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Pontriagin method versus nonlinear programming in applications to structural optimization

by

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Abstract: The problem of optimal redesign of nonlinear elastic columns and frames against buckling was formulated in the paper. It was solved using both Pontriagin method and nonlinear programming. Numerical efficiency of these methods in application to structural design is discussed. Various approaches to the solution of the two-point boundary value problem are demonstrated.

Keywords: structural optimization; numerical methods; Pontriagin method; sensitivity analysis.

1. Introduction

The optimization methods based on Pontriagin maximum principle have been intensively developed and widely implemented in practical problems of optimal control of processes (Bulirsch et al., 1991; Pesch, 1994). When Pontriagin principle is used in structural optimization the role of time is played by the position coordinate x, and therefore applicability is limited to one-dimensional structures. However, when applied to these structures, Pontriagin method demonstrates many advantages, namely easiness of introducing numerous local constraints, considering minimax problems (Szefer, 1985; Mikulski, 1995), obtaining bang-bang solutions (Mikulski, 1995), treating multimodal eigenvalue problems (Gajewski, 1985; Garstecki and Glema; 1991, Glema, 1992). In the last years remarkable improvement of numerical methods of solution for the set of equations and inequalities appearing in Pontriagin method has been observed (Bulirsch et al., 1991; Pesch, 1994; Stryk and Bulirsch, 1992).

The paper discusses the advantages and disadvantages of Pontriagin method in comparison with Nonlinear Programming (NP) employing Finite Element Method (FEM). The authors' experience based on the optimal solution of stability problems of physically nonlinear structures is used. Various approaches to the solution of two-point and multi-point boundary value problems are discussed.

2. Optimal design of structural reinforcement against buckling

Consider a column or a frame structure which is to be redesigned by the addition of structural material in the state of initial load and distortions resulting from manufacturing process. Assume that the material is nonlinear elastic. The goal is to find the optimal distribution of the reinforcing material $s(x)$, which minimizes the volume of material (1) for constrained values of critical loads $\hat{P}^{cr}$ and $\breve{P}^{cr}$ (2). We allow for bimodal buckling, denoting by $\hat{}$ and $\breve{}$ the values associated with those two modes. This problem is governed by an ordinary differential equation of fourth order with coefficients nonlinear in s and load P^0. It can be transformed to a set of first order differential equations (3). We add proper boundary conditions (4) and side conditions (5).
The optimal control problem takes the form

$$F = \int_{x_0}^{x_t} s(x)\, dx \rightarrow \min_s \tag{1}$$

$$\hat{P}^{cr} \geq P^0 \qquad \breve{P}^{cr} \geq P^0 \tag{2}$$

$$\hat{u}_i' = \hat{A}_{ij}\, \hat{u}_j, \qquad \breve{u}_i' = \breve{A}_{ij}\, \breve{u}_j, \qquad i,j = 1,\ldots,4 \tag{3}$$

$$\hat{u}_i(x_0) = \hat{u}_i^0,\ \hat{u}_i(x_t) = \hat{u}_i^t, \qquad \breve{u}_i(x_0) = \breve{u}_i^0,\ \breve{u}_i(x_t) = \breve{u}_i^t \tag{4}$$

$$s_{min} \leq s(x) \leq s_{max} \tag{5}$$

In the case of a column, the nonzero elements of matrices are: $\hat{A}_{12} = \hat{A}_{32} = \hat{A}_{34} = \breve{A}_{12} = \breve{A}_{12} = \breve{A}_{32} = \breve{A}_{34} = 1$, $\hat{A}_{23} = \hat{P}^{cr} L^2/\hat{D}(x)$, $\breve{A}_{23} = \breve{P}^{cr} L^2/\breve{D}(x)$. Here $\hat{D}(x)$ and $\breve{D}(x)$ denote the bending stiffness coefficients, which are nonlinear functions of s and load P^0. Note that $\hat{D}(x) = \breve{D}(x)$ when the two modes are expected to appear in one plane. The Hamilton function takes the form

$$\begin{aligned} H(s\,,\, \hat{u}_j\,,\, \breve{u}_j\, \hat{\lambda}_i\,,\, \breve{\lambda}_i\,,\, \mu_1\,,\, \mu_2\,,\, \nu_1\,,\, \nu_2) &= -C_0\, s(x) + \\ + \hat{\lambda}_i\, \hat{A}_{ij}\, \hat{u}_j + \breve{\lambda}_i\, \breve{A}_{ij}\, \breve{u}_j + \mu_1\, (\hat{P}^{cr} - P^0) \\ + \mu_2\, (\breve{P}^{cr} - P^0) + \nu_1\, (s - s_{min}) - \nu_2\, (s - s_{max}) \end{aligned} \tag{6}$$

The stationarity conditions of (6) with respect to the arguments of H provide necessary conditions for optimal solution:

optimality condition $$\frac{\partial H}{\partial s} = 0 \tag{7}$$

state equations $$\hat{u}_i' = \frac{\partial H}{\partial \hat{\lambda}_i} = \hat{A}_{ij}\hat{u}_j \qquad \breve{u}_i' = \frac{\partial H}{\partial \breve{\lambda}_i} = \breve{A}_{ij}\breve{u}_j \tag{8}$$

adjoint equations $$\hat{\lambda}_i' = -\frac{\partial H}{\partial \hat{u}_i} = -\hat{A}_{ij}\hat{\lambda}_j \quad \breve{\lambda}_i' = -\frac{\partial H}{\partial \breve{u}_i} = -\breve{A}_{ij}\breve{\lambda}_j \tag{9}$$

transversality conditions $$\hat{\lambda}_j \delta \hat{u}_j \Big|_0^t = 0 \qquad \breve{\lambda}_j \delta \breve{u}_i \Big|_0^t = 0 \tag{10}$$

Since the above problem is self-adjoint, $\hat{\lambda}$ and $\breve{\lambda}$ are proportional to $\hat{u}$ and $\breve{u}$ with proportionality factors $\hat{C}$ and $\breve{C}$, respectively.

$$\begin{aligned} &\hat{\lambda}_1 = -\hat{C}\hat{u}_4\,, && \hat{\lambda}_2 = \hat{C}\hat{u}_3\,, && \hat{\lambda}_3 = -\hat{C}\hat{u}_2\,, && \hat{\lambda}_4 = \hat{C}\hat{u}_1\,, \\ &\breve{\lambda}_1 = -\breve{C}\breve{u}_4\,, && \breve{\lambda}_2 = \breve{C}\breve{u}_3\,, && \breve{\lambda}_3 = -\breve{C}\breve{u}_2\,, && \breve{\lambda}_4 = \breve{C}\breve{u}_1\,, \end{aligned} \tag{11}$$

Introducing (6) and (11) into (7) we obtain

$$-C_0 + \hat{C}\,\hat{u}_3^2\,\frac{P^0}{\hat{D}^{b\,2}}\,\frac{\partial \hat{D}^b}{\partial s} + \breve{C}\,\breve{u}_3^2\,\frac{P^0}{\breve{D}^{b\,2}}\,\frac{\partial \breve{D}^b}{\partial s} = 0 \tag{12}$$

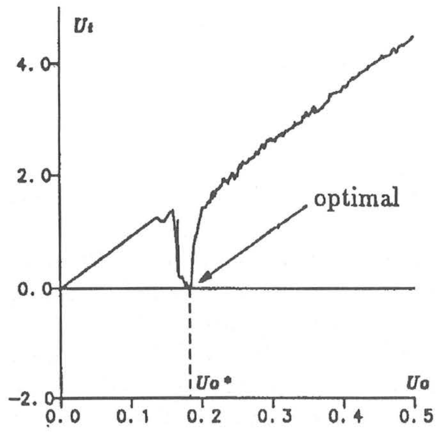
The optimality conditions have the form of Two Point Boundary Value Problem (TPBVP) where we have the set of differential equations (8), (9), (12), mixed initial-terminal boundary conditions (4), (10) and side conditions (5).
In case of implementation of NP method we formulate the problem in similar way as (1) – (5) with the exception that the state equations (3) are usually introduced in the form of higher order differential equation. In practical applications this equation and boundary conditions (4) are used in a computer program for structural analysis, whereas Eqs. (2), (5) play the role of constraints.

3. Numerical aspects

3.1. Pontriagin method

For the solution of the TPBVP the shooting method is widely used. In this method vacant initial conditions are preassumed, the differential equations are solved as a Cauchy problem and the error in terminal conditions, expressed in form of a norm, is minimized by iterative correction of initial conditions. Note that the mapping of initial conditions on terminal ones $(u^t, \lambda^t) = \mathcal{B}(u^0, \lambda^0)$ is a nonlinear and irregular function (Figs. 1a,b,2) (Glema, 1992), because the state equations (8), optimality condition (12) and side inequalities (5) are incorporated in it. Therefore the iteration of boundary conditions requires careful numerical treatment. In the above iteration the derivatives of $\mathcal{B}$ with respect to u^0, λ^0 are used. These sensitivity matrices can be computed either by finite difference method (the secant matrix) or by variational method proposed by Armand (1973) (the tangent matrix). In the latter case the round off errors are smaller because small quantities of the same magnitude appear.

The authors studied numerical efficiency of both methods, combined with Runge-Kutty integration of fourth order with adaptive step, and found that they involve equal computational effort per one iteration. The number of iteration steps depended on the choice of relaxation factor in variational method and on size of steps in both methods. A little more robust appeared the method employing the secant matrix. Substitution of boundary displacements by proportionality constants C (Figs. 1b,2) also can lead to better convergence. Crucial was the choice of starting point. For a bad starting point both methods were



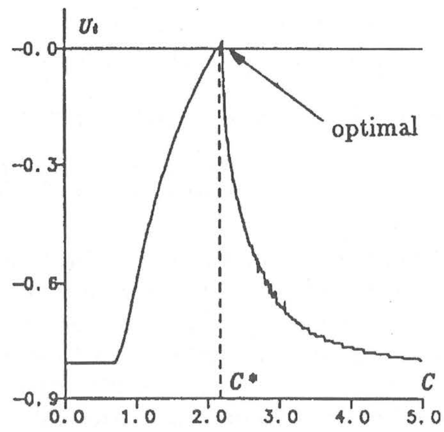


Figure 1. Iteration of boundary condition for a column (Glema, 1992): a) using displacement u^0, b) using proportionality constant C.

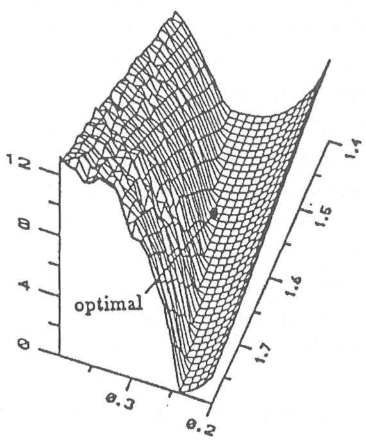


Figure 2. Iteration of boundary condition for bimodal buckling of a column (Glema, 1992)

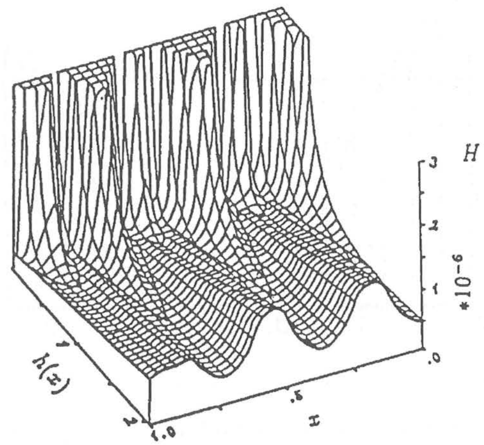


Figure 3. Minimum of Hamiltonian in optimization of load and cross-section for three-span beam (Mikulski, 1995)

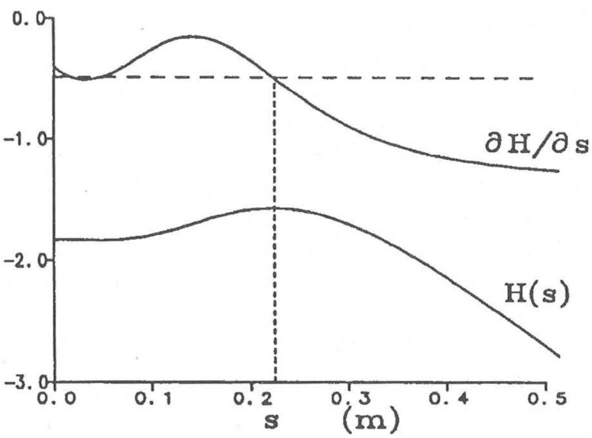


Figure 4. Optimality conditions and Hamiltonian for specified cross-section (Glema, 1992)

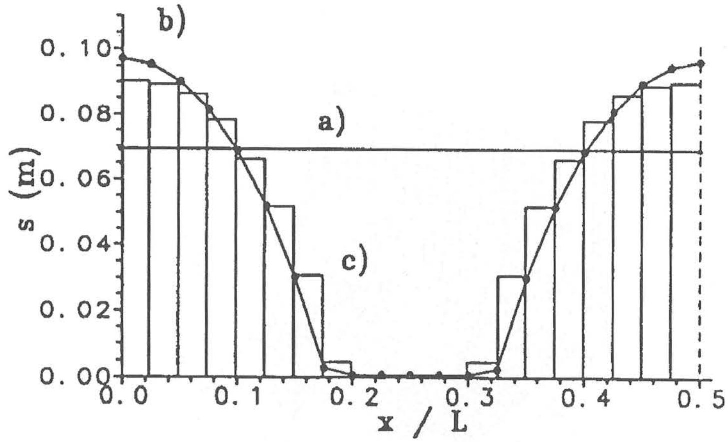


Figure 5. Optimal reinforcement a) constant solution, b) continuous solution by Pontriagin method, c) stepwise solution by NP method (Garstecki and Glema, 1991)

not convergent. To find the proper starting point usually the trial and error method is sufficiently useful.

Until quite lately the above described method was not applicable to Multipoint Boundary Value Problems (MBVP), when additional interior constraints on state variable u were imposed. In structural optimization this problem appeared, for example, in case of continuous beam. Lately, the multiple shooting method was developed and efficiently applied to complex practical control problems (Bulirsch et al., 1991; Pesch, 1994). Mikulski (1995) demonstrated that this method can be effectively used in minimax problems of optimal design of continuous beams with unspecified support positions and load distribution. It consists in independent application of shooting method to intervals of x taking into account continuity conditions. Combination of multiple shooting with direct collocation is also used (Stryk and Bulirsch, 1992).

Note that the Hamiltonian function (6) is not convex in control variable s, Fig. 4 (Glema, 1992), Fig. 3 (Mikulski, 1995), therefore the computation of optimal s cannot be limited to stationarity condition of H expressed in form (7) or (12), but it must be checked whether it is the maximum of H.

3.2. Nonlinear programming method

For comparison, the above formulated problems of optimal redesign were also solved by nonlinear programming (NP) employing FEM. Constant s within elements was assumed. The feasible direction method for constrained optimization was used (Vanderplaats, 1985), where the sensitivity derivatives with respect to

s were computed analytically by employing the adjoint variable method (Cohen et al., 1990; Glema 1992). Obviously the numerical efficiency and robustness of NP method depends strongly on the implemented algorithm for optimization (Schittkowski et al., 1994).

Optimal reinforcement of the clamped-clamped column, obtained independently by the Pontriagin and NP methods, is presented in Fig. 5.

4. Concluding remarks

The problems of optimal redesign of nonlinear elastic structures against buckling were solved using both Pontriagin and NP methods. The Hamiltonian and the mapping of initial conditions onto terminal conditions appeared highly nonlinear and nonconvex. This has not been reported in the literature and has required careful numerical treatment. In this respect the NP method was more robust.

Despite the above described difficulties there are many advantages of Pontriagin method. The optimal control function s can be obtained in continuous form with desired accuracy. In natural way the bimodal solutions can be found. The bang-bang solutions can be computed, too. The mini-max problems can be relatively easily solved.

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