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Optimal procedures of statistical process control in the presence of erroneous process inspections

by

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Abstract: Statistical process control (SPC) is currently the most frequently used method for quality control and quality improvement. This paper presents the methodology for the optimal design of the SPC procedures. In contrast to previously published results, the proposed model describes the situation when the inspection actions are not perfect. Approximate solutions have been proposed in order to simplify the design of SPC procedures.
Keywords: quality control, statistical process control (SPC), control charts, optimal design of control charts.

1. Introduction

In the last two decades quality has become the key factor that determines a market success for every organization. Continuous quality improvement of products or services is now the main strategic goal for every firm striving for the market success. In order to arrive at the required quality levels different tools are used. The range of applicable methods is very wide: from the installation of sophisticated (and very costly) equipment for automatic process control to simple statistical methods that can be used on a shop floor by inexperienced personnel. The experience of organizations which are most successful in the implementation of quality improvement policies shows that the application of relatively simple statistical methods is the most effective way to meet high quality requirements for the majority of firms - especially small ones.

High quality of products or services can be widely recognised if a firm introduces a quality system which fulfills the requirements stated in the International Standards from the ISO 9000 series. In these standards many managerial tools for quality improvement are indicated. However, only one of them — statistical process control (SPC)— has direct technical interpretation. Therefore, it has become the main tool for controlling quality of production processes.

Among various methods which are used in SPC, the control charts, introduced by Walter A. Shewhart in 1920s, are the most popular. In the second section of this paper we describe the notion of control charts, and discuss the problems that are connected with their design and usage. We also try to delimit the area of application of control charts in contrast to the methods of automatic process control (APC). In the third section of the paper we present the problems of the economic design of control charts.

The main original results of this paper are presented in its fourth section. First, we propose a mathematical model that describes the economic consequences of the application of SPC procedures in the presence of erroneous process inspections. Next, we propose some asymptotic simplifications which allow to find approximately optimal SPC procedures in a relatively easy way. These simplifications give also additional insight into complicated relations between different quantities which describe production processes. Finally, in the fifth section, we propose further approximations which allow to find approximate optimal solutions with even smaller computational effort.

2. Statistical process control (SPC)

Statistical process control and related techniques of sampling inspections were developed in 1920s. SPC is a part of statistical quality control (SQC) which is focused rather on the process itself than on the final products. This distinction motivated some people involved in quality activity to contrast SPC with the traditional SQC. This distinction is not correct — as it has been pointed out by Wetherill and Brown (1991). SPC procedures can be used in traditional SQC activities such as final quality inspections (screenings), and traditional SQC methods can also be used as process control procedures. All these procedures are based on the same statistical concepts, and sometimes can be viewed upon as the same statistical tests used for slightly different purposes.

The concept of SPC is based on the assumption that there are two basic sources of the variation of the process. Some variation of the process, called *random variation* is in a certain sense unavoidable, and can be diminished only by fundamental changes of the process (e.g. change of technology, implementation of automatic control, etc.). The remaining part of the total variation of the process is due to some *special* or *assignable* causes, such as failures of an equipment, errors of operators, etc. When only random variation is observed the process is said to be *in statistical control.* The role of SPC procedures is to detect the moment when an assignable cause occurs, and to alarm the process operator. The role of the process operator is to identify the cause of deterioration, and to remove it. Thus, SPC can be looked upon as a collection of simple statistical tools that can be used to:

(1) Provide an evidence of the process performance.
(2) Assess the current process quality level.

(3) Indicate the necessity to investigate the process which has possibly fallen in troubles.
(4) Help in making improvements to the process or product.

It has to be noticed, however, that positive results can be achieved only then, when the statistical procedures are correctly designed and not misused. Therefore, the problem of the design of the SPC procedures is a crucial one, and deserves special attention.

Control charts are the most frequently used SPC procedures. They were introduced by Shewhart in 1920s in a form which is now known as Shewhart charts. In Shewhart charts a sample is periodically taken from a process. A certain statistics (for example, the sample mean $\bar{X}$), is used to evaluate the process performance. Its value is plotted on a chart, and compared to certain control limits. In the case of two-sided quality requirements, two control limits: lower control limit (LCL), and upper control limit (UCL) are used. If the observed value of the statistics of interest is beyond the control limits an alarm signal is generated. A typical control chart is schematically presented in Fig. 1.

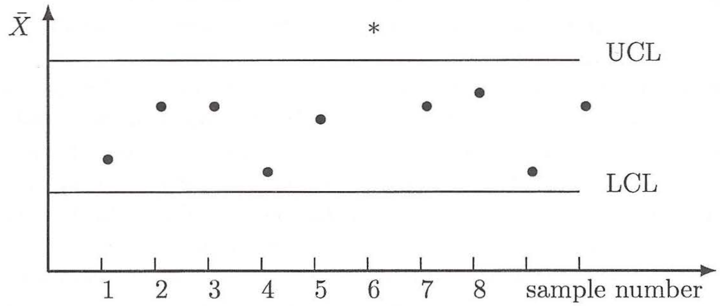

Figure 1. Schematic view of an $\bar{X}$ chart.

In Fig. 1 we see that the process remains under control till the sample No.6. For this sample the sample average falls beyond the upper control limit UCL, and the alarm signal is generated. After removing an assignable cause of this deterioration, the process operates anew under control.

From this brief description of SPC we can see the basic difference between statistical process control (SPC), and automatic process control (APC). The role of SPC is to *monitor* the process, and to indicate moments when it goes out of order due to some assignable causes. The role of APC, as it was stated in Box and Kramer (1992) is to *adjust* the process, or keep it on target. This basic difference is due mainly to the different origins of SPC and APC. SPC was originated in the *parts industry*, where automatic process control was not frequently used. On the other hand, APC was originated in the *process industry*. The goal of the process industry is to reproduce individual items as accurately as possible. The process industries, as it has been indicated in Box and Kramer

(1992), were typically interested in yields of product and were attempting to obtain the highest possible mean values of the characteristics of interest. With the process of automatisation of the production both, SPC and APC, are steadily converging. SPC remains, however, the ultimate means to monitor production processes, even those fully automatised.

3. Economically optimal SPC

Statistical process control has obvious economic consequences. It is not difficult to notice that differently designed control charts, when applied in practice, may lead to different economic effects. In his pioneering paper Duncan (1956) proposed the first mathematical model for the economic design of control charts. Since the publication of Duncan's paper many interesting results have been published on this subject. The recent results on the economic design of control charts have been reviewed in the paper of Ho and Case (1994). Duncan's model has been used as a starting point in many papers. Lorenzen and Vance (1986) proposed a generalisation of Duncan's model for a wide class of control charts. In Del Castillo and Montgomery (1996) a very general model has been proposed which allows to design optimal control charts for short run processes.

Another approach to the problem of economic design of control charts was proposed by von Collani (1981b, 1986), and fully described in von Collani (1989). In contrast to Duncan, who used profit per time unit as his objective function, von Collani proposed to define the objective function in terms of profit per unit produced. As the result, much simpler mathematical models have been proposed with a significantly smaller number of input parameters. The model of von Collani has been generalised by Hryniewicz (1992), and by other authors, mainly in the Research Reports of The Würzburg Research Group on Quality Control. In this paper we propose a further generalisation of the model proposed in Hryniewicz (1992) which is based on von Collani's approach.

According to the unified approach by von Collani (1989) we consider three type of actions connected with the implementation of an SPC procedure:

a) monitoring of the process (sampling),
b) inspection (searching for an assignable cause),
c) renewal of the process.

By monitoring we understand any procedure which allows us to determine the actual state of the considered process basing on its observation. In this paper we identify monitoring with sampling of the process, but other methods of monitoring (e.g. continuous observation) are also possible. The role of monitoring is to generate a signal (an alarm) that the process may not operate in an acceptable STATE I. In the case of such alarm the inspection is performed with the aim to decide whether a renewal of the process is necessary. When a monitoring action generates an alarm signal an inspection begins. By inspection we understand any action which uses non-statistical methods and reveals the actual state of the process. If the results of the inspection indicate that the process is in STATE II a renewal action is undertaken. After the renewal the process operates anew in STATE I.

Let us notice that some profits and losses are connected with the implementation of statistical process control procedures such as control charts. Suppose that there exists a certain acceptable state of the process, and we wish the process to operate in this state as long as possible. Denote this state by STATE I. Every process which operates under statistical control is in STATE I. When the process deteriorates it enters an unacceptable STATE II. Let τ be the expected duration of STATE I, g_1 the average profit from a unit produced while the process operates in STATE I, and g_2 the average profit from a unit produced while the process operates in STATE II. The expected profit from operating a control procedure can be computed as follows

$$P = (g_1 - g_2)\tau \tag{1}$$

Profits from the operation of statistical control procedures are diminished by costs related to monitoring (sampling) of the process, inspections, and renewal actions. Let a_0 be a constant cost of a monitoring action, and a_1 a cost of inspection of one unit from a sample. Denote now by n_1 the expected number of sampled units (expected sample size) during a monitoring action while the process operates in STATE I, and by n_2 the expected number of sampled units during a monitoring action while the process operates in STATE II. It is worth noticing that these expected sample sizes in general may not be equal. In the case of simple control charts we have, of course, $n_1 = n_2 = n$, where n is the sample size. We can now calculate the expected cost of one monitoring action in STATE I

$$S_1 = a_0 + n_1 a_1 \tag{2}$$

and the same expected cost while the process remains in STATE II

$$S_2 = a_0 + n_2 a_1 \tag{3}$$

Denote now by A_1 the expected number of monitoring actions in STATE I, and by A_2 the expected number of monitoring actions in STATE II. We can now calculate the total expected cost of monitoring as

$$C_M = A_1 S_1 + A_2 S_2 \tag{4}$$

As the decisions resulting from the monitoring action are based on random results there exists a certain probability α of a false alarm. Let e^* be the cost of unnecessary inspection caused by a false alarm, and r^* be the total cost of inspection and renewal when the monitoring action reveals the deterioration of the process. The expected cost of inspection and renewal actions can be now calculated as follows

$$C_{IR} = \alpha A_1 e^* + r^* \tag{5}$$

Define now a renewal cycle as a time (measured in unit produced) between two consecutive renewal actions. Let N be the expected duration of the renewal cycle. Then the average profit per unit produced can be calculated from a general formula

$$G = \frac{P - C_M - C_{IR}}{N} \tag{6}$$

To design optimally any statistical process control procedure using von Collani's approach we have to maximise (6) for a given mathematical model of the process.

4. Optimal SPC procedures in the presence of erroneous inspections

In the papers on optimal SPC procedures it is usually assumed that false alarms do not cause renewals of the process. It is assumed that an inspection which follows an alarm signal reveals the true state of the inspected process. Thus, in the case of false alarms inspections confirm that the process remains in the acceptable STATE I, and unnecessary renewal actions are not undertaken. However, in real situations process inspections may not undoubtedly confirm that the process does not require renewal. In such cases the operators of the inspected process may decide to stop it, and to perform renewal actions. The possibility of such unnecessary actions makes the process less profitable, and should be taken into account in the design of SPC procedures. This problem was first considered by von Collani (1981a), who derived the mathematical formulae for the description of the process. In this section we introduce a general mathematical model which allows to design optimal SPC procedures, and we find some approximate solutions to the stated problem.

4.1. Mathematical model

Suppose that the considered production process is described by a random variable X (univariate or multivariate) which in an acceptable STATE I has a certain probability distribution characterised by a vector of parameters $p_1 = (p_{11}, p_{12}, \dots, p_{1k})$. When the process deteriorates, i.e. enters an unacceptable STATE II the vector of parameters describing the probability distribution of X becomes $p_2 = (p_{21}, p_{22}, \dots, p_{2k})$. The moment of transition from STATE I to STATE II is described by a random variable T distributed accordingly to a continuous distribution function $F(t)$, and is not directly observable. Moreover, we assume that the transition from STATE I to STATE II can be achieved only by a special renewal action.

When statistical procedures are used to evaluate the results of monitoring two types of errors are involved:

a) type I error when the monitoring procedure indicates the necessity of inspection while the process is operating in STATE I, and

b) type II error when the monitoring does not indicate the necessity of any action while the process is in an unacceptable STATE II.

Suppose that the statistical procedure used for the evaluation of the results of monitoring is described by a vector of parameters $\gamma = (\gamma_1, \dots, \gamma_m)$. We assume that the probability of type I error α, which is also called the probability of a false alarm, is given by a certain known function $\alpha = \alpha(p_1, \gamma)$. Analogously, we assume that the probability of type II error β is described by a certain known function $\beta = \beta(p_2, \gamma)$. To complete the description of monitoring we assume that monitoring actions are performed periodically after every h produced units. It has to be noted, however, that monitoring processes with varying monitoring intervals can be also considered (see, for example, Rahim and Banerjee, 1993).

In the majority of mathematical models of SPC it is assumed that inspection is perfect, i.e. reveals all false alarms with probability one. In real situations, however, it happens that the technical staff who inspects the process decides to renew it even if some results of inspection indicate that the inspection was caused by a false alarm. In this case we take into account this possibility by introducing the probability ρ of an erroneous inspection which indicates the necessity of renewal when the process is operating in an acceptable STATE I. Knowing the probability of the erroneous inspection ρ, and the probability of the false alarm α we can calculate the probability δ that no renewal action will follow a monitoring action while the process remains in STATE I.

$$\delta = 1 - \alpha\rho \tag{7}$$

As it was indicated in the previous section, in order to describe economic consequences of any SPC procedure it is necessary to evaluate the time when the process remains in STATE I. In contrast to the models where the probability of erroneous inspection is equal to zero, the duration of STATE I does not equal the time to the transition between STATE I and STATE II - denoted here by T. Denote by E the expected value of the duration of STATE I. In the case of perfect inspections $E = \tau = E(T)$. Erroneous inspections can decrease the duration of STATE I and in such a case the value of E is given von Collani (1981a) by

$$E = (1-\delta)\sum_{i=1}^{\infty} ihR(ih)\delta^{i-1} + \sum_{i=1}^{\infty}\delta^{i-1}\int_{(i-1)h}^{ih} tf(t)dt \tag{8}$$

where $R(t) = 1 - F(t)$, and $f(t) = F'(t)$.

The expected number of monitoring actions while the process remains in STATE I can be computed using the formula proposed in the paper of von Collani (1981a)

$$A_1 = \sum_{i=1}^{\infty} i\{(1-\delta)R(ih)\delta^{i-1} + \delta^i[F(i+h) - F(ih)]\} \tag{9}$$

Let us notice that due to the possibility of a renewal action while the process remains in STATE I it is also possible that during one renewal action the considered process does not enter STATE II. We have to take it into account while computing the average run length in STATE II. Following von Collani (1981a) we arrive at the expression for the average number of monitoring actions in STATE II

$$A_2 = \frac{1}{1-\beta}\sum_{i=1}^{\infty}\{F(ih) - F[(i-1)h]\}\delta^{i-1} = \frac{1}{\delta}A_2^*[1-(1-\delta)\sum_{i=0}^{\infty}\delta^i R(ih)] \quad (10)$$

where

$$A_2^* = \frac{1}{1-\beta} \quad (11)$$

is the expected number of monitoring actions in STATE II in the case of perfect inspection($\rho = 0$).

Now, we can calculate the expected profit per one item produced during a renewal cycle by inserting the derived quantities into the general formula (6).

$$G = \frac{g_1 E + g_2[(A_1 + A_2)h - E] - \alpha A_1 e^* - S_1^* A_1 - S_2^* A_2 - r^*}{(A_1 + A_2)h} \quad (12)$$

Let us introduce now some new transformed quantities

$$\omega = \alpha + S_1^*/e^* \quad (13)$$

$$S_2 = S_2^*/e^* \quad (14)$$

$$r = r^*/e^* \quad (15)$$

$$b^* = (g_1 - g_2)/e^* \quad (16)$$

$$G^* = (G - g_2)/e^* \quad (17)$$

The maximisation of the average profit per item G with respect to the length of sampling interval h and the parameters of the sampling procedure γ is equivalent to the maximisation of the standardised profit G^* given by

$$G^* = \frac{Eb^* - \omega A_1 - S_2 A_2 - r}{h(A_1 + A_2)} \quad (18)$$

where A_1 and E are functions of h, A_2 is a function of h and γ, ω and S_2 are functions of γ.

Maximisation of the objective function is very difficult due to very complex relations between the value of the standardised profit G^* and the optimised variables. Moreover, the objective function can be expressed in a closed form only in a few special cases. One of them is the case when the time to the deterioration of the process is described by the exponential distribution $F(t) = 1 - exp(-\lambda t)$. This case, however, is frequently used in practice because of its simplicity, and obvious practical interpretation (pure random mechanism of deteriorations). In the case of exponentially distributed time T the main characteristics in the considered model are given by the following formulae

$$A_1 = (e^{\lambda h} - \delta)^{-1} \tag{19}$$

$$A_2 = A_2^*(e^{\lambda h} - 1)/(e^{\lambda h} - \delta) \tag{20}$$

$$E = \lambda^{-1}(e^{\lambda h} - 1)/(e^{\lambda h} - \delta) \tag{21}$$

We will use these formulae to evaluate numerically the influence of inspection errors on the value of profit per item G.

Suppose that the production process, and the control procedures are characterised by the following parameters: $\lambda = 0.00001$, $g_1 = 1.0$, $g_2 = 0.1$, $S_1 = S_2 = 1.0$, $e^* = 20$, $r^* = 200$, $\alpha = 0.05$, and $\beta = 0.1$. Using the results from Hryniewicz (1992) we can find that in the case of perfect inspection ($\rho = 0$) the optimal sampling interval is $h = 605$. Now, let us analyse how the profit per unit changes in the presence of inspection errors. In Table 1 we present this comparison for a few values of the probability of the erroneous inspection ρ.

ρ	G
0.0	0.9914
0.1	0.9898
0.3	0.9865
0.5	0.9832
0.7	0.9799
1.0	0.9750

Table 1. Dependence of the profit per unit produced G on the probability of the erroneous inspection ρ.

In the considered example the probability of false alarm α is reasonably low, so the frequency of false alarms is also low. Thus, false inspections have not a very large impact on the value of the profit per item produced. However, the loss from neglecting this fact taken for a long production period may be substantial.

4.2. Simplified approximately optimal design

The parameters (γ^*, h^*) of the optimally designed statistical process control procedures can be found by numerical maximisation of (18). From the analysis of numerical examples presented in Hryniewicz (1989) one can find that the economic effectiveness of a control procedure depends mainly upon the value of the sampling interval h. To find the optimal value of h we have to solve the following equation

$$\begin{aligned}&\frac{\partial G^*}{\partial h} = \\ &h\{b^*(A_1+A_2)E' + (A_1'+A_2')(r-b^*E) + (\omega - S_2)(A_1A_2' - A_1'A_2)\} + \\ &-(A_1+A_2)(b^*E - \omega A_1 - S_2A_2 - r) = 0\end{aligned} \tag{22}$$

where A_1', A_2', and E' are the derivatives with respect to h of A_1,A_2, and E, respectively.

The equation (22) can be solved only numerically. To obtain its approximate solution that could be used as an initial point of an iteration process we transform it to the following form

$$\phi(h) + \omega = 0 \tag{23}$$

where

$$\phi(h) = \phi_1 - \phi_2\phi_3 - \phi_4\phi_5 \tag{24}$$

and,

$$\phi_1 = hb^*E'/(A_1+A_2) \tag{25}$$
$$\phi_2 = b^*E - r + A_2(\omega - S_2) \tag{26}$$
$$\phi_3 = (A_1+A_2)^{-1} + hA_1'/(A_1+A_2)^2 \tag{27}$$
$$\phi_4 = hA_2'/(A_1+A_2)^2 \tag{28}$$
$$\phi_5 = b^*E - r - A_1(\omega - S_2) \tag{29}$$

It is easy to notice that in the case of perfect inspection (i.e. when $\rho = 0$), the form (23) reduces to

$$\omega - \phi_2\phi_3 = 0 \tag{30}$$

which is equivalent to the equation already obtained in Hryniewicz (1992). It has been shown in Hryniewicz (1992) that this equation has the following approximate solution

$$h_0 = \tau\sqrt{\frac{2(\alpha + S_1)}{(2A_2^* - 1)[b + A_2^*(\alpha + S_1 - S_2)]}} \tag{31}$$

where

$S_1 = S_1^*/e^*$

$b = b^*\tau - r$

In practical situations the probability of the erroneous inspection ρ is close to zero and h_0 can be used as the first approximation for the optimal value of the monitoring (sampling) interval. However, in many cases this approximation may not be sufficient, so we can calculate the approximation of the second order from the following equation

$$h_1 = h_0 - \frac{\omega + \phi(h_0)}{\phi'(h_0)} \tag{32}$$

where

$$\phi'(h) = \phi_1' - (\phi_2'\phi_3 + \phi_2\phi_3') - (\phi_4'\phi_5 + \phi_4\phi_5') \tag{33}$$

and

$$\begin{aligned}
\phi_1' &= b^*\{E^*/(A_1 + A_2) + h[E''/(A_1 + A_2) + \\
&\quad -E'(A_1' + A_2')/(A_1 + A_2)^2]\} && (34)\\
\phi_2' &= b^*E' + A_2'(\omega - S_2) && (35)\\
\phi_3' &= h\{A_1''/(A_1 + A_2)^2 - 2(A_1')^2/(A_1 + A_2)^3\} && (36)\\
\phi_4' &= A_2'/(A_1 + A_2)^2 + h\{A_2''/(A_1 + A_2)^2 + \\
&\quad -2A_2'(A_1' + A_2')^2/(A_1 + A_2)^3\} && (37)\\
\phi_5' &= b^*E' - A_1'(\omega - S_2) && (38)
\end{aligned}$$

E'',A_1'', and A_2'' are the second derivatives with respect to h of E, A_1, and A_2, respectively.

The accuracy of this approximation can be evaluated numerically. In the case of exponentially distributed times to the deterioration of the process the optimal value h_{opt} can be found relatively easily thanks to the existence of closed formulae for E, A_1, and A_2. In Table 2 we present the results of such evaluation for the case when $\lambda = 0.00001$, $g_1 = 1.0$, $g_2 = 0.1$, $S_1 = S_2 = 1.0$, $e^* = 20$, $r^* = 200$, $\alpha = 0.05$, and $\beta = 0.1$.

Similar results have been obtained for different sets of input parameters for exponential and Weibull distributions. The approximation seems to be very accurate. Slightly worse results might be expected when the profit from monitoring is not much greater than the cost of renewal. Moreover, it is worth to notice that even in the case of differences between the values of h_1 and h_{opt}, the differences between the values of the objective function can be neglected.

To simplify the optimisation procedure for the set of parameters γ is a much more difficult task. First, notice that for a reasonably designed procedure the duration of STATE II should be much smaller than the duration of STATE I. Thus, it is reasonable to assume that $A_1 >> A_2$. Second, we can also assume that the profit $P = E(g_1 - g_2)$ is much greater than the cost of renewal r^*. Moreover, for a well designed procedure we can expect that $h < \tau$. Thus,

ρ	h_1	G(h_1)	h_{opt}	G(h_{opt})
0.01	622	0.99123	620	0.99123
0.05	691	0.99062	677	0.99062
0.1	764	0.98992	742	0.98992
0.3	963	0.98759	958	0.98759
0.5	1131	0.98570	1135	0.98570
0.7	1296	0.98407	1288	0.98407
1.0	1540	0.98194	1489	0.98194

Table 2. Dependence of the profit per unit produced G upon the length of the sampling interval (approximately optimal h_1, and optimal h_{opt}).

taking into account that $A_1 h < E < A_2 h$ we can assume that $E \approx A_1 h$. We can now notice that

$$G^* = \frac{Eb^* - \omega A_1 - S_2 A_2 - r}{h(A_1 + A_2)} = \frac{Eb^* - r}{h(A_1 + A_2)} - \frac{\omega A_1 - S_2 A_2}{h(A_1 + A_2)} \approx b^* - \frac{\omega}{h} \tag{39}$$

Hence, for a fixed value of h the maximisation of G^* is approximately equivalent to the minimisation of a much simpler objective function

$$G^{**} = \frac{\alpha + S_1^*/e^*}{h} \tag{40}$$

In the case of perfect inspections we can insert (31) into (40) and the minimised objective function is now expressed as

$$G_0^{**} = (\alpha + S_1^*/e^*)(2A_2^* - 1) \tag{41}$$

This result was already obtained in Hryniewicz (1992). Now we can propose the following algorithm for optimisation of γ.

1° Find the first approximate solution γ_0^* by the minimisation of (41).

2° From (31) find the approximation for the optimal value of h.

3° Insert the approximate value of the optimal h into (40), and find the approximation for the optimal value of γ.

5° From (32) find the approximate sampling interval h.

6° Repeat 3° to 5°

7° Stop the algorithm when there is no significant improvement in the value of the objective function.

Numerical experiments have revealed that the proposed algorithm converges in only few steps to the result which is usually very close to the optimal one. The accuracy of this approximation depends upon the degree to which the assumptions used for the derivation of (40) are fulfilled.

5. Further useful approximations

The computation of the optimal parameters of the monitoring procedures requires, in a general case, multiple computations of infinite sums, such as in (8), (9), and (10). The expressions for E, A_1, A_2, and their derivatives are very seldom in a closed form. Numerical experiments have revealed, however, that in many practical cases these infinite sums converge rather slowly. Thus, there is a need to take into account a large number of their terms. It usually results in long lasting computations even on fast microcomputers. Therefore, it is of interest to find some relatively accurate approximations which can be expressed in a closed form.

To find such approximations we assume that the monitoring interval is small enough that the following equality holds approximately

$$R(x+h) \simeq R(x) - f(x)h + 0.5f'(x)h^2 \tag{42}$$

We use this approximation to present A_1 given by (9) in the following form

$$\begin{aligned} A_1 &= \frac{1}{\delta}\sum_{i=1}^{\infty} i\delta^i[R(ih) - \delta R((i+1)h)] \\ &\simeq \frac{1-\delta}{\delta}h^{-2}\sum_{i=1}^{\infty}(ih)e^{[\ln\delta/h]ih}R(ih)h \\ &\quad + h^{-1}\sum_{i=1}^{\infty}(ih)e^{[\ln\delta/h]ih}f(ih)h+ \\ &\quad + 0.5\sum_{i=1}^{\infty}(ih)e^{[\ln\delta/h]ih}f'(ih)h \end{aligned} \tag{43}$$

For small values of h we can approximate infinite sums in (43) by corresponding intervals, arriving at the following expression

$$A_1 \approx \frac{1-\delta}{\delta}h^{-2}\int_0^{\infty} xe^{ax}R(x)dx + h^{-1}\int_0^{\infty} xe^{ax}f(x)dx + 0.5\int_0^{\infty} xe^{ax}f'(x)dx \tag{44}$$

where

$$a = h^{-1}\ln\delta \tag{45}$$

Let

$$M(x) = \int_0^{\infty} e^{xt}f(t)dt \tag{46}$$

and

$$M'(x) = \frac{d}{dz}M(z)\bigg|_{z=x} \tag{47}$$

δ	$\epsilon(A_1)$	$\epsilon(A_2)$	$\epsilon(E)$
0.9	0.27	0.18	0.09
0.99	<0.01	0.01	<0.01
0.999	<0.01	<0.01	<0.01

Table 3. Relative errors for the approximations of E, A_1, and A_2.

After some transformations we find the following approximation

$$\tilde{A}_1 \approx \frac{1-\delta}{\delta} h^{-2}\{a^{-2}[1-M(a)] + a^{-1}M'(a)\} + h^{-1}M'(a) - 0.5[M(a) + aM'(a)] \quad (48)$$

Using similar derivations we obtain the approximations for A_2, and E.

$$\tilde{A}_2 \approx A_2^* M(a) \left\{1 - \frac{\ln\delta}{2}(1 - \frac{\ln\delta}{3})\right\} \quad (49)$$

$$\tilde{E} \approx \frac{1}{\ln\delta}\frac{1-\delta}{\delta}\{a^{-1}[1-M(a)] + M'(a)\} + M'(a) - \frac{\ln\delta}{2}M'(a) \quad (50)$$

The accuracy of these approximations has been investigated numerically for several probability distributions $F(t)$. Let us adopt the following notation: X - the quantity to be approximated, $\tilde{X}$ - its approximation. The relative error of an approximation is evaluated from the following formula

$$\epsilon(X) = \frac{|X - \tilde{X}|}{X} 100\% \quad (51)$$

In Table 3 we present the relative errors of approximations for the exponential distribution such that $\lambda h = 0.1$, and for different values of δ.

For shorter monitoring intervals (such that $\lambda h < 0.1$) the accuracy of these approximations is significantly better. Even for hardly possible sets of parameters (e.g. $\delta = 0.5$) the relative errors of these approximations have not exceeded 5%. The approximations for the derivatives of E, A_1, and A_2 can be obtained using the same approach. However, we can find very good approximations just by differentiating $\tilde{E}$, $\tilde{A}_1$, and $\tilde{A}_2$. Such approximations have been also found to be very accurate.

If we use $\tilde{E}$, $\tilde{A}_1$, $\tilde{A}_2$, and their derivatives instead of the exact values of these functions the results of optimisation do not change very much. This is especially true when we compare the values of the objective functions for exact and approximate solutions. We have to notice, however, that our approximation cannot be used in certain circumstances. First of all, it cannot be used in the case of the very popular Weibull distribution, as in this case the integral (46) does not exist. Moreover, when $F(t)$ is characterised by a very small variance the approximation of infinite sums by corresponding integrals may not be accurate.

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