamped-simply supported beam with unknown position of the load and of the intermediate elastic support, Lekszycki and Lombardi (1995). The objective function of both the optimization and the anti-optimization problems is nonlinear and the behavior of the convergence history will be shown. In this case an analytical solution is obtained for both problems. A general routine for nonlinear optimization problem, *donlp* by P. Spellucci from the SLATEC Common Mathematical Library is used.

5. Example problem 1: composite plate

The optimization of composite laminate plate for minimum weight under strict displacement constraints is considered. The structure is subject to several static loads, $\mathbf{L} = \{L_k\}$, which are not known in magnitude. The design load combination(s) is(are) also unknown. The design variables of the optimization problem are the thicknesses t_i of laminate plies, while some load multipliers c_{jk} are the variables of the anti-optimization problems. The response of the structure is assumed to be linear. An initial structure is optimized for a given objective function, say minimum weight, under an initial combination of the considered loads. Constraints are represented by limits on the value of some displacement components w_j of a given set of points of the structure.

$$\begin{aligned} \min_{t_i} \quad & W \\ \text{s.t.} \quad & w_j^{min} \le w_j \le w_j^{max} \qquad j = 1, \ldots, N_d \\ & t_i^L \le t_i \le t_i^U \qquad\qquad\quad i = 1, \ldots, N \end{aligned} \tag{16}$$

The optimal structure is then analyzed to find the value of displacement w_{jk} under the k^{th} load L_k of the considered set. Each displacement w_j can be obtained by summing the contribution given by each load, so that in general if the load magnitude is assumed to vary between given bounds (which includes the case of unknown loads position), the displacements w_j will be given by

$$w_j = \sum_k c_{jk} w_{jk} \tag{17}$$

where load multipliers c_{jk} account for the variation of the loads magnitude. For each constraint the set of load multipliers $\mathbf{c}^{(j)} = \{c_{jk}\}$ maximizing the displacement w_j can be found by solving the LP problem

$$\begin{aligned} \max_{c_{jk}} \quad & w_j \\ \text{s.t.} \quad & [A]\mathbf{c}^{(j)} \le \{\mathbf{c}^0\} \end{aligned} \tag{18}$$

The load multipliers c_{jk} cannot vary arbitrarily, but are subject to the set of linear inequality constraints in (18), which may give the maximum value for each single load as well as the limits on the combinations of two or more loads. Once the worst load combination $\mathbf{L}^{(j)}$ for each considered displacement is obtained, the structure is re-optimized, and the design that fulfills each constraint under the corresponding set of loads $\mathbf{L}^{(j)}$ is found. The scheme is repeated until there are no more changes in the load combinations and the change in the value of the objective function is smaller than a given tolerance.

The geometry of the plate is shown in Fig. 2. The three-span layout and the distribution of the considered loads does not allow for easy analytical solution of the plate problem, which is solved by finite element analysis. The laminate is made of eight unidirectional graphite-epoxy plies ($E_1 = 133500$Mpa, $E_2 = 8730$MPa, $\nu_{12} = 0.304$, $G_{12} = 4410$MPa, $t = 0.125$mm and $\rho = 1.55$ 10^{-5}N/mm^3) with a stacking sequence of ($0°$, $90°$, $-45°$, $+45°$)$_S$. The ply thicknesses are the only design variables of the problem. Although variation of ply orientation angles could also be included, in practical applications usable angles of lamination are often restricted to $0°$, $90°$ and $\pm45°$ angles, which are all represented in the considered stacking sequence. The finite element model of the plate, Fig. 2, is composed of 16 blocks of two four-node plate elements. Each block has a different stacking sequence, defined by four ply variables, for a total of 64 ply-variables. Solution of the problem will give, for each block, the ply thicknesses for the four angles of lamination which will subsequently be converted in an integer number of plies of the considered material. The thicknesses, initially set to 0.25mm, i.e. two plies, are limited between 0.01mm, which is equivalent to no-ply, and 1.0mm, corresponding to 8 plies. It is worth mentionning that usually the number of contiguous iso-oriented plies is limited to four in order to reduce the risk of matrix cracks. Since this limit is not considered here, an interesting problem could then be to find the optimal re-distribution of the plies of the optimal solution, in order to fulfill this practical requirement with minimum loss of stiffness of the plate.

The displacement constraints prescribe 1mm limit to the deflection of the plate at the mid-span and at the cantilever tips. The vertical displacement distribution has maximum values at points 1 and 5, Fig. 2, and thus the constraints are to be verified only at these two points. Different distributed and concentrated loads are considered, to deform the plate in both bending and torsion. The six load sets are shown in Fig. 3. Each force of the concentrated-type loads can vary between -1N and 1N, while the maximum value of the distributed loads is such that the resulting force for each element of the mesh is again between -1N and 1N. Moreover the load combinations must satisfy the requirement that in each node of the mesh the total force is between -1N and 1N. This leads to only few relations between the coefficients of the load sets. The optimization and the anti-optimization problems are then formulated as follows:

optimization

$$\begin{aligned} \min_{t_i} \quad & W \\ \text{s.t.} \quad & |w_1| \le 1.0 \\ & |w_5| \le 1.0 \\ & 0.01 \le t_i \le 1.0 \end{aligned} \tag{19}$$

***kth* anti-optimization**

$$\begin{aligned} \max_{c_{jk}} \quad & w_j = \textstyle\sum_{k=1}^{6} c_{jk} w_{jk} \quad j = 1,5 \\ \text{s.t.} \quad & 0.5c_{j1} + 0.25c_{j2} + c_{j4} \le 1.0 \\ & c_{j1} + 0.5c_{j2} \le 1.0 \\ & c_{j1} + 0.25c_{j2} + 0.5c_{j3} + c_{j6} \le 1.0 \\ & c_{j1} + c_{j3} + c_{j5} + c_{j6} \le 1.0 \\ & |c_{jk}| \le 1.0 \quad k = 1, \ldots, 6 \end{aligned} \tag{20}$$

The weight of the plate vs. the iteration number, where iteration here is the total number of analyses performed during the optimization phases, is plotted

Block #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0°	3	6	7	8	8	8	8	8	8	8	8	8	8	7	6	3
90°	1	1	2	3	3	4	4	4	4	4	4	3	3	2	1	1
+45°	-	1	2	3	4	4	5	6	6	5	4	4	3	2	1	-
-45°	1	2	3	3	4	5	5	6	6	5	5	4	3	3	2	1

Table 1. Integer number of ply design corresponding to the optimal design of the plate example (only upper part of the laminate is shown)

in Fig. 4. The optimal solution is found in less than five cycles. The diamonds on the curve represent the points where a new set of loads has been found by the anti-optimization, and the beginning of a new optimization. Initially the weight of the plate is 1.98 N, and both displacements are violating the constraint, with $w_1 = 11.2$ mm and $w_5 = 4.2$ mm. The design is largely improved during the first optimization with the displacements inside the limit values and almost no change in the weight. The application of the worst load sets brings the design back into the infeasible zone with displacements value of $w_1 = 33.2$ mm and $w_5 = 20.2$ mm. A new feasible design is obtained during the second optimization, paid with a large increase in the value of the plate weight, which reaches the final value of 4.79 N. The optimal design was converted into a more practical integer-number-of-ply design, INP, in which optimal values of ply thickness are transformed into the nearest multiple of the thickness of the material used for the plate (i.e. 0.125 mm). The final INP design is shown for each block of the plate in Table 1. Recall for comparison that the symmetric top part of the initial design had only eight plies per block, two for each lamination angle. The final design presents near the tips of the plate slightly unbalanced stacking sequences (i.e. the number of +45° plies is not equal to the number of −45° plies). The reason lies in the fact that the laminate tries to resist the torsional loads and to limit displacements at the tip of the plate with a stacking sequence that produces coupling between bending and torsion.

6. Example problem 2: beam with elastic support

As a second application a simple beam is considered, Fig. 5, clamped at one end and simply supported at the other end, loaded with a concentrated force of undetermined position. An additional elastic support, also of undetermined position, limits the displacements of the beam. Both the force and the elastic support are free to move along the length of the beam. The beam displacement distribution is then a function of the position of both the spring and the load, i.e. $w = w(x, x_0, s)$. Optimal position of the support minimizes the displacement of the beam under the force $w(x_0)$. The uncertainty here lies in the position of the force. In order to design the beam one needs to know the value of the moment at the clamped edge (which can be assumed to be the maximum over the beam), affected by the position of the force on the beam. This dependence can be eliminated by placing the force so that the moment is maximized in the most stressed section of the beam. In this sense anti-optimal position of the force will be the one that maximizes the moment in that particular section. The two problems can be stated as

optimization

$$\begin{aligned} \min_{s} \quad & w|_{x=x_0}(x_0, s) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & 0 \leq s \leq 1.0 \end{aligned} \tag{21}$$

anti-optimization

$$\begin{aligned} \max_{x_0} \quad & M|_{x=0}(x_0, s) \\ \text{s.t.} \quad & s = \bar{s} \\ & 0 \leq x_0 \leq 1.0 \end{aligned} \tag{22}$$

The force position x_0 is a given parameter in (21) and the only design variable in (22). Conversely, the spring position s is the design variable in (21) and fixed parameter in (22). The problem cannot actually be considered as an example of the described method for nonlinear cases, since the objective function of the anti-optimization is not a constraint of the optimization problem. However, the case when the objective function of (22) is a constraint of (21) may be seen as a subcase of the present problem, and then its solution can give an idea about the algorithm behavior in case of non linear functions, where the search of the optimal solution is no longer restricted to the corners of the uncertainty domain.

The analytical solution, which can be easily found, allows plotting of the displacement under the force $w(x_0)$ and the moment $M(0)$ for the whole domain of variation of the spring and the force position. It is then possible to find graphically the optimal placement of the spring and of the force and verify the analytical solution. The 3-D plots of the displacement and of the moment are shown in Fig. 6, while Fig. 7 shows the contour levels of the functions. Comparison of the contour maps with the 3-D plots helps locate maximum and minimum points. The large dashed lines in Fig. 7 represent the set of optimal solution of problems (21) and (22) for all values of the parameters x_0 and s respectively.

Numerical optimal solution was sought from several different starting points, and found in close agreement with the graphical one, i.e. at the intersection of the two *optimal* lines shown in the plots of Fig. 7, in which one of the paths to the optimal point is shown. Figure 8 shows initial and final configuration of the beam. Initial displacement and moment, for $x_0^{initial} = 0.7$ and $s^{initial} = 0.2$, are

$w(x_0) = 0.00673$ and $M(0) = -0.019$. Final values of the objective functions are $w(x_0) = 0.00027$ and $M(0) = -0.059$, corresponding to the optimal placement of the force and the spring $x_0^{opt} = 0.14$ and $s^{opt} = 0.24$. The convergence history of the displacement and moment value are shown in Fig. 9. Here iteration is the the number of analyzed beam configurations during each problem i.e. the left graph shows the number of analyzed beams during the optimization phases, while the right one – the number of beam configurations of the anti-optimization phase. The diamonds in both graphs represent starting points of a new iteration of the corresponding problem, and the jumps right before the diamonds correspond to the application of the optimal solution of the other problem, e.g. the value of the moment which has been maximized during the anti-optimization by finding optimal placement of the force, is greatly reduced when the spring is placed in its optimal position, which has been found in the optimization phase. Convergence is reached in few cycles.

7. Concluding remarks

The method shown here may represent a useful tool that can be used in problems with non-deterministic data. Its main advantages lie in its simplicity, and in the fact that it allows one to use different programs for the analysis of the structure and for the solution of the optimization problems. It also seemed to be very robust in the cases shown here and in others not included in this note.

Currently, the research on this topic is focused on the definition of a general theoretical framework for these problems, e.g. conditions for the existence of the solution, and new schemes for the search for optimal points, and on the use of the algorithm for the solution of more complex non-linear problems, e.g. optimization of dynamic properties of structures.

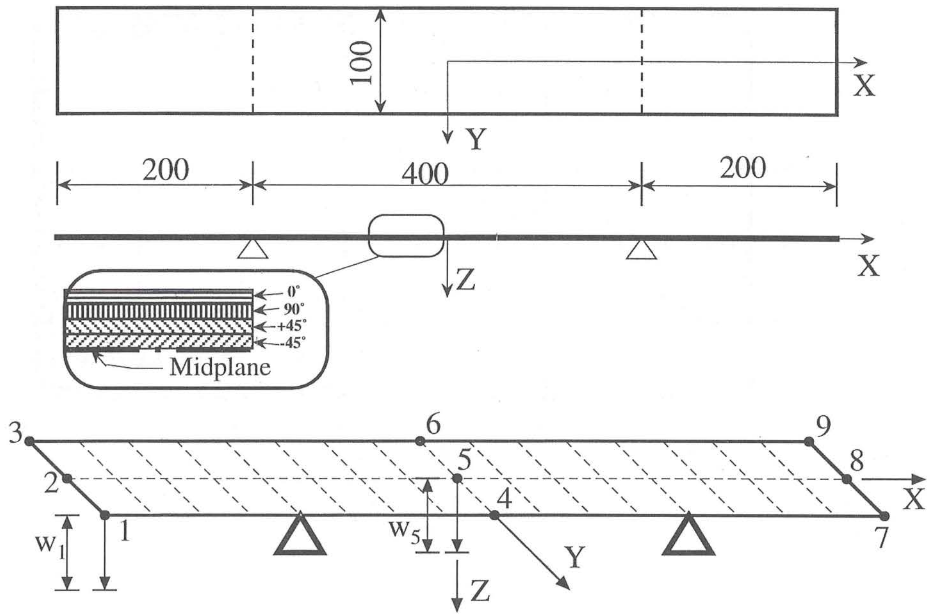

Figure 2. Geometry, finite element mesh and displacement constrained points of the plate.

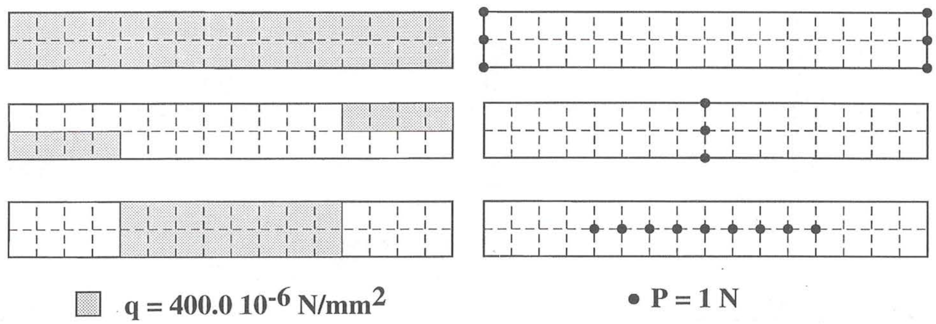

Figure 3. The plate load distributions considered.

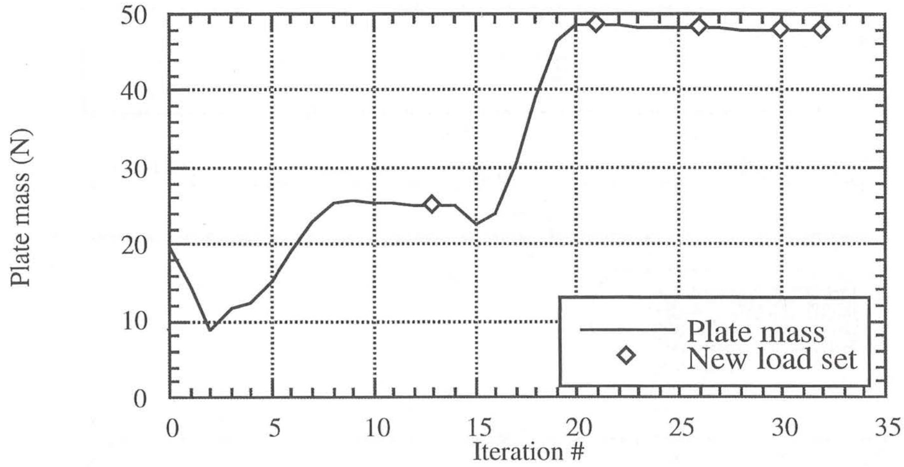

Figure 4. Objective function history for the plate example.

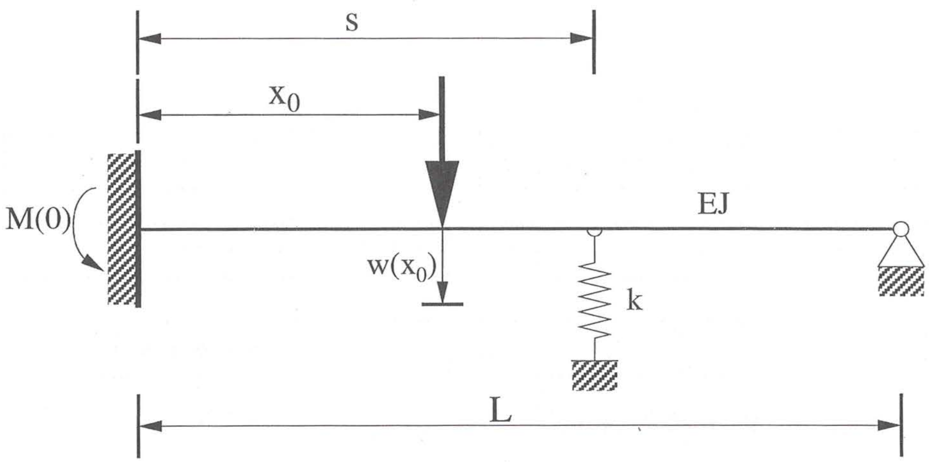

Figure 5. Geometry of the beam.

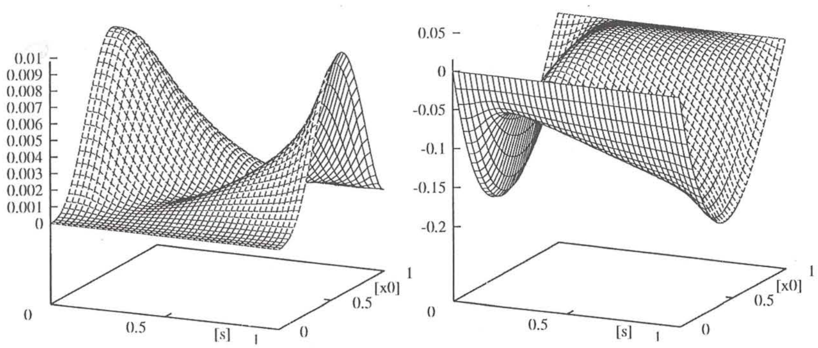

Figure 6. 3-D view of the displacement and of the moment functions.

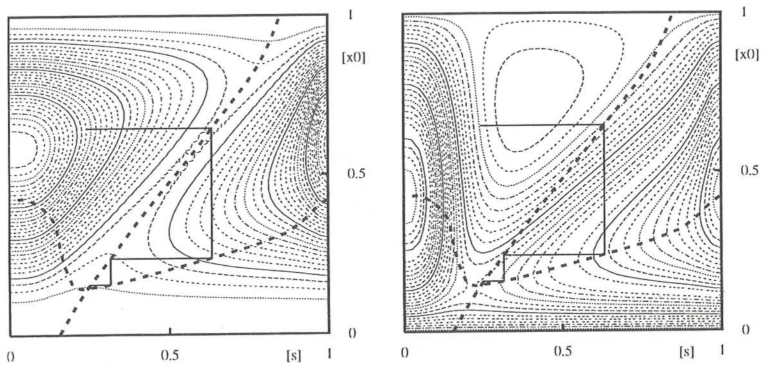

Figure 7. Contour level plots of the displacement and the moment. Thick dashed curves represent the set of optimal solutions for all values of the parameters. The thick step line is a possible path to the optimal solution.

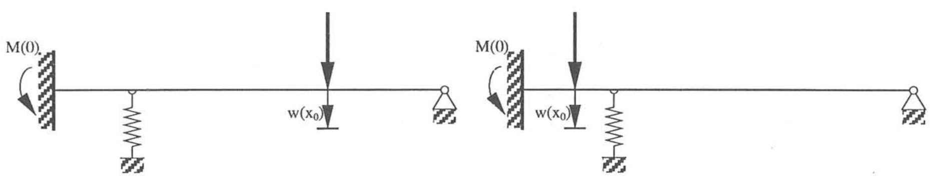

Figure 8. Initial and final placement of force and spring.

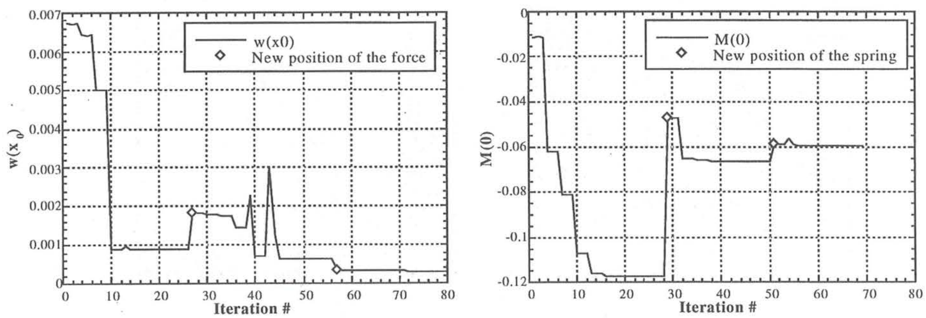

Figure 9. Iteration history of displacement and moment of the beam.

References

ADALI, S., RICHTER, A. and VERIJENKO, V.E. (1995) Minimum weight design of symmetric angle-ply laminates under multiple uncertain loads. *Structural Optimization*, **9**, 89-95.

BEN-HAIM, Y. (1994) A Non-probabilistic Concept of Reliability. *Structural Safety*, **14**, 227-245.

BEN-HAIM, Y., ELISHAKOFF, I.E. (1990) *Convex Models of Uncertainties in Applied Mechanics.* Studies In Applied Mechanics n. 25, Elsevier Science Publisher, The Netherlands.

ELISHAKOFF, I.E. (1990) An Idea on the Uncertainty Triangle. *The Shock and Vibration Digest*, **22**, 10, 1.

ELISHAKOFF, I.E., CAI, G.Q. and STARNES, JR., J.H. (1994) Non-linear buckling of a column with initial imperfection via stochastic and non-stochastic convex models. *Int. J. Non-Linear Mechanics*, **29**, 71-82.

ELISHAKOFF, I.E., LI, Y.W. and STARNES, JR., J.H. (1994) A deterministic method to predict the effect of unknown-but-bounded elastic moduli on the buckling of composite structures. *Comp. Meth. Appl. Mech. Eng.*, **111**, 155-167.

FONG, K.W., JEFFERSON, T.H., SUYEHIRO, T., WALTON, L. (1992) *Guide to the SLATEC Comon Mathematical Library.* Internet URL http://www.netlib.org.

GANGADHARAN, S., NIKOLAIDIS, E., LEE, K., HAFTKA, R.T. (1993) The Use of Anti-optimization to Compare Alternative Structural Models. In: *Proceedings of the 34th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, Part 1, La Jolla, CA, 534-543.

KELLY, P.J., WEISS, M.L. (1979) *Geometry and Convexity: A Study in Mathematical Methods*, Wiley, New York.

LEE, J., HAFTKA, R.T., GRIFFIN, JR., O.H., WATSON, L.T., SENSMEIER, M.D. (1994) Detecting Delaminations in a Composite Beam Using Anti-Optimization. *Structural Optimization*, **8**, 93-100.

LEKSZYCKI, T., LOMBARDI, M. (1995) *Anti-optimization: some solution in structural design.* Internal report, Department of Structural Mechanics, University of Pavia, Italy.

LOMBARDI, M. (1995) *Ottimizzazione Strutturale: metodo a due passi e applicazioni a strutture in materiale composito.* Doctorate Thesis, Polytechnic of Milan, Italy (in Italian).

LOMBARDI, M., CINQUINI, C., CONTRO, R., HAFTKA, R.T. (1995) Anti-optimization Technique for designing Composite Structures. *Proceedings of the First World Conference of Structural and Multidisciplinary Optimization*, Goslar, Germany, 28 May - 2 June.

VAN WAMELEN, A., JOHNSON, E.R., HAFTKA, R.T. (1993) *Optimal design of laminated specimens to evaluate composite failure criteria.* Presented at the ASC 8th technical Conference on Composite Materials, Cleveland, Ohio, 19-21 October.