Control and Cybernetics

vol. **25** (1996) No. 6

A control problem with state constraints for a phase-field model

by

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Abstract: The Penrose-Fife model for phase transitions consists of a system of quasi-linear parabolic PDE. After reviewing previous results on existence and uniqueness for the state equations an optimal control problem for this system is introduced. This problem involves local state constraints. An observation operator corresponding to this optimal control problem is introduced, and its regularity properties are studied. Finally, these regularity properties are used to derive first order optimality conditions.

1. Introduction

In this article we consider the following initial-boundary value optimal control problem

$$\phi_t = \Delta\phi - s_0'(\phi) - \frac{\lambda(\phi)}{T} \qquad \text{in} \quad Q, \tag{1}$$

$$T_t = -\Delta\left(\frac{1}{T}\right) - \lambda(\phi)\phi_t + v \qquad \text{in} \quad Q, \tag{2}$$

$$\frac{\partial T}{\partial n} = -(T - w) \qquad \text{on} \quad \partial Q \tag{3}$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on} \quad \partial Q, \tag{4}$$

$$\phi(x,0) = \phi_0(x), \qquad T(x,0) = T_0(x) \qquad \text{in} \quad \Omega, \tag{5}$$

where Ω is a domain in $\mathbf{R}^3$ with a smooth boundary Γ, $Q = \Omega \times (0, t^*)$ for some $t^* > 0$, and $\partial Q = \Gamma \times (0, t^*)$.

These equations arise in a model for phase transitions introduced by Penrose and Fife (1990). The phase transitions depend only on the temperature T. ϕ is a non-conserved order parameter which indicates the present phase at a point $(x,t) \in Q$. The function s_0 is a double-well potential, whose minima correspond to the states when there is only a single phase present. v indicates a heat source.

The mathematical treatment of this and similar initial-boundary value problems can be found, for example, in Amann (1993), Horn, Sprekels, Zheng (1996), Sprekels, Zheng (1993), Colli, Sprekels (1995), Horn, Laurençot, Sprekels (1996), Laurençot (1994) and requires some additional assumptions. In this article we will use the same assumptions as in Horn, Sprekels, Zheng (1996), Sprekels, Zheng (1993). Namely, for the potential s_0 we will assume that either

- **(A)** $s_0 \in C^3(\mathbf{R})$ and there exists a constant $C > 0$ such that $s_0''(\phi) > -C$ for all $\phi \in \mathbf{R}$.
  or
- **(B)** $s_0 = \phi \log \phi + (1-\phi)\log(1-\phi)$.
- $\lambda(\phi) = a\phi + b$, for a positive constant a. To simplify notations we will, without loss of generality, use $a = 1$ and $b = 0$, i.e. use $\lambda(\phi) = \phi$.

We will also need the following regularity assumptions.

(H1) $\phi_0 \in H^4(\Omega); \frac{\partial \phi}{\partial n}(x) = 0, \forall x \in \Gamma;$
$\frac{\partial}{\partial n}\left(-s_0'(\phi_0) + \frac{\phi_0}{T_0} + \Delta\phi_0\right)(x) = 0, \forall x \in \Gamma.$

(H2) $T_0 \in H^3(\Omega); \tilde{T}(x) = \frac{\partial T_0}{\partial n}(x) + T_0(x) > 0, \forall x \in \Gamma; T_0(x) > 0, \forall x \in \overline{\Omega}.$

Finally, we introduce some Banach spaces which will be widely used throughout this article.

$$
\begin{aligned}
X_1 &= C([0,t^*]; H^4(\Omega)) \cap C^1([0,t^*]; H^2(\Omega)) \cap C^2([0,t^*]; L^2(\Omega)),\\
X_2 &= C([0,t^*]; H^3(\Omega)) \cap C^1([0,t^*]; H^1(\Omega)) \cap H^{4,2}(Q),\\
V &= H^2(0,t^*; L^2(\Omega)) \cap H^1(0,t^*; H^2(\Omega)),\\
W &= H^2(0,t^*; H^{\frac{3}{2}}(\Gamma)).
\end{aligned}
$$

Throughout the paper, we let $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$

In this setting, the main existence result (cf. Horn, Sprekels, Zheng, 1996; Sprekels, Zheng, 1993) is

Proposition 1 *Suppose* **(H1)** *and* **(H2)** *are satisfied and that $(v,w) \in V \times W$. Then there exists a unique global smooth solution $(\phi, T) \in X_1 \times X_2$ to the initial-boundary value problem (1)–(5). Furthermore, there exists a constant $c_{t^*} > 0$ such that $T(x,t) \geq c_{t^*}$ for all $(x,t) \in \overline{Q}$, and in the case* **(B)** *there exist constants $0 < a_{t^*} < b_{t^*} < 1$, such that $a_{t^*} \leq \phi(x,t) \leq b_{t^*}$ for all $(x,t) \in \overline{Q}$.*

In Section 2 of this article we will state and discuss the optimal control problem with state constraints. In Section 3 we will investigate the related observation operator and prove its differentiability in the setting of Section 2. Finally, we will derive the necessary conditions for optimality in Section 4 of this paper.

2. Optimal control problems with state constraints

The state equations (1)–(2) give rise to several interesting optimal control problems. Here we want to control the state (ϕ, T) by using the source term v in (2) and the boundary term w in (4). Such a control problem was introduced in Sprekels, Zheng (1992). In the present article we want to put local constraints on the state as well (see Sokołowski, Sprekels, 1994, for a similar problem). These constraints will ensure that the solutions of the state equation ϕ and θ stay in a desirable range.

There are several reasons to add state constraints on the temperature. Perhaps the most important reason is that these constraints will force the temperature to stay in a range in which the model is a valid description of the underlying physical processes. State constraints on the phase have been added here for the sake of the completeness of the description.

In order to formulate this problem in a precise manner we need to introduce some additional notation. We start by defining the cost functional

$$\begin{aligned} I(\phi, T; v, w) \quad &= \quad \frac{\alpha_1}{2} \left\| \phi(t^*) - \hat{\phi}(t^*) \right\|^2_{L^2(\Omega)} + \frac{\alpha_2}{2} \left\| T - \hat{T} \right\|^2_{L^2(Q)} \\ &\quad + \frac{\alpha_3}{2} \|v\|^2_{L^2(Q)} + \frac{\alpha_4}{2} \int_0^{t^*} \|w(t)\|^2_{L^2(\Gamma)} \, dt, \end{aligned} \tag{6}$$

for given target functions $\hat{\phi} \in X_1$ and $\hat{T} \in X_2$. Next let

$$\begin{aligned} \tilde{W} \quad = \quad \Big\{ w \in W : \quad & w(x, 0) = \tilde{T}(x), \quad \forall x \in \Gamma; \\ & w(x,t) \geq \beta, \quad |w_t(x,t)| < k, \quad \forall (x,t) \in \partial Q \Big\}, \end{aligned}$$

where $\tilde{T}$ is the function introduced by **(H2)** and β and k are suitably chosen positive constants. We use this set to introduce

$$K = V \times \tilde{W}.$$

The set $\mathcal{U}_{\text{ad}}$ of admissible controls is a closed, convex and bounded subset of K.

To state the local state constraints we use the constants $0 < K_1 < K_2$ and $K_3 < K_4$ to define the set of admissible states.

$$\mathcal{Y}_{\text{ad}} = \left\{ (\phi, T) \in X_1 \times X_2 : K_1 \leq T \leq K_2 \wedge K_3 \leq \phi \leq K_4, \quad \text{in} \quad \overline{Q} \right\}. \tag{7}$$

$\mathcal{Y}_{\text{ad}}$ has a non-empty interior, since $X_1 \times X_2 \subset (C(\overline{Q}))^2$.

We can now state the optimal control problem under consideration.

Optimal Control Problem (CP)

Minimize $I(\phi, T; v, w)$ under the following conditions:

1. (ϕ, T) satisfies the state equations (1)–(2) and the initial and boundary conditions (3)–(5).
2. $(v, w) \in \mathcal{U}_{\text{ad}}$.
3. $(\phi, T) \in \mathcal{Y}_{\text{ad}}$.

In the study of the control problem **(CP)** it is useful to introduce the observation operator S. To do this we define the space of observations B by

$$B = (C([0, t^*]; H^2(\Omega))) \times (C([0, t^*]; H^2(\Omega))). \tag{8}$$

Next define

$$S \ : \ K \to B, \tag{9}$$

$$S \ : \ (v, w) \mapsto (\phi, T), \tag{10}$$

i. e. S assigns to every pair $(v, w) \in K$ the pair (ϕ, T) which solves (1)–(5) for the given v and w. Since $X_1 \times X_2 \subset B$ and by virtue of Proposition 1 this operator S is well defined. Using this operator one sees that the cost functional $I(\phi, T; v, w)$ depends only on the controls v and w, i.e. we can rewrite it as

$$J(v, w) = I(\phi, T; v, w)|_{(\phi,T)=S(v,w)} \, .$$

In the following section we will study the properties of the operator S. In Section 4 these properties will be used to derive the necessary conditions of optimality. To do this we will use similar methods as in Casas (1993).

Remark: Since the authors of Sprekels, Zheng (1992) did not consider state constraints, they could use a larger space of observations with a coarser topology.

3. Differentiability of the observation operator

This section is devoted to a closer investigation of the observation operator S. The proofs in this section are rather technical. However, the techniques are largely straightforward and many of the details have been omitted.

To begin, we state that S is well-defined, and – also due to Proposition 1 – there exist positive constants α and γ such that

$$\|\phi\|_{X_1} + \|T\|_{X_2} \ \leq \ \alpha, \quad \forall (v, w) \in \mathcal{U}_{\text{ad}}, \tag{11}$$

$$T(x, t) \geq \gamma \ > \ 0, \quad \forall (x, t) \in \overline{Q}. \tag{12}$$

Moreover, if $s_0(\phi)$ is of the form given in case **(B)**, there exist constants $0 < \hat{a}_{t^*} < \hat{b}_{t^*} < 1$ such that

$$\hat{a}_{t^*} \leq \phi(x, t) \leq \hat{b}_{t^*}, \quad \forall (x, t) \in \overline{Q}. \tag{13}$$

In order to prove differentiability of the observation operator S one has to first improve the stability result of Sprekels, Zheng (1992). To do this we let $(\phi_i, T_i) = S(v_i, w_i)$, $i = 1,2$ and $(v_i, w_i) \in \mathcal{U}_{\rm ad}$. We define $\overline{\phi} = \phi_1 - \phi_2$, $\overline{T} = T_1 - T_2$, $\overline{v} = v_1 - v_2$, and $\overline{w} = w_1 - w_2$. Using these notations we have the following result.

Proposition 2 *There exists a constant $C > 0$ such that*

$$\max_{0\le t\le t^*}\left(\|\overline{\phi}_t(t)\|^2_{H^1} + \|\overline{\phi}(t)\|^2_{H^3} + \|\overline{T}(t)\|^2_{H^2} + \|\overline{T}_t(t)\|^2\right)$$
$$+\int_0^{t^*}\left(\|\overline{\phi}_t(t)\|^2_{H^1} + \|\overline{T}_t(t)\|^2_{H^1}\right)dt + \int_0^{t^*}\|\overline{\phi}_{tt}(t)\|^2\,dt \le C\overline{G}(\overline{v},\overline{w}), \tag{14}$$

where

$$\overline{G}(\overline{v},\overline{w}) = \int_0^{t^*}\left(\|\overline{w}_t(t)\|^2_{L^2(\Gamma)} + \|\overline{v}_t(t)\|^2 + \|\overline{v}(t)\|^2\right)dt$$
$$+\|\overline{v}(0)\|^2 + \|\overline{w}\|^2_{H^1(0,t^*;L^2(\Gamma))} + \max_{0\le t\le t^*}\|\overline{w}(t)\|^2_{H^{\frac{1}{2}}(\Gamma)}. \tag{15}$$

Proof: From Theorem 2.1 of Sprekels, Zheng (1992) we know that there exists a constant $\hat{C} > 0$ such that

$$\max_{0\le t\le t^*}\left(\|\overline{\phi}_t(t)\|^2_{H^1} + \|\overline{\phi}(t)\|^2_{H^3} + \|\overline{T}(t)\|^2_{H^1}\right)$$
$$+\int_0^{t^*}\left(\|\overline{\phi}_{tt}(t)\|^2 + \|\overline{T}_t(t)\|^2\right)dt + \int_0^{t^*}\left(\|\overline{\phi}_t(t)\|^2_{H^1} + \|\overline{T}(t)\|^2_{H^2}\right)dt$$
$$\le \hat{C}G(\overline{v},\overline{w}), \tag{16}$$

where

$$G(\overline{v},\overline{w}) = \int_0^{t^*}\|\overline{v}(t)\|^2\,dt + \|\overline{w}\|^2_{H^1(0,t^*;L^2(\Gamma))}. \tag{17}$$

Similar to that paper $\overline{T}$ satisfies the following linear parabolic boundary value problem.

$$\overline{T}_t - \Delta\left(\overline{T}\zeta\right) = \phi_{1,t}\overline{\phi} - \phi_2\overline{\phi}_t + \overline{v}, \tag{18}$$

$$\left.\frac{\partial\overline{T}}{\partial n} + \overline{T}\right|_\Gamma = \overline{w}|_\Gamma, \qquad \overline{T}(x,0) = 0, \quad \forall x \in \overline{\Omega}, \tag{19}$$

where $\zeta = (T_1T_2)^{-1}$. Since $T_i \in X_2$ we have that $\zeta \in C([0,t^*];H^2(\Omega))$ and $\nabla\zeta_t \in L^2(Q)$. We can now differantiate (18) and (19) with respect to time to obtain

$$\overline{T}_{tt} - \Delta\left(\overline{T}\zeta\right)_t = \phi_{1,tt}\overline{\phi} - \phi_2\overline{\phi}_{tt} + \overline{\phi}_t^2 + \overline{v}_t, \tag{20}$$

$$\overline{T}_{tt} - \Delta\left(\overline{T}\zeta\right)_t = f, \tag{21}$$

$$\left.\frac{\partial\overline{T}_t}{\partial n} + \overline{T}_t\right|_\Gamma = \overline{w}_t|_\Gamma. \tag{22}$$

For the initial values of $\overline{T}_t$

$$\overline{T}_t(x,0) = \left(\Delta\left(\overline{T}\zeta\right) + \phi_{1,t}\overline{\phi} + \phi_2\overline{\phi}_t\right)(x,0) + \overline{v}(x,0) = \overline{v}(x,0).$$

Furthermore, we observe that

$$\int_0^{t^*} \|f(t)\|^2 \, dt \le c_1 G(\overline{v},\overline{w}) + \int_0^{t^*} \|\overline{v}_t(t)\|^2 \, dt, \tag{23}$$

by the previous results. To continue with our proof we multiply (20) by $\overline{T}_t$ and integrate the resulting equation over Ω to arrive at

$$\frac{1}{2}\frac{d}{dt}\left\|\overline{T}_t(t)\right\|^2 + \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta(t)\overline{T}(t)\right)_t dx - \int_\Gamma \overline{T}_t(t)\frac{\partial\left(\zeta(t)\overline{T}(t)\right)_t}{\partial n}\,dx$$
$$\le \frac{\delta_1}{2}\|f(t)\|^2 + \frac{1}{2\delta_1}\left\|\overline{T}_t(t)\right\|^2,$$

after applying (23) and Young's inequality. The value of δ_1 will be determined later. Next we observe that

$$\begin{aligned}\int_\Omega \nabla\overline{T}_t(t)\nabla\left(\zeta(t)\overline{T}(t)\right)_t dx &= \int_\Omega \zeta(t)\left|\nabla\overline{T}_t(t)\right|^2 dx + \int_\Omega \overline{T}(t)\nabla\overline{T}_t(t)\nabla\zeta_t(t)\,dx\\ &+ \int_\Omega \zeta_t(t)\nabla\overline{T}_t(t)\nabla\overline{T}(t)\,dx + \int_\Omega \overline{T}(t)\nabla\overline{T}_t(t)\nabla\zeta(t)\,dx\\ &= \int_\Omega \zeta(t)\left|\nabla\overline{T}_t(t)\right|^2 dx + I_1(t) + I_2(t) + I_3(t).\end{aligned}$$

Using standard esitmates, we can treat the terms on the right of this last inequality individually as follows

$$\begin{aligned}|I_1(t)| &\le \frac{\delta_2}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_2}\|\nabla\zeta_t(t)\|^2_{H^1(\Omega)}\left\|\overline{T}(t)\right\|^2_{H^1(\Omega)},\\ |I_2(t)| &\le \frac{\delta_3}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_3}\|\zeta_t(t)\|^2_{H^1(\Omega)}\left\|\overline{T}(t)\right\|^2_{H^2(\Omega)},\\ |I_3(t)| &\le \frac{\delta_4}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_4}\|\nabla\zeta(t)\|^2_{L^\infty(\Omega)}\left\|\overline{T}_t(t)\right\|^2.\end{aligned}$$

In each of these inequalities one can estimate the integral over t of the second term on the right via $G(\overline{v},\overline{w})$. The values for δ_i will be determined later. For the boundary term we observe that

$$\begin{aligned}-\int_\Gamma \overline{T}_t\frac{\partial}{\partial t}\frac{\partial\left(\zeta\overline{T}\right)}{\partial n}dx &= \int_\Gamma \zeta\overline{T}_t^2\,dx - \int_\Gamma \overline{T}_t\left(\overline{w}-\overline{T}\right)\zeta_t\,dx - \int_\Gamma \zeta\overline{T}_t\overline{w}_t\,dx\\ &+ \int_\Gamma \overline{T}_t^2\left(\zeta^2\left(T_1\left(w_2-T_2\right)+T_2\left(w_1-T_1\right)\right)\right)dx\\ &+ \int_\Gamma \overline{T}_t\overline{T}\left(\zeta^2\left(T_1\left(w_2-T_2\right)+T_2\left(w_1-T_1\right)\right)\right)_t dx\\ &= \int_\Gamma \zeta\overline{T}_t^2\,dx + J_1(t) + J_2(t) + J_3(t) + J_4(t).\end{aligned}$$

We can again estimate the terms individually as follows

$$
\begin{aligned}
|J_1(t)| &\leq \frac{\delta_5}{2} \left\|\overline{T}_t(t)\right\|^2_{L^2(\Gamma)} \\
&+ \frac{c_5}{2\delta_5}\left(\|\overline{w}(t)\|^2_{L^4(\Gamma)} + \left\|\overline{T}(t)\right\|^2_{L^4(\Gamma)}\right)\|\zeta_t\|^2_{L^4(\Gamma)}, \\
|J_2(t)| &\leq \frac{\delta_6}{2} \left\|\overline{T}_t(t)\right\|^2_{L^2(\Gamma)} + \frac{c_6}{2\delta_6}\|\overline{w}_t(t)\|^2_{L^2(\Gamma)}, \\
|J_3(t)| &\leq c_7 \left\|\overline{T}_t(t)\right\|^2_{L^2(\Gamma)} \leq \frac{\delta_7}{2}\left\|\nabla \overline{T}_t(t)\right\|^2 + \hat{c}_7 \left\|\overline{T}_t(t)\right\|^2, \\
|J_4(t)| &\leq \frac{\delta_8}{2} \left\|\overline{T}_t(t)\right\|^2_{L^2(\Gamma)} \\
&+ \frac{1}{2\delta_8}\left\|\overline{T}\right\|^2_{L^4(\Gamma)} \left\|\left(\zeta^2\left(T_1\left(w_2 - T_2\right) + T_2\left(w_1 - T_1\right)\right)\right)_t\right\|^2_{L^4(\Gamma)}.
\end{aligned}
$$

From the trace theorem and the Sobolev imbedding theorem (see, for example, Adams (1984) for the Sobolev theorem for fractional exponents) we have the following continuous imbeddings

$$
\left\{v : v = u|_{\Gamma}\, ; u \in H^1(\Omega)\right\} \hookrightarrow H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^4(\Gamma). \tag{24}
$$

Using this we can bound the time integrals of the second terms on the right by $\overline{G}(\overline{v}, \overline{w})$. After choosing the δ_i's sufficiently small we combine all the estimates to get after integration over t

$$
\begin{aligned}
\frac{1}{2}\left\|\overline{T}_t(t)\right\|^2 + \hat{c}\int_0^t \left\|\overline{T}_t(s)\right\|^2_{H^1(\Omega)} ds &\leq C_1\overline{G}(\overline{v}, \overline{w}) + \frac{1}{2}\left\|\overline{T}_t(0)\right\|^2 \\
&\leq C_2\overline{G}(\overline{v}, \overline{w}).
\end{aligned}
$$

The result now immediately follows from elliptic regularity estimates. ∎

In order to formulate the next result we introduce the sets

$$
K^{\pm}(v, w) = \left\{(h, k) \in V \times W : \exists\lambda > 0 \ni (v \pm \lambda h, w \pm \lambda k) \in \mathcal{U}_{\mathrm{ad}}\right\}, \tag{25}
$$

for $(v, w) \in \mathcal{U}_{\mathrm{ad}}$.

Proposition 3 *Suppose* **(H1)** *and* **(H2)** *hold and* $(v, w) \in \mathcal{U}_{\mathrm{ad}}$. *Then the observation operator*

$$
S : K \to B,
$$

has a directional derivative $(\psi, \theta) = D_{(h,k)}S(v, w)$ *in the direction* $(h, k) \in K^+$. *Furthermore, at* $S(v, w) = (\phi, T)$, *this directional derivative* $(\psi, \theta) \in X_1 \times X_2$ *is the unique solution of the linear parabolic initial-boundary value problem*

$$
\psi_t - \Delta\psi = -\psi\left(\frac{1}{T} + s_0''(\phi)\right) + \frac{\phi}{T^2}\theta,
$$

$$
\begin{aligned}
\theta_t - \Delta\left(\frac{\theta}{T^2}\right) &= -(\phi\psi)_t + h,\\
\frac{\partial\psi}{\partial n} = 0, \qquad & \frac{\partial\theta}{\partial n} + \theta = k, \quad \text{on } \Gamma,\\
\psi(0,x) &= \theta(0,x) = 0, \quad \text{on } \overline{\Omega}
\end{aligned}
$$

The same result also holds for the directional derivative $D_{(-h,-k)}S(v,w)$ at (v,w) in direction $(h,k) \in K^-(v,w)$.

Proof: As in Sprekels, Zheng (1992) we let

$$(\phi^\lambda, T^\lambda) = S(v + \lambda h, w + \lambda k).$$

Furthermore, we use the notation of the previous Proposition and let

$$\overline{\phi} = \phi^\lambda - \phi; \quad \overline{T} = T^\lambda - T; \quad \zeta = \frac{1}{TT^\lambda}.$$

Finally, define

$$p = \overline{\phi} - \lambda\psi; \quad q = \overline{T} - \lambda\theta.$$

It is clear that the linear parabolic system in the statement admits a unique solution $(\psi,\theta) \in X_1 \times X_2$. To continue, suppose that $(h,k) \in K^+(v,w)$ and suppose that there is a $\overline{\lambda} > 0$ such that $(v+\lambda h, w+\lambda k) \in \mathcal{U}_{\mathrm{ad}}, \quad \forall \lambda \in (0,\overline{\lambda})$. We have to show

$$\|(p,q)\|_B = o(\lambda), \quad \text{as } \lambda \to 0^+. \tag{26}$$

Using our notation p and q observe the following system of linear parabolic boundary value problems.

$$
\begin{aligned}
p_t - \Delta p &= s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi \\
&\quad -\frac{p}{T} + \frac{\phi}{T^2}q - \frac{\phi}{T}\overline{T}^2\zeta + \overline{\phi}\,\overline{T}\zeta, \qquad (27)\\
q_t - \Delta\left(\frac{q}{T^2}\right) &= -\phi_t p - \phi p_t - \overline{\phi}\,\overline{\phi}_t - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right), \qquad (28)\\
\frac{\partial p}{\partial n} = 0; \qquad & \frac{\partial q}{\partial n} + q = 0; \quad \text{on } \partial Q, \qquad (29)\\
0 &= p(x,0) = q(x,0). \qquad (30)
\end{aligned}
$$

We prove (26) in several steps.
Step 1: In Sprekels, Zheng (1992) the authors show that

$$
\begin{aligned}
&\max_{0\le t\le t^*}\left(\|p(t)\|^2_{H^1} + \|q(t)\|^2\right)\\
&\quad + \int_0^{t^*}\left(\|p_t(s)\|^2 + \|q(s)\|^2_{H^1} + \|p(s)\|^2_{H^2}\right) ds \;\le\; C\lambda^4, \qquad (31)
\end{aligned}
$$

for a suitable constant $C > 0$. We continue from there by multiplying (28) by $\left(\frac{q}{T^2}\right)_t$. After integrating the resulting equation over $\Omega \times [0,t]$ we obtain

$$\int_0^t \left\|\frac{q_t}{T}\right\|^2 ds + \frac{1}{2}\left\|\nabla\left(\frac{q}{T^2}\right)(t)\right\|^2 - \int_0^t \int_\Gamma \left(\frac{q}{T^2}\right)_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right) dxds \quad (32)$$
$$= \int_0^t \int_\Omega f\left(\frac{q_t}{T^2} - 2\frac{qT_t}{T^3}\right) dxds - 2\int_0^t \int_\Omega \frac{q_t qT_t}{t^3} dxds,$$

where f is given by

$$-\phi_t p - \phi p_t - \overline{\phi}\,\overline{\phi}_t - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right).$$

From Proposition 1 and the earlier estimates we see that

$$\int_0^{t^*} \|f(s)\|^2 ds \leq C_1\lambda^4,$$

for a suitable constant C_1. Furthermore, we have

$$\int_0^{t^*} \left\|\frac{qT_t}{T^3}(s)\right\|^2 ds \leq C_2\lambda^4,$$

due to earlier estimates. For the boundary term we observe

$$\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right) = \frac{q}{T^2}\left(\frac{w}{T} - 1\right).$$

Therefore we have

$$\left|\int_0^t \int_\Gamma \left(\frac{q}{T^2}\right)_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right) dxds\right| = \left|\int_0^t \int_\Gamma \left(\frac{q}{T^2}\right)_t \frac{q}{T^2}\left(1 - \frac{w}{T}\right) dxds\right|$$
$$\leq c_1 \left\|\frac{q}{T^2}(t)\right\|^2_{L^2(\Gamma)} + c_2 \int_0^t \int_\Gamma q^2 \left|\left(\frac{w}{T}\right)_t\right|_{L^2(\Gamma)} dxds$$
$$\leq c_1\delta \left\|\nabla\left(\frac{q}{T^2}(t)\right)\right\|^2 + c_3\|q(t)\|^2 + c_2\int_0^t \|q(s)\|^2_{L^4(\Gamma)} \left\|\left(\frac{w}{T}\right)_t\right\| ds$$
$$\leq c_1\delta \left\|\nabla\left(\frac{q}{T^2}(t)\right)\right\|^2 + c_4\lambda^4 + c_5\int_0^t \|q(s)\|^2_{H^1} ds.$$

In the last line of this estimate we used (24). Combining these estimates, using Young's inequality and chosing δ sufficiently small we obtain

$$\max_{0\leq t\leq t^*} \left\|\nabla\left(\frac{q}{T^2}\right)(t)\right\|^2 + \int_0^{t^*} \left\|\frac{q_t}{T}\right\|^2 ds \leq C_3\lambda^4.$$

It immediately follows

$$\max_{0\leq t\leq t^*} \|q(t)\|^2_{H^1} + \int_0^{t^*} \|q_t\|^2 ds \leq C_4\lambda^4. \quad (33)$$

Step 2: In this step, we take the derivative of (27) with respect to t to get

$$\begin{aligned} p_{tt} - \Delta p_t &= \left(s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi\right)_t \\ &\quad + \left(\frac{p}{T} - \frac{\phi}{T^2}q + \frac{\phi}{T}\overline{T}^2\zeta - \overline{\phi}\,\overline{T}\zeta\right)_t \\ &= F_{1,t} + F_{2,t}. \end{aligned} \tag{34}$$

We observe that

$$\begin{aligned} |F_{1,t}| &= \left|\left(s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi\right)_t\right| \\ &\leq \left|\phi_t\left(s_0''(\phi) - s_0''(\phi^\lambda) - s_0'''(\phi)\overline{\phi}\right)\right| + \left|s_0'''(\phi)\phi_t p\right| \\ &\quad + \left|s_0''(\phi)p_t\right| + \left|\left(s_0''(\phi^\lambda) - s_0''(\phi)\right)\overline{\phi}_t\right|. \end{aligned}$$

Using the mean-value theorem one easily sees that

$$\int_0^{t^*} \|F_{1,t}(s)\|^2 \, ds \leq c_5\lambda^4, \tag{35}$$

for a suitable constant c_8. Next we observe that

$$\begin{aligned} F_{2,t} &= \frac{p_t}{T} - \frac{pT_t}{T^2} - \frac{\phi_t}{T^2}q + 2\frac{\phi T_t}{T^3}q - \frac{\phi}{T^2}q_t + \frac{\phi_t}{T}\overline{T}^2\zeta - \frac{\phi}{T^2}T_t\overline{T}_2\zeta \\ &\quad + 2\frac{\phi}{T}\overline{T}\,\overline{T}_t\zeta + \frac{\phi}{T}\overline{T}^2\zeta_t - \overline{\phi}_t\overline{T}\zeta - \overline{\phi}\,\overline{T}_t\zeta - \overline{\phi}\,\overline{T}\zeta_t \end{aligned}$$

Since both ϕ_t and T_t are elements of $C([0,t^*];H^1(\Omega))$ we see that

$$\int_0^{t^*} \|F_{2,t}(s)\|^2 \, ds \leq c_6\lambda^4, \tag{36}$$

for a suitable constant c_9. So if one multiplies (34) by p_t and integrates the result over $\Omega \times [0,t]$ one gets immediately

$$\max_{0\leq t\leq t^*} \|p_t(t)\|^2 + \int_0^{t^*} \|p_t(s)\|_{H^1}^2 \, ds \leq C_5\lambda^4. \tag{37}$$

We can now apply the standard elliptic regularity estimates to get

$$\max_{0\leq t\leq t^*} \|p(t)\|_{H^2}^2 \leq C_6\lambda^4. \tag{38}$$

Furthermore, we can multiply (34) by p_{tt}, integrate the result over $\Omega\times[0,t]$ and use (35) and (36) again to get

$$\max_{0\leq t\leq t^*} \|p_t\|_{H^1}^2 + \int_0^{t^*} \|p_{tt}(s)\|^2 \, ds \leq C_7\lambda^4, \tag{39}$$

for a suitable constant C_7.

Step 3: To continue we take the time derivative of (28) to obtain

$$q_{tt} - \Delta\left(\frac{q}{T^2}\right)_t = F_{3,t}(x,t), \tag{40}$$

where

$$F_{3,t} = \left(\phi_t p + \phi p_t + \overline{\phi}\phi_t - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right)\right)_t.$$

To simplify notations we introduce $\hat{\zeta} = \frac{\zeta}{T}$, which has the same properties as ζ. We observe that

$$\begin{aligned}
\Delta\left(\overline{T}^2\hat{\zeta}\right)_t &= 2\overline{T}_t\hat{\zeta}\Delta\overline{T} + 4\hat{\zeta}\nabla\overline{T}\nabla\overline{T}_t + 4\overline{T}_t\nabla\overline{T}\nabla\hat{\zeta} + 2\overline{T}\hat{\zeta}\Delta\overline{T}_t \\
&\quad + 4\overline{T}\nabla\overline{T}_t\nabla\hat{\zeta} + 2\overline{T}\,\overline{T}_t\Delta\hat{\zeta} + 2\left|\nabla\overline{T}\right|^2\hat{\zeta}_t + 2\hat{\zeta}_t\overline{T}\Delta\overline{T} \\
&\quad + 4\overline{T}\nabla\overline{T}\nabla\hat{\zeta}_t + \overline{T}^2\Delta\hat{\zeta}_t.
\end{aligned}$$

Using the results of Proposition 2, we can bound $\left\|\overline{T}(t)\right\|_{H^2}$ by $c_9\lambda$ for a sufficiently large constant c_9. Furthermore, we know that $\overline{T}$ has the same regularity as $\hat{\zeta}$ which enables us to bound terms of the form

$$\int_0^{t^*}\left\|\overline{T}\right\|_{H^2}^2 ds, \text{ and } \quad \max_{0\le t\le t^*}\left\|\overline{T}_t(t)\right\|_{H^1}$$

by constants. Combining these properties we see that

$$\int_0^{t^*}\left\|\Delta\left(\overline{T}^2\hat{\zeta}\right)_t(s)\right\|^2 ds \le c_{10}\lambda^2$$

for a suitable constant c_{10}. It follows that

$$\int_0^{t^*}\left\|F_{3,t}(s)\right\|^2 ds \le c_{11}\lambda^2. \tag{41}$$

We multiply (40) by q_t and integrate the result over $\Omega\times[0,t]$ to get

$$\begin{aligned}
\frac{1}{2}\left\|q_t(t)\right\|^2 &+ \int_0^{t^*}\int_\Omega \nabla q_t\cdot\nabla\left(\frac{q}{T^2}\right)_t dxds - \int_0^{t^*}\int_\Gamma q_t\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)_t dxds \\
&\le \left(\int_0^{t^*}\left\|F_{3,t}(s)\right\|^2 ds\right)^{\frac{1}{2}}\left(\int_0^{t^*}\left\|q_t(s)\right\|^2 ds\right)^{\frac{1}{2}} \le c_{12}\lambda^3,
\end{aligned}$$

for a suitable constant c_{13}. We next observe that

$$\begin{aligned}
&\int_0^{t^*}\int_\Omega \nabla q_t\nabla\left(\frac{q}{T^2}\right)_t dxds = \int_0^{t^*}\int_\Omega \nabla q_t\nabla\left(\frac{q_t}{T^2} - 2\frac{qT_t}{T^3}\right) dxds \\
&= \int_0^{t^*}\left\|\frac{\nabla q_t}{T}(s)\right\|^2 ds - 2\int_0^{t^*}\int_\Omega \nabla q_t\left(\frac{q_t}{T^3}\nabla T + \frac{T_t}{T^3}\nabla q\right) dxds \\
&+2\int_0^{t^*}\int_\Omega \nabla q_t\left(3\frac{qT_t}{T^4}\nabla T - \frac{q}{T^3}\nabla T_t\right) dxds.
\end{aligned}$$

One sees that the mixed terms on the right can be treated via Young's inequality, and that we can use the fact that

$$\int_0^{t^*} \|q\|_{H^2}^2 \, ds \le c_{14}\lambda^4,$$

and the other earlier estimates on q. Finally we observe

$$\int_0^{t^*}\int_\Gamma q_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)_t dx ds = \int_0^{t^*}\int_\Gamma q_t \left(\frac{1}{T^2}\frac{\partial q_t}{\partial n} - 2\frac{q_t}{T^3}\frac{\partial T}{\partial n}\right) dx ds$$
$$+2\int_0^{t^*}\int_\Gamma q_t \left(3\frac{qT_t}{T^4}\frac{\partial T}{\partial n} - \frac{T_t}{T^3}\frac{\partial q}{\partial n} - \frac{q}{T^3}\frac{\partial T_t}{\partial n}\right) dx ds$$
$$= -\int_0^{t^*}\int_\Gamma \frac{q_t^2}{T^2}\left(1+2\frac{1}{T}\frac{\partial T}{\partial n}\right) dx ds + 2\int_0^{t^*}\int_\Gamma \frac{qq_t}{T^3}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) dx ds.$$

In the first term we observe that

$$1+2\frac{1}{T}\frac{\partial T}{\partial n} \in L^\infty(\partial Q).$$

In the second term one has

$$\frac{1}{T}\left(T_t - \frac{\partial T_t}{\partial n} + 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) \in L^2(0,t^*;L^\infty(\Gamma)).$$

Using this we get

$$\left|2\int_0^{t^*}\int_\Gamma \frac{qq_t}{T^3}\left(T_t - \frac{\partial T_t}{\partial n} + 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) dx ds\right| \le C_9\left(\int_0^{t^*}\left\|\frac{q_t}{T}\right\|_{L^2(\Gamma)}^2 \|q\|_{L^2(\Gamma)}^2 \, ds\right)^{\frac{1}{2}}$$

Observe that

$$\|q(t)\|_{L^2(\Gamma)}^2 \le C_{10}\lambda^4.$$

This implies that we are left to treat estimate of the form

$$\int_0^{t^*}\left\|\frac{q_t}{T}\right\|_{L^2(\Gamma)}^2 ds.$$

We do this by using

$$\int_0^{t^*}\|g(s)\|_{L^2(\Gamma)}^2 \, ds \le \delta\int_0^{t^*}\|\nabla g(s)\|^2 \, ds + \hat{C}\int_0^{t^*}\|g(s)\|^2 \, ds,$$

for a suitable constant $\hat{C}$. We can now combine all these estimates and use the properties of T to conclude

$$\max_{0\le t\le t^*}\|q_t(t)\| + \int_0^{t^*}\|\nabla q_t\|^2 \, ds \le C_{11}\lambda^3, \tag{42}$$

for a suitable constant C_9. From elliptic regularity estimates it follows, that the same estimate holds for

$$\max_{0\le t\le t^*} \|q(t)\|^2_{H^2}\,.$$

This finishes the proof of the proposition. ∎

4. Optimality conditions

We return to the optimal control problem **(CP)** of Section 2. We introduced the non-linear observation operator S (9)–(10). We can write S in components (S_1, S_2) as follows.

$$S(v,w) = \begin{pmatrix} S_1(v,w) \\ S_2(v,w) \end{pmatrix} = \begin{pmatrix} \phi \\ T \end{pmatrix}. \tag{43}$$

Proposition 3 states that this operator is Gateaux differentiable with Gateaux derivative

$$DS(v,w)(h,k) = \begin{pmatrix} DS_1(v,w)(h,k) \\ DS_2(v,w)(h,k) \end{pmatrix} = \begin{pmatrix} \psi \\ \theta \end{pmatrix}, \tag{44}$$

given by the following system of linearized equations

$$\psi_t - \Delta\psi = \psi\left(\frac{1}{T} - s_0''(\phi)\right) - \frac{\phi}{T^2}\theta, \tag{45}$$

$$\theta_t - \Delta\left(\frac{\theta}{T^2}\right) = (\phi\psi)_t + h, \tag{46}$$

$$\frac{\partial\psi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} + \theta = k, \quad \text{on } \Gamma, \tag{47}$$

$$\psi(0,x) = \theta(0,x) = 0, \quad \text{on } \overline{\Omega}. \tag{48}$$

We can apply the Lagrange multiplier rule (see, for example, Thm. 5.2 of Casas, 1993, for a proof) to conclude that there exist $\lambda \ge 0$ and Borel measures $\mu_1, \mu_2, \mu_3, \mu_4$, with the properties:

$$\mu_i(\{(x,t) \in \overline{Q} | T(x,t) \ne K_i\}) = 0, \; i = 1, 2, \tag{49}$$

$$\mu_i(\{(x,t) \in \overline{Q} | \phi(x,t) \ne K_i\}) = 0, \; i = 3, 4, \tag{50}$$

such that

$$\lambda + |\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| > 0,$$

where $|\mu_i|, i = 1, ..., 4$, denotes the norm of the measure μ_i.

The constants K_i are the same as in the state constraints (7). To continue, we denote $\mu = \mu_1 - \mu_2$, $\nu = \mu_3 - \mu_4$.

The optimality system given in the abstract case in Casas (1993), formulae (5.1)-(5.3) at page 1001, can be specified for the control problem under consideration in the form of two inequalities marked below by $(*)$ and $(**)$. The first condition takes the form

$$(*) \qquad \forall (\zeta, \eta) \in \mathcal{Y}_{ad} : \int (\eta - T) d\mu + \int (\zeta - \phi) d\nu \leq 0,$$

where $(\phi, T) = S(v, w)$ is a solution to the state equations for optimal controls $(v, w) \in \mathcal{U}_{\text{ad}}$.

For the second condition, we need to introduce some notation. We denote by $I(\phi, T; v, w)$ the cost functional i.e. $J(v, w) = I(S_1(v, w), S_2(v, w); v, w)$, then the gradient of the cost functional, with respect to the controls takes the form

$$\begin{aligned} &\langle DJ(v,w), (h,k)\rangle = \langle D_1 I(\phi, T; v, w), D_1 S(\phi, T)(h,k)\rangle \\ &+\langle D_2 I(\phi, T; v, w), D_2 S(\phi, T)(h,k)\rangle + \langle D_3 I(\phi, T; v, w), h\rangle + \langle D_4 I(\phi, T; v, w), k\rangle. \end{aligned}$$

The second optimality condition is of the form

$$\begin{aligned} (**) \qquad &\lambda \langle DJ(v,w), (h-v, k-w)\rangle + \langle [DS_2(v,w)]^*(h-v, k-w)], \mu\rangle \\ &+\langle [DS_1(v,w)]^*(h-v, k-w)], \nu\rangle \geq 0, \end{aligned}$$

for all $(h, k) \in \mathcal{U}_{\text{ad}}$, where $[DS_i(v, w)]^*$ denotes the adjoint to $[DS_i(v, w)], i = 1, 2$.

Assuming that the Slater condition is satisfied, we can take $\lambda = 1$ (see, e.g. Casas, 1993). In the present case the Slater condition **(S)** means that there exists an optimal control $(h_0, k_0) \in \mathcal{U}_{\text{ad}}$ such that for all $(x, t) \in \overline{Q}$

$$\begin{aligned} K_1 &< T(x,t) + [DS_2(v,w)(h_0 - v, k_0 - w)](x,t) < K_2 \\ K_3 &< \phi(x,t) + [DS_1(v,w)(h_0 - v, k_0 - w)](x,t) < K_4 \end{aligned}$$

Furthermore, an adjoint state is introduced in order to simplify the latter optimality condition. To this end, we rewrite the linearized equations in the form

$$\mathcal{L}_{11}(\psi) + \mathcal{L}_{12}(\theta) = 0, \tag{51}$$

$$\mathcal{L}_{21}(\psi) + \mathcal{L}_{22}(\theta) = h, \tag{52}$$

with boundary conditions on Γ

$$\mathcal{L}_3(\psi, \theta) = \mathcal{L}_{32}(\theta) = k, \tag{53}$$

$$\mathcal{L}_4(\psi, \theta) = \mathcal{L}_{41}(\psi) = 0, \tag{54}$$

where

$$\mathcal{L}_{11}(\psi) = \psi_t - \Delta\psi - \psi\left(\frac{1}{T} - s_0''(\phi)\right), \tag{55}$$

$$\mathcal{L}_{12}(\theta) = \frac{\phi}{T^2}\theta, \tag{56}$$

$$\mathcal{L}_{21}(\psi) = -(\phi\psi)_t\,, \tag{57}$$

$$\mathcal{L}_{22}(\theta) = \theta_t - \Delta\left(\frac{\theta}{T^2}\right), \tag{58}$$

$$\mathcal{L}_{32}(\theta) = \frac{\partial\theta}{\partial n} + \theta, \tag{59}$$

$$\mathcal{L}_{41}(\psi) = \frac{\partial\psi}{\partial n}\,. \tag{60}$$

We denote by

$$\begin{aligned}
&\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2\,,\\
&\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3 \times \mathcal{V}_4\,,\\
&\mathcal{L}\ : \mathcal{W} \mapsto \mathcal{V}\,,\\
&\mathcal{L}[(\psi,\theta)] = (\mathcal{L}_1(\psi,\theta), \mathcal{L}_2(\psi,\theta), \mathcal{L}_3(\psi,\theta), \mathcal{L}_4(\psi,\theta))\,,\\
&\mathcal{L}_i(\psi,\theta) \in \mathcal{V}_i\,, \quad i = 1, ..., 4,
\end{aligned}$$

and we assume that the operator $\mathcal{L}$ is an isomorphism. The space $\mathcal{V}_2$ can be selected as the space V defined in section 1. We assume that $\mathcal{W}_1 \times \mathcal{W}_2 \subset [C(\overline{Q})]^2$ with continuous embedding. The linearized equation takes the form

$$\text{Find } (\psi,\theta) \in \mathcal{W}_1 \times \mathcal{W}_2 \text{ such that } \mathcal{L}[(\psi,\theta)] = (0,h,k,0) \text{ in } \mathcal{V},$$

where $h \in \mathcal{V}_2$, $k \in \mathcal{V}_3$ are given elements.

For any continuous linear form $\Psi[(\cdot,\cdot)]$ on $\mathcal{W}$ there exists a unique element $\overline{v}$ in $\mathcal{V}$ such that

$$(\mathcal{L}[(\zeta,\eta)], \overline{v})_{\mathcal{V}} = \Psi[(\zeta,\eta)] \quad \forall(\zeta,\eta) \in \mathcal{W},$$

since for an element $(\zeta,\eta) \in \mathcal{W}$ we have $(\zeta,\eta) = \mathcal{L}^{-1}[(p,q,r,s)]$ for the unique element $(p,q,r,s) \in \mathcal{V}$.

We select the following linear form on $\mathcal{W}$

$$\Psi[(\zeta,\eta)] = \langle D_1 I(\phi,T;v,w), \zeta\rangle + \int \zeta d\nu + \langle D_2 I(\phi,T;v,w), \eta\rangle + \int \eta d\mu,$$

which is continuous under our assumptions.

Then there exixts the unique adjoint state $\overline{v} = (\overline{q},\overline{p},\overline{r},\overline{s}) \in \mathcal{V}$ such that the following adjoint state equation is satisfied

$$(\mathcal{L}[(\zeta,\eta)], \overline{v})_{\mathcal{V}} = \Psi[(\zeta,\eta)] \quad \forall(\zeta,\eta) \in \mathcal{W}.$$

For any solution (ψ,θ) of the linearized equation we have

$$\begin{aligned}
\Psi[(\psi,\theta)] &= \langle D_1 I(\phi,T;v,w), \psi\rangle + \langle D_2 I(\phi,T;v,w), \theta\rangle\\
&\quad + \int \theta d\mu + \int \psi d\nu\\
&= (\mathcal{L}[(\psi,\theta)], \overline{v})_{\mathcal{V}}\\
&= (h,\overline{p})_{\mathcal{V}_2} + (k,\overline{r})_{\mathcal{V}_3}.
\end{aligned}$$

Using the above construction, it follows that for $\lambda = 1$ the necessary optimality conditions can be rewritten in the following form.

Theorem 1 *Assume that condition (S) is satisfied. Then there exist μ, ν and the adjoint state $(\overline{p}, \overline{q}, \overline{r}, \overline{s}) \in \mathcal{V}$ such that the optimality system for the control problem includes the state equation, the adjoint state equation, and the condition $(*)$, as well as the following condition*

$$\begin{aligned} &\langle D_3 I(\phi, T; v, w), h - v\rangle + (h - v, \overline{p})_{\mathcal{V}_2} \\ &+\langle D_4 I(\phi, T; v, w), k - w\rangle + (k - w, \overline{r})_{\mathcal{V}_3} \geq 0 \end{aligned}$$

for all $(h, k) \in \mathcal{U}_{\mathrm{ad}}$.

5. Concluding remarks

In this paper first order optimality conditions were derived for a control problem for a system described by quasi-linear parabolic equations. These results for a control problem with state constraints seem to be new.

In particular our analysis shows that the mapping "control $\mapsto$ state" is Gateaux differentiable. Further analysis of the obtained optimality system could eventually provide additional regularity results for optimal solutions of the state equations. These results would be useful for the stability analysis of optimal controls. A next step would be the derivation of second order optimality conditions for our problem.

The control problem is well-posed provided the controls are sufficiently regular. This is a very restrictive requirement in applications which usually use controls which are square integrable functions. However, such regularity assumptions seem to be unavoidable in systems governed by non-linear PDE's. Finally, the analysis of numerical methods for our problem would be of big interest.

References

ADAMS, R.A. (1984) *Sobolev Spaces.* Academic Press, New York.

AMANN, H. (1993) Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. Preprint.

CASAS, E. (1993) Boundary control of semilinear elliptic equations with pointwise state constraints. *SIAM J. Control Optim.*, **31**, 993–1006.

COLLI, P., SPREKELS, J. (1995) On a Penrose-Fife model with zero interfacial energy leading to a phase-field system of relaxed Stefan type. *Ann. Math. Pure Appl.*, **4**, to appear.

HORN, W. (1994) Mathematical aspects of the Penrose-Fife phase-field model. *Control and Cybernetics*, **23**, 4, 677–690.

HORN, W., LAURENÇOT, PH., SPREKELS, J. (1996) *Global solutions to a Penrose-Fife phase-field model under flux boundary conditions for the inverse temperature.* (Submitted).

HORN, W. SPREKELS, J. ZHENG, S. (1996) Global smooth solutions to the Penrose-Fife model for ising ferromagnets. *Adv. Math. Sci. Appl.*, to appear.

KENMOCHI, N., NIEZGÓDKA, M. (1993) Systems of nonlinear parabolic equations for phase change problems. *Adv. Math. Sci. Appl.*, **3**, 89–117.

LAURENÇOT, PH. (1994) Solutions to a Penrose-Fife model of phase-field type. *J. Math. Anal. Appl.*, **185**, 262–274.

MAZUMDAR, T. (1994) Generalized projection theorem and weak noncoercive evolution problems in Hilbert spaces. *J. Math. Anal. Appl.*, **46**, 143–168.

PENROSE, O., FIFE, P.C. (1990) Thermodynamically consistent models of phase-field type for the kinetics of phase transitions. *Physica*, **D 43**, 44–62.

SOKOŁOWSKI, J., SPREKELS, J. (1994) Control problems for shape memory alloys with constraints on the shear strain. *Lecture Notes in Pure and Applied Mathematics*, **165**, Marcel Dekker, G. Da Prato, L. Tubaro, eds., 189–196.

SOKOŁOWSKI, J., ZOLESIO, J.-P. (1992) *Introduction to Shape Optimization.* Springer–Verlag, Heidelberg.

SPREKELS, J., ZHENG, S. (1992) Optimal Control problems for a thermodynamically consistent model of phase-field type for phase transitions. *Adv. in Math. Sci. and Appl.*, **1**, 1, 113–125.

SPREKELS, J., ZHENG, S. (1993) Global smooth solutions to a thermodynamically consistent model of phase-field type in higher space dimensions. *J. Math. Anal. Appl.*, **176**, 200–223.