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Some remarks on the market model of interest rates[1]

by

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Abstract: The paper deals with pricing of interest rate derivative securities. It can be considered as a proposal of unification of the general Heath, Jarrow and Morton framework with the practice of Black formula and Delta hedging. We give analytical formulas for European interest rate derivatives and numerical methods for American and exotic products.

1. Introduction

In this note we give short account of the Market Model by Miltersen et al. (1995) and Brace et al. (1995). The aim of this paper is to present some ideas behind the Market Model, rather then giving exact pricing formulae as in Brace et al. (1995) and Brace (1996).

Let $B(t,T)$ be the price at time t of zero-coupon bond maturing at time T. Define the family of forward rates $r : \mathbf{R}_+^2 \to \mathbf{R}$ by

$$r(t,x) = -\frac{\partial}{\partial x} \log B(t, t+x)$$

provided $B(t,T)$ are differentiable with respect to T. If the prices of zero-coupon bonds are deterministic then obviously

$$B(t,T) = \exp\left\{ -\int_t^T r(s,0)ds \right\}. \tag{1}$$

The function $r(t,0)$ is here called short rate of the economy. But in reality prices of bonds are **not deterministic!** In consequence the exist interest rate options, caps and floors among them.

Let $\delta > 0$. Let the forward LIBOR rate be defined by

$$\delta L(t,x) = \frac{B(t,t+x)}{B(t,t+x+\delta)} - 1.$$

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In a forward cap (res. floor) on principal 1 settled in arrears at times $T_j = j\delta$, $j = 1, \ldots, n$ the cash flows at times T_j are $(L(T_{j-1}, 0) - \kappa)^+\delta$ (res. $(\kappa - L(T_{j-1}, 0))^+\delta$). In most cases caps are priced at time $t \le T_0$ by the following Black and Scholes formula:

$$\begin{aligned} Cap(t) &= \sum_{j=1}^{n} \delta B(t, T_j) \\ &\quad (L(t, T_{j-1} - t) N(h(t, T_{j-1})) - \kappa N(h(t, T_{j-1}) - \zeta(t, T_{j-1}))), \end{aligned}$$

where

$$\begin{aligned} h(t,T) &= \frac{\log \frac{L(t,T-t)}{\kappa} + \frac{1}{2}\zeta^2(t,T)}{\zeta(t,T)}, \\ \zeta^2(t,T) &= (T-t)\gamma^2 \end{aligned}$$

and $\gamma > 0$ is the volatility parameter. Recalling the derivation of the Black and Scholes formula we notice that the interest rate is taken here as both risky and riskless asset. Hence this methodology seems to be based on two logically inconsistent assumptions:

1. The short rate $r(s, 0)$ is deterministic on the interval $[t, T_n]$ (as riskless asset).
2. The short rate $r(s, 0)$ is random on the interval $[T_0, T_n]$ with positive volatility parameter γ (as risky asset).

The aim of this note is to show that the market methodology can be made mathematically rigorous.

2. Term structure models

We say that a probability measure P on the given probability space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is arbitrage-free if

$$B(t,T) \exp\left\{-\int_0^t r(s,0)ds\right\}$$

are $\mathcal{F}_t$-martingales for $0 \le t \le T$ under the measure P. This is equivalent to the following equality

$$B(t,T) = E\left(\exp\left\{-\int_t^T r(s,0)ds\right\} \middle| \mathcal{F}_t\right). \tag{2}$$

Assume that

$$\mathcal{F}_t = \sigma(W(s), s \le t),$$

where W is a d-dimensional Wiener process. This means that all the randomness of the model is generated by d factors and is not very restrictive.

If there exists a stochastic process $\sigma(t, x)$, differentiable with respect to x, such that r satisfies

$$dr(t,x) = \frac{\partial}{\partial x}\left\{\left(r(t,x) + \frac{1}{2}|\sigma(t,x)|^2\right) dt + \sigma(t,x)\cdot dW(t)\right\} \tag{3}$$

then the model is arbitrage-free (Heath, Jarrow and Morton, 1992, Brace and Musiela, 1994). In this approach construction of the model is reduced to the specification of volatility parameter σ.

First idea: $r(t) = r(s, 0)$ is itself a Markov process.

This was the main assumption in various papers, e.g. Vasiček (1977), Richard (1978), Dothan (1978), Brennan and Schwartz (1979), Langetieg (1980), Cox, Ingersoll and Ross (1985), Longstaff (1989), Artzner and Delbaen (1989), Hull and White (1990), Black, Derman and Toy (1990), Black and Karasiński (1991), Sandmann and Sondermann (1991). Let us notice the essential property of this class of models which makes their practical use questionable.

By the Feynman-Kac formula, (2) becomes

$$B(t,T) = E\left(\exp\left\{-\int_t^T r(s)ds\right\}\bigg| r(t)\right) = f(t,T,r(t))$$

for certain deterministic function $f : \mathbf{R}_+^2 \times \mathbf{R} \to \mathbf{R}_+$. Hence all the bond prices are determined by the current position of one random parameter - this is a strongly unrealistic property.

Second idea: σ deterministic: as Gaussian process admits negative rates and not consistent with previously described market practice of using Black formula. This model is usually called Heath, Jarrow and Morton model.

Third idea: σ'_x linear,

$$\sigma'_x(t,x) = \gamma(t,x)r(t,x).$$

The following theorem contradicts also the hypothesis above:

Theorem 1 *(Morton, 1988) Let $d = 1$ and $\gamma(t,x) = \gamma > 0$. Then the rate r explodes!*

This is also an unrealistic property. There exist different functional volatility specifications

$$\sigma'_x(t,x) = \gamma(t,x,r(t,x))$$

such that the rate r does not explode (Heath, Jarrow and Morton, 1992, Goldys, Musiela and Sondermann, 1994, Carverhill, 1995). The idea of Miltersen et al, 1995 and Brace et al, 1995 is based on different concept: forward LIBOR rate rather than short rate.

3. Forward LIBOR specification

The equation (3) gives non-arbitrage conditions in terms of continuously compounded forward rates r. Let us derive analogous equation for forward LIBOR rates L. Let

$$z(t,x) = \int_x^{x+\delta} r(t,s)ds.$$

Notice that

$$\delta L(t,x) = e^{z(t,x)} - 1.$$

By the Ito formula z and L satisfy

$$\begin{aligned} dz(t,x) &= \left(\frac{\partial}{\partial x} z(t,x) + \frac{1}{2}|\sigma(t,x+\delta)|^2 - \frac{1}{2}|\sigma(t,x)|^2\right) dt \\ &\quad + (\sigma(t,x+\delta) - \sigma(t,x)) \cdot dW(t) \end{aligned}$$

and

$$\begin{aligned} dL(t,x) &= \delta^{-1} e^{z(t,x)} \left(dz(t,x) + \frac{1}{2}|\sigma(t,x+\delta) - \sigma(t,x)|^2 dt \right) \\ &= \left(\frac{\partial}{\partial x} L(t,x) + \left(L(t,x) + \delta^{-1}\right) \sigma(t,x+\delta) \cdot (\sigma(t,x+\delta) - \sigma(t,x)) \right) dt \\ &\quad + \left(L(t,x) + \delta^{-1}\right)(\sigma(t,x+\delta) - \sigma(t,x)) \cdot dW(t). \end{aligned}$$

We make the hypothesis of lognormality of rates, namely

$$\left(L(t,x) + \delta^{-1}\right)(\sigma(t,x+\delta) - \sigma(t,x)) = L(t,x)\gamma(t,x). \tag{4}$$

As we show in the next section, this will give consistence of the model with the market practice. Hence

$$\begin{aligned} dL(t,x) &= \left(\frac{\partial}{\partial x} L(t,x) + L(t,x)\sigma(t,x+\delta) \cdot \gamma(t,x) \right) dt \\ &\quad + L(t,x)\gamma(t,x) \cdot dW(t). \end{aligned}$$

By (4)

$$\sigma(t,x+\delta) = \frac{\delta L(t,x)}{1 + \delta L(t,x)} \gamma(t,x) + \sigma(t,x). \tag{5}$$

This is a recursive formula depending on some initial condition. Putting $\sigma(t,x) = 0$ for $0 \leq x \leq \delta$ we derive the equation:

$$dL(t,x) = \left(\frac{\partial}{\partial x} L(t,x) + L(t,x)\Xi(t,x) \cdot \gamma(t,x) \right) dt + L(t,x)\gamma(t,x) \cdot dW(t), \tag{6}$$

where

$$\Xi(t,x) = \sigma(t,x+\delta) = \sum_{k=0}^{[\delta^{-1}x]} \frac{\delta L(t,x-k\delta)}{1+\delta L(t,x-k\delta)}\gamma(t,x-k\delta).$$

Let

$$M(t,x) = \int_0^t \gamma(s,x+t-s)\cdot dW(s).$$

Lemma 1 *(Brace et al. 1995) Assume that γ and M are continuous. Let y be a solution of the following equation:*

$$dy(t,x) = y(t,x)\gamma(t,x-t)\cdot\left(\frac{\delta y(t,x)}{1+\delta y(t,x)}\gamma(t,x-t)+\xi(t,x)\right)dt$$

$$+y(t,x)\gamma(t,x-t)\cdot dW(t), \tag{7}$$

$$y(0,x) = y_0(x) \geq 0,$$

where $\xi(t,x)\in \mathbf{R}^d$ is an adaptive and bounded stochastic process for any $x\geq 0$. Then there exists a unique nonnegative solution to the equation (7). If moreover $y_0\in C(\mathbf{R}_+)$ and $\xi\in C(\Delta)$ then $y\in C(\Delta)$, where

$$\Delta = \{x,t\geq 0 : t\leq x\}.$$

In addition, if y_0,γ,M and ξ are differentiable with respect to x then y is differentiable with respect to x.

Theorem 2 *Assume that $M(t,\cdot)$ and γ are differentiable functions satisfying $\gamma(t,0) = \gamma'_x(t,0) = 0$. Then the equation (6) admits a unique nonnegative differentiable with respect to x strong solution $L\in C(\mathbf{R}_+\times\mathbf{R})$ for any nonegative differentiable initial condition $L(\cdot,0)\in C(\mathbf{R})$.*

Corollary 1 *The process L is a Markov process on the state space $C^1(\mathbf{R}_+)$.*

Proof of the theorem. Let $0<x\leq\delta$. Let $y(t,x) = L(t,x-t)$. For any $0\leq t\leq x$, $y(t,x)$ satisfies

$$dy(t,x) = \left(\frac{\delta y^2(t,x)}{1+\delta y(t,x)}|\gamma(t,x-t)|^2\right)dt + y(t,x)\gamma(t,x-t)\cdot dW(t).$$

By Lemma 1 (putting $\xi=0$) this equation admits a unique nonnegative solution for any $0\leq t\leq x$. For any $x\geq\delta$ and $0\leq t\leq x$

$$dy(t,x) = y(t,x)\gamma(t,x-t)\cdot\left(\frac{\delta y(t,x)}{1+\delta y(t,x)}\gamma(t,x-t)+\xi(t,x)\right)dt+$$

$$+y(t,x)\gamma(t,x-t)\cdot dW(t), \tag{8}$$

where

$$\xi(t,x)=\sigma(t,x-t)=\sum_{k=1}^{[\delta^{-1}(x-t)]}\frac{\delta y(t,x-k\delta)}{1+\delta y(t,x-k\delta)}\gamma(t,x-k\delta-t).$$

By Lemma 1, the equation (8) admits a unique nonnegative solution for any $\delta\le x\le 2\delta$ and $0\le t\le x$. By induction the equation (8) admits a unique solution for any $x>0$ and $0\le t\le x$. Since $\gamma(t,0)=0$, $\xi(t,x)$ is continuous for $x\le k\delta$, provided y is continuous for $x\le(k-1)\delta$. By Lemma 1, y is continuous for $0\le t\le x\le\delta$. Therefore ξ is continuous for $0\le t\le x\le 2\delta$. Again by Lemma 1, y is continuous for $0\le t\le x\le 2\delta$. By induction $\xi,y\in C(\Delta)$. Returning to L we notice that $L\in C(\mathbf{R}^2_+)$ and $L\ge 0$.

Since $\gamma'_x(t,0)=0$, ξ is differentiable for $x\le k\delta$, provided y is differentiable for $x\le(k-1)\delta$. Again by Lemma 1 and induction y is differentiable. Hence L is differentiable and therefore the solution of (6) is strong.

■

Since $\sigma(t,x)=\Xi(t,x-\delta)$, $\sigma(t,\cdot)$ is differentiable. Therefore the equations (3) and (6) are consistent and L defines arbitrage-free dynamics in the sense of Heath, Jarrow and Morton.

Since $L\ge 0$,

$$\begin{aligned} L(t,x-t) &= L(0,x)\exp\left(M(t,x-t)\right)\\ &\quad\times\exp\left(\int_0^t\left(\Xi(s,x-s)-\frac{1}{2}\gamma(s,x-s)\right)\cdot\gamma(s,x-s)ds\right). \end{aligned}$$

Hence $L_1\le L\le L_2$, where

$$\begin{aligned} L_1(t,x) &= L(0,x+t)\exp\left(M(t,x)\right)\\ &\quad\times\exp\left(-\int_0^t\left(\Xi_2(t,x+t-s)+\frac{1}{2}|\gamma(s,x+t-s)|^2\right)ds\right),\\ L_2(t,x) &= L(0,x+t)\exp\left(M(t,x)\right)\\ &\quad\times\exp\left(\int_0^t\left(\Xi_2(t,x+t-s)-\frac{1}{2}|\gamma(s,x+t-s)|^2\right)ds\right) \end{aligned}$$

and

$$\Xi_2(t,x)=\left|\gamma(t,x)\cdot\sum_{k=0}^{[\delta^{-1}x]}\gamma(t,x-k\delta)\right|.$$

Therefore $EL(t,x)\le EL_2(t,x)<\infty$ and the Eurodollar future price is well defined.

4. Cap pricing

The cap price at time $t \leq T_0$ is equal

$$Cap(t) = \sum_{j=1}^{n} E\left(\frac{\beta(t)}{\beta(T_j)} (L(T_{j-1},0) - \kappa)^+ \delta \Big| \mathcal{F}_t \right)$$

$$= \sum_{j=1}^{n} B(t,T_j) E_{T_j} \left((L(T_{j-1},0) - \kappa)^+ \delta \Big| \mathcal{F}_t \right), \tag{9}$$

where

$$\beta(t) = \exp\left\{ \int_0^t r(s,0)ds \right\}$$

and E_T stands for the expectation under the forward measure P_T defined by

$$P_T = B(0,T)^{-1}\beta(T)^{-1}P.$$

The process

$$K(t,T) = L(t,T-t)$$

satisfies for $0 \leq t \leq T$

$$\begin{aligned} dK(t,T) &= K(t,T)\gamma(t,T-t) \\ &\quad \cdot \left(\left(\frac{\delta K(t,T)}{1+\delta K(t,T)} \gamma(t,T-t) + \sigma(t,T-t) \right) dt + dW(t) \right) \\ &= K(t,T)\gamma(t,T-t) \cdot \left(\sigma(t,T+\delta-t)dt + dW(t) \right). \end{aligned}$$

Moreover, the process

$$W_T(t) = W(t) + \int_0^t \sigma(s,T-s)ds$$

is a Brownian motion under P_T. Consequently

$$dK(t,T) = K(t,T)\gamma(t,T-t) \cdot dW_{T+\delta}(t)$$

and hence $K(t,T)$ is lognormally distributed under $P_{T+\delta}$. Let X satisfy linearized version of (6)

$$dX(t,x) = \frac{\partial}{\partial x} X(t,x)dt + X(t,x)\gamma(t,x) \cdot dW(t) \tag{10}$$

admitting closed-form solution

$$X(t,x) = X(s,x+t-s)$$

$$\times \exp\left\{-\frac{1}{2}\int_s^t |\gamma(u,t-u+x)|^2 du + \int_s^t \gamma(u,t-u+x)\cdot dW(u)\right\}.$$

By (9)

$$Cap(t) = \sum_{j=1}^{n} B(t,T_j)E\Big((X(T_{j-1},0)-\kappa)^+\delta \big| \, X(t) = L(t)\Big).$$

It follows that

$$\begin{aligned} E_{T+\delta}\Big((L(T,0)-\kappa)^+\Big|\mathcal{F}_t\Big) &= E\Big((X(T,0)-\kappa)^+\Big|X(t)=L(t)\Big) \\ &= E_{T+\delta}\Big((K(T,T)-\kappa)^+\Big|\mathcal{F}_t\Big) \\ &= K(t,T)N(h(t,T)) - \kappa N(h(t,T)-\zeta(t,T)), \end{aligned}$$

where

$$\begin{aligned} h(t,T) &= \frac{\log\frac{K(t,T)}{\kappa} + \frac{1}{2}\zeta^2(t,T)}{\zeta(t,T)}, \\ \zeta^2(t,T) &= \int_t^T |\gamma(s,T-s)|^2\, ds \end{aligned}$$

and hence we have the following result.

Proposition 1 *The cap price at time $t \le T_0$ is*

$$Cap(t) = \sum_{j=1}^{n} \delta B(t,T_j)\Big(K(t,T_{j-1})N(h(t,T_{j-1})) - \kappa N(h(t,T_{j-1}) - \zeta(t,T_{j-1}))\Big).$$

Remark 1 *The above $Cap(t)$ formula corresponds to the market Black futures formula with discount from the settlement date.*

Corollary 2 *There exists a term structure model for which the market practice is arbitrage-free.*

5. Discrete approximation

Define the skeleton

$$\gamma_n(t) = \gamma(t, n\delta - t),$$

$$Y_n(t) = L(t, n\delta - t).$$

By the Ito formula Y_n satisfy the following closed system of stochastic equations:

$$dY_n(t) = Y_n(t)\left(\sum_{j>\frac{t}{\delta}}^{n} \frac{\delta Y_j(t)}{1+\delta Y_j(t)}\gamma_j(t)\cdot\gamma_n(t)\right)dt + Y_n(t)\gamma_n(t)\cdot dW(t) \quad (11)$$

for $0 < t < n\delta$. This property is useful in constructing finite-dimensional approximations. For simplicity we take the approximation step equal δ and assume that $d = 1$. Let us start by noticing the following property of the process Y, resulting from (11) and the Ito formula:

$$\begin{aligned} Y_n((k+1)\delta) &= Y_n(k\delta)\exp\left\{\int_{k\delta}^{(k+1)\delta}\gamma_n(s)dW(s)\right\}\times \\ &\exp\left\{\int_{k\delta}^{(k+1)\delta}\gamma_n(s)\left(\sum_{j>\frac{s}{\delta}}^{n}\frac{\delta Y_j(s)\gamma_j(s)}{1+\delta Y_j(s)}-\frac{\gamma_n(s)}{2}\right)ds\right\} \end{aligned} \tag{12}$$

for any $k < n$. We approximate:

1. Functions $\gamma_n(s)$ by $\gamma_n(k\delta)$.
2. Processes $Y_n(s)$ by $Y_n(k\delta)$ (on the right-hand side).
3. The Wiener process W by sums of i.i.d. random variables.

Therefore we derive the following approximation formula:

$$\begin{aligned} \tilde{y}_n((k+1)\delta) &= \tilde{y}_n(k\delta)\exp\left\{\gamma_n(k\delta)\left(\xi_k\sqrt{\delta}-\frac{\delta\gamma_n(k\delta)}{2}\right)\right\}\times \\ &\exp\left\{\gamma_n(k\delta)\left(\sum_{j=k+1}^{n}\frac{\delta^2\tilde{y}_j(k\delta)\gamma_j(k\delta)}{1+\delta\tilde{y}_j(k\delta)}\right)\right\}, \end{aligned} \tag{13}$$

where $(\xi_k)_{k=0,1,2,\ldots}$ are i.i.d. random variables such that

$$P(\xi_k = 1) = P(\xi_k = -1) = \frac{1}{2}.$$

We preserve the following properties:

$$\sqrt{\delta}E\xi_k = E\int_{k\delta}^{(k+1)\delta} dW(s) = 0$$

and

$$\delta E\xi_k^2 = E\left(\int_{k\delta}^{(k+1)\delta} dW(s)\right)^2 = \delta.$$

The process $\tilde{y}$ is a discrete-time Markov process on $\mathbf{R}_+^N$. Our approximate European option pricing problem consists of finding

$$V(x,0) = E^{0,x}\left\{F(\tilde{y}(n\delta),n\delta)\prod_{j=0}^{n-1}(1+\delta y_j(j\delta))^{-1}\right\},$$

where F is intrinsic value of the interest rate or swap option (for example swaption). Analogously, the value of American option is given by

$$\tilde{V}(x,0) = \sup_{0\leq n\leq N} E^{0,x}\left\{F(\tilde{y}(n\delta),n\delta)\prod_{j=0}^{n-1}(1+\delta y_j(j\delta))^{-1}\right\},$$

where n is a (random) stopping time with values in $\mathbf{N}$. We set $\prod_{j=0}^{-1} a_j = 1$.

Let $x \in \mathbf{R}_+^N$. For any $0 \leq k < N$ and $0 \leq n \leq N$, we define the following two nonlinear transformations $\mathbf{R}_+^N \to \mathbf{R}_+^N$ by

$$T_{\text{up}}^k(x)_n = x_n \exp\left\{\gamma_n(k\delta)\left(\sum_{j=k+1}^{n}\frac{\delta^2 x_j\gamma_j(k\delta)}{1+\delta x_j}+\sqrt{\delta}-\frac{\delta\gamma_n(k\delta)}{2}\right)\right\} \tag{14}$$

and

$$T_{\text{down}}^k(x)_n = x_n \exp\left\{\gamma_n(k\delta)\left(\sum_{j=k+1}^{n}\frac{\delta^2 x_j\gamma_j(k\delta)}{1+\delta x_j}-\sqrt{\delta}-\frac{\delta\gamma_n(k\delta)}{2}\right)\right\}. \tag{15}$$

We can also define set operator $S^k : 2^{\mathbf{R}_+^N} \to 2^{\mathbf{R}_+^N}$ by

$$\begin{aligned} S^k(A) \quad = \quad &\Big\{x \in \mathbf{R}_+^N : \text{ there exists } y \in A \text{ such that} \\ &x = T_{\text{up}}(y) \text{ or } x = T_{\text{down}}(y)\Big\}. \end{aligned}$$

Starting from any initial term structure $x \in \mathbf{R}_+^N$ we can build a binary tree up to level N. Namely: Let $x \in \mathbf{R}_+^N$. Put $E(0) = \{x\}$ and recursively $E(n+1) = S^n(E(n))$.

Let $C \in C\left(\mathbf{R}_+^N \times \{\delta, 2\delta, \ldots, N\delta\}; \mathbf{R}\right)$. We define the following linear operator

$$T : C\left(\mathbf{R}_+^N \times \{\delta, 2\delta, \ldots, N\delta\}; \mathbf{R}\right) \to C\left(\mathbf{R}_+^N \times \{0, \delta, 2\delta, \ldots, (N-1)\delta\}; \mathbf{R}\right)$$

by

$$TC(x,k\delta) = \frac{1}{2}C\left(T_{\text{up}}^k(x),(k+1)\delta\right) + \frac{1}{2}C\left(T_{\text{down}}^k(x),(k+1)\delta\right).$$

Notice that:

1. $TC(\cdot,k\delta)$ depends only on $C(\cdot,(k+1)\delta)$.
2. $TC(\cdot,k\delta)$ is determined on $E(k)$ if $C(\cdot,(k+1)\delta)$ is determined on $E(k+1)$.
3. T is the transition operator for the discrete-time Markov process $\tilde{y}$, namely
$$E^{k\delta,x}C(\tilde{y}((k+1)\delta),(k+1)\delta) = TC(x,k\delta).$$

By standard backward induction procedure for discrete time optimal stopping problems we can give recursive computation formula for approximate value functions:

$$\begin{aligned} V(y, N\delta) &= F(y, N\delta) \text{ for } y \in E(N), \\ V(y, n\delta) &= \frac{TV(y, n\delta)}{1+\delta y_n} \text{ for } 0 \le n < N \text{ and } y \in E(n) \end{aligned}$$

for European options and

$$\begin{aligned} \tilde{V}(y, N\delta) &= F(y, N\delta) \text{ for } y \in E(N), \\ \tilde{V}(y, n\delta) &= \max\left\{ F(y, n\delta), \frac{T\tilde{V}(y, n\delta)}{1+\delta y_n} \right\} \text{ for } 0 \le n < N \text{ and } y \in E(n) \end{aligned}$$

for American options. The value $V(x, 0)$ is the approximate price of European option and value $\tilde{V}(x, 0)$ is the approximate price of American option, given initial term structure $x \in \mathbf{R}_+^N$.

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