

Smart materials and flexible structures¹

by

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Abstract: Starting from constitutive relationships of piezo-electric materials, we construct a coupled model between a flexible structure and its actuators in order to be able to optimize the control procedure. A basic difficulty arises because of the singular effect of the piezo-devices. Hence the classical framework for the linear wave equation must be adapted. Then a necessary condition for the controllability of the vibration is developed. Because it is almost impossible that this condition be fulfilled, we do not discuss whether it is or is not sufficient (it is not). Then, several regulation laws based on optimal control strategy are presented. Finally, optimum design methods are used in order to define a strategy for improving the regulation law.

Keywords: control, modelling, piezoelectric structures

1. Introduction

The piezo-electric phenomenon² is now quite old and well understood. The first contributions are certainly those by the Curie brothers. But a real breakthrough was done by Langerien few years later. The main application was for quartz. More recently, Kawai (1969), has suggested to consider the reversed phenomenon (i.e. to produce a strain by applying a voltage to a piezo-electric material). This new idea opens a new area for applications: it is the possibility to prescribe the structure to rest. The use of PVDF (polyvinyl-difluor) is the cheapest way but ceramics are certainly more efficient (higher piezo-electric constants). Nevertheless, the mathematical analysis is the same and therefore we do not discuss further this material aspect.

Our goal in this paper is to answer to the following questions:

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²Piezoelectricity is a physical property of a material which can be summarised as follows: "a deformation induces a difference of potential and conversely".

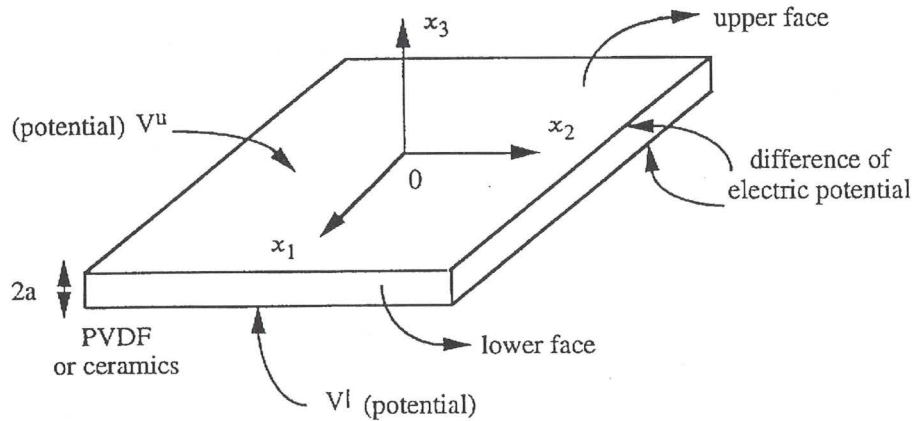


Figure 1. The geometry of the piezo-device

1. How to construct coupled model for beams, plates and shells equipped with piezo-electric wafers?
2. Does there exist a solution to the coupled model?
3. Can one control perfectly (exact control in finite time) the vibrations of a structure using the piezo-electric wafers?
4. Which kind of stabilization law can be used for the voltage on the piezo-electric devices?
5. How to optimize the shape of the wafer in order to increase the efficiency of the system?

2. A coupled model for structures equipped with piezo-devices

2.1. The constitutive relationships

Let us first consider a thin layer made of a piezo-electric material like PVDF or ceramics. The coordinate system we are using is such that the transverse coordinate is denoted by x_3 , while x_1, x_2 are varying in the plane of the layer (see Fig. 1).

The displacement of the particles is transversal to the piezo-electric sheet. Hence there is only one component parallel to the direction x_3 and it is denoted by D . Because the thickness is very small compared to the length or/and the width, we can assume that there is only inplane strain (Kirchhoff-Love assumptions). They are represented by the strain tensor $\gamma_{\alpha\beta}(u)$ or briefly by $\gamma_{\alpha\beta}$, the definition of which is:

$$\gamma_{\alpha\beta} = \frac{1}{2}(\partial_{\alpha}u_{\beta} + \partial_{\beta}u_{\alpha}) \quad (\alpha, \beta \in \{1, 2\})$$

u_α being the components of the inplane displacement and ∂_α standing for the partial derivatives with respect to the coordinates x_α . The inplane stress field is $\sigma_{\alpha\beta}$ and the electric field through the thickness is E . It is clearly parallel to the direction x_3 and also D (the electric displacement).

Then a first realistic assumption is to set a linear relationship between the observable variables $\gamma_{\alpha\beta}$ and D on the one hand and the corresponding thermodynamical forces $\sigma_{\alpha\beta}$ and E on the other hand. Hence we write (recall that Greek indices take values from $\{1, 2\}$, and when they are repeated, summation over them is implied):

$$\begin{cases} \sigma_{\alpha\beta} = R_{\alpha\beta\mu\lambda}^P \gamma_{\mu\lambda} + h_{\alpha\beta} D, \\ E = k_{\alpha\beta} \gamma_{\alpha\beta} + c D, \end{cases} \quad (1)$$

where $R_{\alpha\beta\mu\lambda}^P$ is the plane-stress stiffness tensor of the piezo-device, $h_{\alpha\beta}$ and $k_{\alpha\beta}$ are piezo-electric constants and c is a capacitor coefficient.

The components of the stiffness tensor satisfy the usual symmetry and positivity properties:

$$\begin{cases} R_{\alpha\beta\mu\lambda}^P = R_{\mu\lambda\alpha\beta}^P = R_{\beta\alpha\mu\lambda}^P \\ \forall \tau_{\alpha\beta} = \tau_{\beta\alpha}; R_{\alpha\beta\mu\lambda}^P \tau_{\alpha\beta} \tau_{\mu\lambda} \geq c_0 \tau_{\alpha\beta} \tau_{\alpha\beta} \end{cases} \quad (2)$$

where c_0 is a strictly positive constant which is independent of $\tau_{\alpha\beta}$. If the piezo-electric material is isotropic, with a Young modulus E and a Poisson coefficient ν , one has:

$$R_{\alpha\beta\mu\lambda}^P = \frac{Ea^3}{3(1-\nu^2)} \{ (1-\nu)(\delta_{\alpha\mu}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\mu}) + 2\nu\delta_{\alpha\lambda}\delta_{\mu\lambda} \}$$

By applying the first law of thermodynamics one can write (e denoting the internal energy and ρ the mass density).

$$\rho \dot{e} = \sigma_{\alpha\beta} \dot{\gamma}_{\alpha\beta} + E \dot{D} \quad (3)$$

where the upper dot stands for the time derivative.

Heat exchange is neglected in such a modelling. Then, because e is a state function (i.e. it only depends on the state variables $\gamma_{\alpha\beta}$ and D), one deduces from (3) that (analogously to Clapeyron formula):

$$\sigma_{\alpha\beta} = \rho \frac{\partial e}{\partial \gamma_{\alpha\beta}}, \quad E = \rho \frac{\partial e}{\partial D} \quad (4)$$

and therefore:

$$\rho e = \frac{1}{2} R_{\alpha\beta\mu\lambda}^P \gamma_{\alpha\beta} \gamma_{\mu\lambda} + h_{\alpha\beta} \gamma_{\alpha\beta} D + z(D)$$

z being a function of D . Then from:

$$E = \rho \frac{\partial e}{\partial D},$$

$$\begin{cases} \rho e = \frac{1}{2} R_{\alpha\beta\mu\lambda}^P \gamma_{\alpha\beta} \gamma_{\mu\lambda} + h_{\alpha\beta} \gamma_{\alpha\beta} D + \frac{c}{2} D^2 + cte, \\ \sigma_{\alpha\beta} = \rho \frac{\partial e}{\partial \gamma_{\alpha\beta}} = R_{\alpha\beta\mu\lambda}^P \gamma_{\mu\lambda} + h_{\alpha\beta} D, \\ E = \rho \frac{\partial e}{\partial D} = h_{\alpha\beta} \gamma_{\alpha\beta} + cD. \end{cases}$$

Table 1. The constitutive relationships of a piezo-electric material

we obtain that for any $\gamma_{\alpha\beta}$ and D :

$$h_{\alpha\beta} \gamma_{\alpha\beta} + \frac{\partial z}{\partial D} = k_{\alpha\beta} \gamma_{\alpha\beta} + cD$$

and finally:

$$h_{\alpha\beta} = k_{\alpha\beta} \text{ and } z = \frac{c}{2} D^2 + cte.$$

Results given before are summarized in Table 1.

There is another way to write the mechanical constitutive equation, using the electric potential V . Because one has:

$$E = -\frac{\partial V}{\partial x_3}$$

(V is the difference of electric potential between the upper and the lower faces of the piezo-sheet), we can write:

$$\sigma_{\alpha\beta} = \left[R_{\alpha\beta\mu\lambda}^P - \frac{h_{\alpha\beta} h_{\mu\lambda}}{c} \right] \gamma_{\mu\lambda} + \frac{h_{\alpha\beta}}{c} \frac{\partial V}{\partial x_3}. \quad (5)$$

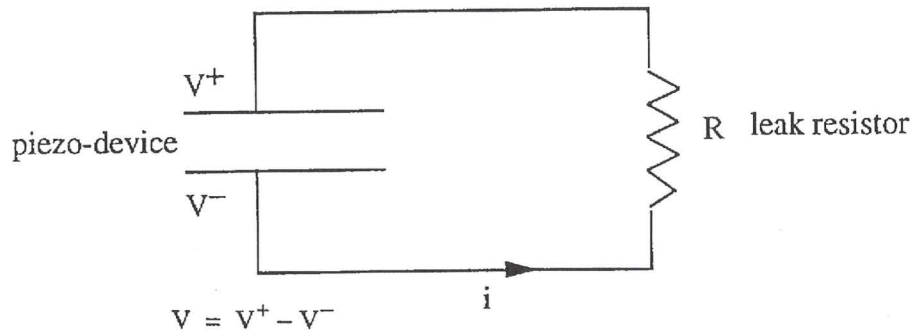
By integrating this relationship over the thickness of the piezo-sheet, one has also:

$$\sigma_{\alpha\beta} = \int_{-a}^a \sigma_{\alpha\beta} dx_3 = 2a \left[R_{\alpha\beta\mu\lambda}^P - \frac{h_{\alpha\beta} h_{\mu\lambda}}{c} \right] \gamma_{\mu\lambda} + V \frac{h_{\alpha\beta}}{c} \quad (6)$$

(we assume that everything is constant with respect to the coordinate x_3).

A basic remark. The smart-structure principle is the following one. The potential V (as a matter of fact the difference of potential), is adjusted so that the mechanical energy (elastic energy and/or kinetic-energy) is transformed into electric energy. Then there are two complementary possibilities for eliminating it. The first one consists in using a leak-resistor which is set in parallel to the capacitor defined by the piezo-sheet. The damping is then passive (Joule effect) but limited (see Fig. 2). Obviously no electronic device is required in such a configuration.

A second strategy for evacuating the mechanical energy is to use an active electronic device. For instance a current generator which enables one to prescribe a voltage on the piezo-electric wafer. Let us for instance consider the very simplified coupled system represented in Fig. 3.



$$V = V^+ - V^-$$

$$\begin{cases} -i = \frac{dq}{dt} = C \frac{dV}{dt} \\ V = Ri = -RC \frac{dV}{dt} \end{cases}$$

$$V(t) = V(0)e^{-\frac{t}{RC}}$$

Figure 2. The passive damping

Finally, let us mention that the so-called spill-over phenomenon is very important for this kind of control (see Ballas, 1982, and Destuynder, et al., 1990).

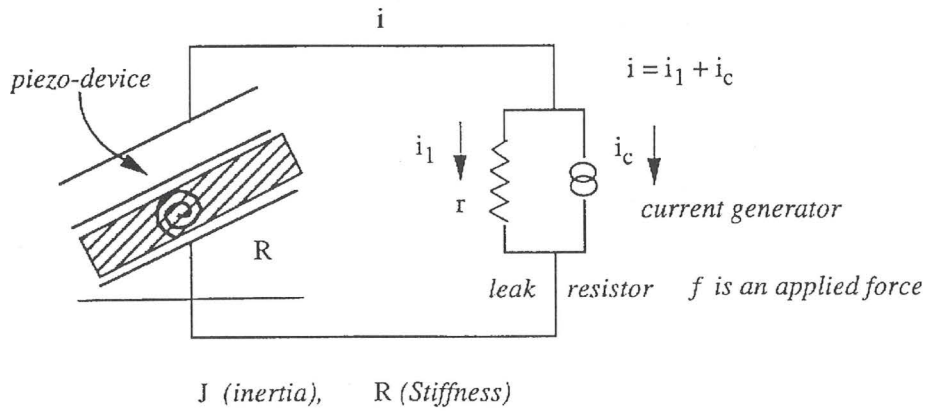
2.2. Coupling between piezo-electric devices and a plate or a shell structure

For sake of simplicity we do not discuss the validity of a shell theory. Only the Kirchhoff-Love model is used in this paper. Furthermore, we point out the particular formulations obtained for beams and plates and arches. We assume that the reader is familiar with these concepts. If not, we suggest him to have some readings on these quite complex mechanical theories (see for instance Destuynder, 1985).

2.2.1. The shell formulation

Let us consider a smooth surface ω embedded into \mathbf{R}^3 and which is represented by a mapping ϕ from $\hat{\omega} \subset \mathbf{R}^2$ into \mathbf{R}^3 . For our purpose and in order to avoid any difficulty, we assume that ϕ is $C^3(\bar{\omega})$. The vectors:

$$\tilde{a}_\alpha = \partial_\alpha \phi = \frac{\partial \phi}{\partial \xi_\alpha}, \quad \xi = (\xi_\alpha) \in \hat{\omega}, \quad \alpha = 1, 2 \quad (7)$$



$$\begin{cases} J\ddot{\alpha} = R\alpha + HV = f & \text{(mechanical equation)} \\ i = -\frac{d\alpha}{dt} = -c\frac{dV}{dt} = \frac{V}{r} + i_c & \text{(control system)} \end{cases}$$

$$\hat{\alpha} = \frac{\hat{\rho}}{R - \omega^2 J} + \frac{H}{(R - \omega^2 J)}(1 + i r c \omega) \hat{i}_c \quad \text{(Fourier transform)}$$

The best control is obtained for:

$$\hat{i}_c(\omega) = -\frac{(1 + i r c \omega)}{H} \hat{f}(\omega).$$

Figure 3. Symbolic representation of an active damping system

define the tangent plane to the surface ω at each point $m = \phi(\xi_1, \xi_2)$. Then the unit normal to ω is defined by.

$$N = \frac{\tilde{a}_1 \wedge \tilde{a}_2}{\|\tilde{a}_1 \wedge \tilde{a}_2\|} \quad (8)$$

and the mapping ϕ is supposed to be such that $\tilde{a}_1 \wedge \tilde{a}_2 \neq 0$. The displacement u (vector in \mathbf{R}^3) of the point m of ω can be split into tangential components u_α and a normal one – say u_3 – such that:

$$u = u^\alpha \tilde{a}_\alpha + u_3 N. \quad (9)$$

Furthermore if we introduce the metric tensor $g_{\alpha\beta}$ by:

$$g_{\alpha\beta} = (\tilde{a}_\alpha, \tilde{a}_\beta), (\cdot, \cdot) \text{ being the scalar product in } \mathbf{R}^3, \quad (10)$$

and its inverse $g^{\alpha\beta}$ by:

$$g^{\alpha\beta} = (\tilde{a}^\alpha, \tilde{a}^\beta) \quad (11)$$

where the dual vectors \tilde{a}^α are defined by:

$$(\tilde{a}^\alpha, a_\beta) = \delta^\alpha_\beta, \text{ (Kronecker symbol).}$$

It is also convenient to introduce the covariant components u_α such that:

$$u_\alpha = g_{\alpha\beta} u^\beta. \quad (12)$$

Finally the Christoffel symbols and the curvature are defined by:

$$\begin{cases} \Gamma^\lambda_{\alpha\beta} = (\tilde{a}^\lambda, \tilde{a}_{\alpha,\beta}) = \left(a^\lambda, \frac{\partial^2 \rho}{\partial \xi_\alpha \partial \xi_\beta} \right), \\ b_{\alpha\beta} = (N, \tilde{a}_{\alpha,\beta}) = \left(N, \frac{\partial^2 \phi}{\partial \xi_\alpha \partial \xi_\beta} \right) \end{cases} \quad (13)$$

and we set:

$$u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} u_\lambda. \quad (14)$$

Let us now introduce the change of inplane metric tensor (linearized) by:

$$\gamma_{\alpha\beta} = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3 \quad (15)$$

and the change of curvature by:

$$\rho_{\alpha\beta} = \frac{1}{2} (\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + b^\lambda_\alpha u_{\beta|\lambda} + b^\lambda_\beta u_{\alpha|\lambda} - b^\lambda_\alpha b_{\beta\lambda} u_3 \quad (16)$$

where the rotation $\theta = \theta_\alpha \tilde{a}^\alpha$ is defined by:

$$\theta_\alpha = -u_{3,\alpha} - b^\lambda_\alpha u_\lambda.$$

The classical Koiter model consists then in finding an element $u = (u_\alpha, u_3)$ in a functional space, which will be defined later on, such that:

$$\forall v, v = (v_\alpha, v_3) \in V \quad m(\ddot{u}, v) + p(v) = l(v) \quad (17)$$

where V is a space of admissible displacements and the bilinear form $m(\cdot, \cdot)$ and the linear one, $p(\cdot)$ being given, for arbitrary elements u, v in V , by:

$$\begin{aligned} m(u, v) &= \int_{\omega} \rho^s u^\alpha v_\alpha + \int_{\omega} I \theta \mu_\alpha^\beta \\ p(v) &= \int_{\omega} n^{\alpha\beta} \gamma_{\alpha\beta}(v) + \int_{\omega} m^{\alpha\beta} \rho_{\alpha\beta}(v) \end{aligned} \quad (18)$$

where $n^{\alpha\beta}$ and $m^{\alpha\beta}$ are the contravariant components of respectively the membrane and the bending moment stress tensors. These two are expressed in terms of the inplane stress-tensor of the shell by:

$$n^{\alpha\beta} = \int_{-\epsilon}^{\epsilon} \sigma^{\alpha\beta} dx_3, \quad m^{\alpha\beta} = \int_{-\epsilon}^{\epsilon} x_3 \sigma^{\alpha\beta} dx_3. \quad (19)$$

Finally the linear form $l(\cdot)$ represents the so-called external loading, and is for instance given by (for any $v = (v_\alpha, v_3)$, lying in the space V):

$$l(v) = \int_{\omega} f_3 v_3 + \int_{\omega} f^\alpha v_\alpha. \quad (20)$$

Let us recall briefly some mathematical results which will be used in the following. First of all let us consider the case of an elastic material (we remain in the linearized theory). The generalized Hooke law can be written in the shell:

$$\delta^{\alpha\beta} = R^{\alpha\beta\mu\lambda} [\gamma_{\mu\lambda}(u) + x_3 \rho_{\mu\lambda}(u)]$$

and therefore (assuming for the sake of simplicity that the material is homogeneous):

$$\begin{cases} n^{\alpha\beta} = 2\epsilon R^{\alpha\beta\mu\lambda} \gamma_{\mu\lambda}(u), \\ m^{\alpha\beta} = \frac{2\epsilon^3}{3} R^{\alpha\beta\mu\lambda} \rho_{\mu\lambda}(u). \end{cases} \quad (21)$$

The Koiter model can then be formulated as follows:

Find $u, u = (u_\alpha, u_3) \in W$ (function of time and space to be defined later on), such that:

$$\forall v \in V, \quad m(\ddot{u}, v) + a(u, v) = l(v) \quad (22)$$

where the bilinear form $a(\cdot, \cdot)$ is defined for arbitrary elements $u = (u_\alpha, u_3)$ and $v = (v_\alpha, v_3)$ lying in the space V (to be precised) by ($V \subset W$!):

$$a(u, v) = \int_{\omega} 2\epsilon R^{\alpha\beta\mu\lambda} \gamma_{\mu\lambda}(u) \gamma_{\alpha\beta}(v) + \int_{\omega} \frac{2\epsilon^3}{3} R^{\alpha\beta\mu\lambda} \rho_{\mu\lambda}(u) \rho_{\alpha\beta}(v). \quad (23)$$

Setting for instance:

$$\begin{aligned} V &= \{v|v = (v_\alpha, v_3), v_\alpha \in H^1(\omega), v_3 \in H^2(\omega) \\ v_\alpha &= v_3 = 0 = \partial_\alpha v_3 \text{ on } \gamma_0 \subset \partial\omega\}, \end{aligned} \quad (24)$$

one has (see Bernadou, Ciarlet, 1976):

$$\exists c_0 > 0, \forall v \in V, a(v, v) \geq c_0 \|v\|_V^2$$

where V is equipped with the norm:

$$v \in V \rightarrow \|v\|_V = \left\{ \sum_{\alpha=1,2} \|v_\alpha\|_{1,\omega}^2 + \|v_3\|_{2,\omega}^2 \right\}^{\frac{1}{2}}.$$

Then, existence and uniqueness of a solution to the wave equation for shells (22), can be obtained for ad hoc regularity assumptions on the external loads and the initial conditions, using spectral theory (see for instance Brezis, 1983).

Let us define by (λ^n, W^n) , the eigenvalues and the eigenvectors of the bilinear form $a(\cdot, \cdot)$ with respect to $m(\cdot, \cdot)$ and such that:

$$\forall v \in V, a(W^n, v) = \lambda^n m(W^n, v). \quad (25)$$

From classical spectral theory we know that there exists a countable sequence of solutions to (25) satisfying:

$$m(W^n, W^m) = \delta_m^n, \text{ (Kronecker symbol).}$$

Furthermore $\left\{ \frac{W^n}{\sqrt{\lambda^n}} \right\}$ is an Hilbert basis in V and the multiplicity of each eigenvalue is finite (i.e. the dimension of the eigenspace of each eigenvalue λ^n is finite).

Let us now introduce the following spaces which are the right ones for the analysis of solutions of the wave equation. For any real numbers larger or equal to -1 , we set:

$$D_S(\omega) = \left\{ v|v \in V', \sum_{n \geq 1} |\langle v, W^n \rangle|^2 (\lambda^n)^s < \infty \right\} \quad (26)$$

where \langle, \rangle denotes the duality between V and its dual space V' .

A simple and classical exercise leads to the characterization of some of these functional spaces. Let us summarize the main results in the following statement:

THEOREM 2.1 i) $\forall s_1 > s_2 > -1$ one has: $D_{S_1}(\omega) \subset D_{S_2}(\omega)$ and in particular:
ii) $D_{-1}(\omega) = V'$, $D_0(\omega) = (L^2(\omega))^2 \times H$ $D_1(\omega) = V$

where:

$$H = \{v|v \in H^1(\omega), v = 0 \text{ on } \gamma_0 \subset \partial\omega\}.$$

Proof:

i) First of all, let us point out that:

$$\left. \begin{array}{l} \forall n \geq 1 \\ \text{and } \lambda^n \geq 1 \end{array} \right\} \Rightarrow (\lambda^n)^{s_2} \leq (\lambda^n)^{s_1}.$$

Because the sequence λ^n tends to the infinity, only a limited number of eigenvalues can be smaller than 1. Hence there exists a constant c_0 for any function v in $D_{s_1}(\omega)$ such that:

$$c_0 \sum_{n \geq 1} |\langle v, w^n \rangle|^2 (\lambda^n)^{s_1} \geq c_0 \sum_{n \geq 1} |\langle v, w^n \rangle|^2 (\lambda^n)^{s_2}$$

and therefore:

$$v \in D_{s_2}(\omega).$$

ii) Let us now consider an element l in V' .

It is possible to define z in V such that:

$$\forall v \in V, a(z, v) = l(v) = \langle l, v \rangle.$$

(due to the V coerciveness of $a(\cdot, \cdot)$ on V).

But from the spectral theory we know that $\left\{ \frac{W^n}{\sqrt{\lambda^n}} \right\}$ is an Hilbert basis of V and therefore:

$$z = \sum_{n \geq 1} \left\{ \frac{\alpha^n}{\sqrt{\lambda^n}} \right\} W^n$$

with on the one hand:

$$\alpha^n = \frac{\langle \rho, W^n \rangle}{\sqrt{\lambda^n}}$$

and on the other hand:

$$\|z\|_V^2, \sum_{n \geq 1} |\alpha^n|^2 = \sum_{n \geq 1} \frac{|\langle l, W^n \rangle|^2}{\lambda^n} < \infty.$$

Finally we proved that $l \in D_{-1}(\omega)$. Conversely, let us assume that l is an element of $D_{-1}(\omega)$. Then for any element v in V one can write:

$$v = \sum_{n \geq 1} \frac{\alpha^n}{\sqrt{\lambda^n}} W^n, \text{ and } \sum_{n \geq 1} |\alpha^n|^2 < \infty.$$

Hence, from:

$$\frac{\langle l, \alpha^n W^n \rangle}{\sqrt{\lambda^n}} = \alpha^n \frac{\langle l, W^n \rangle}{\sqrt{\lambda^n}} \leq |\alpha^n| \frac{|\langle l, W^n \rangle|}{\sqrt{\lambda^n}},$$

we deduce that (Schwarz inequality):

$$\sum_{n \geq 1} \frac{\langle l, \alpha^n W^n \rangle}{\sqrt{\lambda^n}} \leq \|v\|_V \left(\sum_{n \geq 1} \frac{|\langle l, W^n \rangle|^2}{\lambda^n} \right)^{\frac{1}{2}} < \infty$$

and finally that l is an element of V' . The two other inclusions being easy to prove, they are left to the reader. ■

Let us now give a complementary result for the wave equation. For each time interval: $]OT[$, we introduce the spaces (see Lions and Magenes, 1968):

$$L^2(OT; D_s(\omega)) \text{ and } H^1(OT; D_s(\omega))$$

One has the following classical theorem which will be used for the coupled model (see again Lions, Magenes, 1968):

THEOREM 2.2 *Let l be an element of the space $L^2(OT; D_{-1}(\omega)) = L^2(OT; V')$ and u^0, u^1 be two elements lying, respectively, in the space $D_0(\omega)$ and $D_{-1}(\omega) = V'$. Then we introduce:*

i) *The operator A from V into V' by:*

$$u \in V; u = \sum_{n \geq 1} \frac{\alpha^n}{\sqrt{\lambda^n}} W^n, \quad Au = \sum_{n \geq 1} \sqrt{\lambda^n} \alpha^n W^n \in V'$$

ii) *The operator M from $L^2(\omega) \times L^2(\omega) \times H$ into itself defined by:*

$$(Mu, v) \stackrel{\text{def}}{=} m(u, v), \quad \forall (u, v)$$

where the space H is defined by:

$$H = \{v \in H^1(\omega), v = 0 \text{ on } \gamma_0\}$$

and (\cdot, \cdot) is the scalar product in $(L^2(\omega))^2 \times H$ defined precisely by $m(\cdot, \cdot)$.

Let us consider the following system:

find $u \in H^1(OT; V') \cap L^2(OT; D_0(\omega))$ such that:

$$M\ddot{u} + Au = 1$$

$$u(x, 0) = u^0, \quad \dot{u}(x, 0) = u^1.$$

Then the previous problem has a unique solution which satisfies:

$$\int_0^T \left\{ \|\dot{u}\|_{V'}^2 + \|u\|_{D_0(\omega)}^2 \right\} \leq C_1 \left\{ \|u^0\|_{D_0(\omega)}^2 + \|u^1\|_{V'}^2 + \int_0^T \|l\|_{V'}^2 \right\}.$$

Remark. In the proof given below the functional spaces used are not usual. Nevertheless, the proof can also be obtained as usual by analyzing the convergence of the series:

$$u^N = \sum_{n \geq 1} \sum_{n \leq N} \frac{\alpha^n(t)}{\sqrt{\lambda^n}} W^n(x) \quad (27)$$

where:

$$\begin{aligned} \alpha^n(t) &= \langle u^0, W^n \rangle \cos(\sqrt{\lambda^n} t) + \frac{\langle u^1, W^n \rangle}{\sqrt{\lambda^n}} \sin(\sqrt{\lambda^n} t) \\ &+ \frac{1}{\sqrt{\lambda^n}} \int_0^t \langle l(x, s), W^n(x) \rangle \sin(\sqrt{\lambda^n}(t-s)) ds. \end{aligned} \quad (28)$$

Proof: In order to establish that $u(x, t) \in L^2(OT; D_0(\omega))$, and $\dot{u}(x, t) \in L^2(OT; V')$, it is necessary and sufficient to prove that:

$$\sum_{n=1, \infty} \int_0^T |\alpha^n(t)|^2 dt < +\infty, \quad \text{and} \quad \sum_{n=1, \infty} \int_0^T |\dot{\alpha}^n(t)|^2 dt < +\infty.$$

But from:

$$|\alpha^n(t)|^2 \leq c_1 \left\{ \langle u^0, W^n \rangle^2 + \frac{1}{\lambda^n} \langle u^1, W^n \rangle^2 \right\} + \frac{c_2}{\lambda^n} \int_0^t \langle l, W^n \rangle^2.$$

we deduce that:

$$\sum_{n=1, \infty} \int_0^T |\alpha^n(t)|^2 dt \leq c_3 \left\{ \|u^0\|_{D_0(\omega)}^2 + \|u^1\|_{V'}^2 + \|l\|_{L^2(OT; V')}^2 \right\}.$$

We have also:

$$\begin{aligned} \dot{\alpha}^n(t) &= -\sqrt{\lambda^n} \langle u^0, W^n \rangle \sin(\sqrt{\lambda^n} t) + \langle u^1, W^n \rangle \cos(\sqrt{\lambda^n} t) \\ &+ \int_0^t \langle l(\chi, s), W^n(\chi) \rangle \cos(\sqrt{\lambda^n}(t-s)) ds. \end{aligned}$$

Then:

$$|\dot{\alpha}^n(t)|^2 \leq c_4 \left\{ \langle u^0, W^n \rangle^2 + \frac{1}{\lambda^n} \langle u^1, W^n \rangle^2 \right\} + \frac{c_5}{\lambda^n} \int_0^t \langle l, W^n \rangle^2.$$

Finally we have:

$$\sum_{n=1, \infty} \int_0^T |\dot{\alpha}^n(t)|^2 dt \leq c_3 \left\{ \|u^0\|_{D_0(\omega)}^2 + \|u^1\|_{V'}^2 + \|l\|_{L^2(OT; V')}^2 \right\}.$$

Remark. The uniqueness is almost obvious except if there are corners on the boundary of ω . But using the local expansion of the solution it is possible to deduce the uniqueness.

2.2.2. The smart-shell model (coupling between a shell model and piezo-devices)

Let us consider a shell, as the one represented on Fig. 4, equipped with one or more piezo-electric wafers which are glued on the upper or/and lower face.

The constitutive relationship through the thickness of the shell can be formulated as follows (see Fig. 4):

$$\begin{cases} -\epsilon < x_3 < \epsilon \Rightarrow \sigma_{\alpha\beta} = R_{\alpha\beta\mu\lambda} [\gamma_{\mu\lambda}(u) + x_3 \rho_{\mu\lambda}(u)] \\ \epsilon < x_3 < 2a + \epsilon \Rightarrow \\ \sigma_{\alpha\beta} = \left[R_{\alpha\beta\mu\lambda}^p - \frac{h_{\alpha\beta} h_{\mu\lambda}}{c} \right] [\gamma_{\mu\lambda}(u) + (\epsilon + a) \rho_{\mu\lambda}(u)] + \frac{h_{\alpha\beta}}{c} \frac{\partial V}{\partial x_3}. \end{cases} \quad (29)$$

It should be noticed that we assume that because of the very small thickness of the piezo-wafer (i.e. $a \ll \epsilon$), the plane-strain is chosen to be constant, equal to its value in the middle of the piezo-device, for $\epsilon < x_3 < \epsilon + 2a$. Let us now consider the resultant inplane stress and the bending moment. One obtains (ω_p corresponds to the position of a wafer):

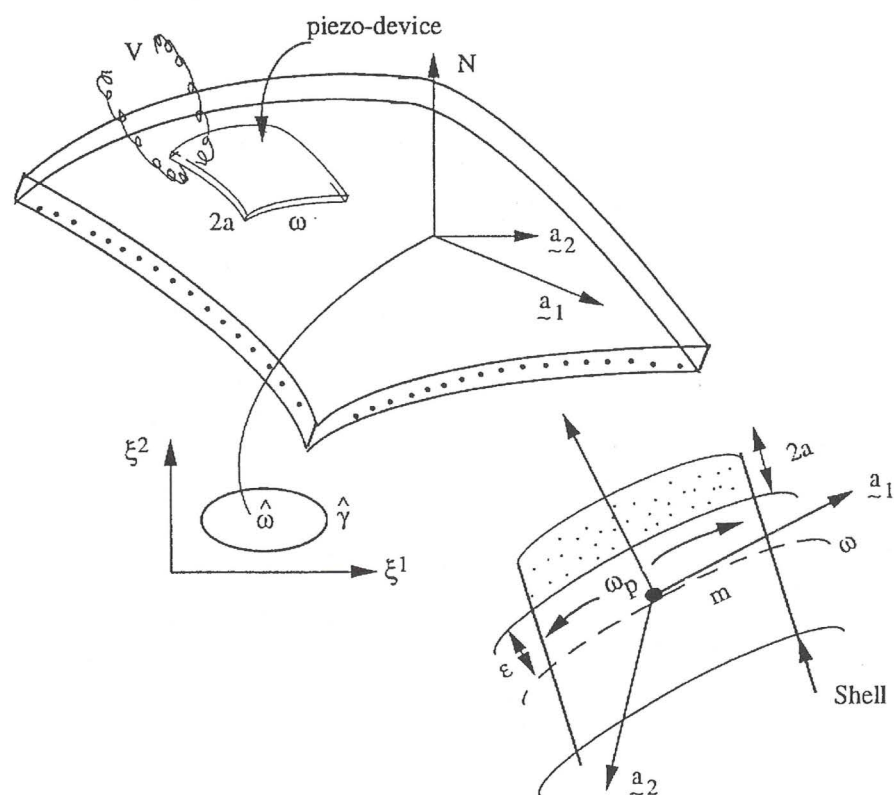


Figure 4. The smart shell

i) $m \notin \omega_p$;

$$\begin{cases} n_{\alpha\beta} = \int_{-\epsilon}^{\epsilon} \sigma_{\alpha\beta} dx_3 = 2\epsilon R_{\alpha\beta\mu\lambda} \gamma_{\mu\lambda}(u), \\ m_{\alpha\beta} = \int_{-\epsilon}^{\epsilon} x_3 \sigma_{\alpha\beta} dx_3 = \frac{2\epsilon^3}{3} R_{\alpha\beta\mu\lambda} \rho_{\mu\lambda}(u). \end{cases}$$

ii) $m \in \omega_p$;

$$\begin{aligned} n_{\alpha\beta} &= \int_{-\epsilon}^{\epsilon+2a} \sigma_{\alpha\beta} dx_3 = 2\epsilon R_{\alpha\beta\mu\lambda} \gamma_{\mu\lambda}(u) \\ &+ 2a \left[R_{\alpha\beta\mu\lambda}^p - \frac{h_{\alpha\beta} h_{\mu\lambda}}{c} \right] [\gamma_{\mu\lambda}(u) + (\epsilon + a) \rho_{\mu\lambda}(u)] \\ &+ \frac{h_{\alpha\beta}}{c} V \\ m_{\alpha\beta} &= \int_{-\epsilon}^{\epsilon+2a} x_3 \sigma_{\alpha\beta} dx_3 = \frac{2\epsilon^3}{3} R_{\alpha\beta\mu\lambda} \rho_{\mu\lambda}(u) \\ &+ 2a(\epsilon + a) \left[R_{\alpha\beta\mu\lambda}^p - \frac{h_{\alpha\beta} h_{\mu\lambda}}{c} \right] [\gamma_{\mu\lambda}(u) + (\epsilon + a) \rho_{\mu\lambda}(u)] \\ &+ \frac{(\epsilon + a)}{c} h_{\alpha\beta} V \end{aligned}$$

(in the wafer, the coordinate x_3 has been again approximated by $\epsilon + a$). We point out that V is the difference of potential between the outer face and the inner one. Furthermore the formula obtained would be also true if the piezo-wafer would be glued on the lower face of the shell, but replacing $\epsilon + a$ by $-\epsilon - a$ and V by $-V$.

Remark (a practical one). In most applications the term $2a \left[R_{\alpha\beta\mu\lambda}^p - \frac{h_{\alpha\beta} h_{\mu\lambda}}{c} \right]$ is neglectible in comparison with the shell coefficients $2\epsilon R_{\alpha\beta\mu\lambda}$. The same is obviously true for the bending terms. Hence they will be omitted in the following of this text.

Then, if we come back to the equation (17) with the previous expression for $n_{\alpha\beta}$ and $m_{\alpha\beta}$, we deduce the following expressions:

$$\forall v \in V, m(\ddot{u}, v) + a(u, v) = l(v) \quad (30)$$

where:

$$l(v) = -\frac{h_{\alpha\beta}}{c} V \int_{\omega_p} \gamma_{\alpha\beta}(v) - (\epsilon + a) \frac{h_{\alpha\beta}}{c} V \int_{\omega_p} \rho_{\alpha\beta}(v) \quad (31)$$

and the bilinear forms $m(\cdot, \cdot)$ and $a(\cdot, \cdot)$ being those defined in (18) and (23).

Remark. The bilinear form $a(\cdot, \cdot)$ would be slightly different if the stiffness of the piezo-electric device would not been neglected. But it does not induce any additional difficulty. For sake of simplicity we neglect these terms. Numerical values of $R_{\alpha\beta\mu\lambda}^p$ and $\frac{h_{\alpha\beta} h_{\mu\lambda}}{c}$ that corresponds to real materials justify completely such an hypothesis.

Remark. One can consider that the space V is the one defined in (25). But we will see in the next section that this is not fully correct.

The system (30) is the so-called coupled shell piezo-electric model. Some difficulties arise for defining correctly the equation (30). But first of all let us try to give a mechanical interpretation of the effect of the piezo-electric wafer.

2.2.3. Mechanical interpretation of the piezo-effect

From Stokes formula on the open set ω_p , it is possible to rewrite the expression of the linear form $l(\cdot)$ as follows (with $v = (v_\alpha, v_3, \mu_\alpha) \in V$):

$$\begin{aligned} l(v) = & -\frac{h^{\alpha\beta}}{2c}V \left[\int_{\partial\omega_p} (v_\alpha \nu_\beta + v_\beta \nu_\alpha) - 2 \int_{\omega_p} \Gamma_{\alpha\beta}^\lambda v_\lambda - 2 \int_{\omega_p} b_{\alpha\beta} v_3 \right] \\ & - \frac{(\epsilon + a)}{2c} h^{\alpha\beta} V \left[\int_{\partial\omega} (\mu_\alpha \nu_\beta + \mu_\beta \nu_\alpha) + \int_{\partial\omega_p} b_\alpha^\lambda v_\alpha \nu_\lambda + b_\alpha^\lambda v_\alpha \nu_\lambda \right. \\ & \left. - 2 \int_{\omega_p} b_\alpha^\lambda b_{\lambda\beta} v_3 - 2 \int_{\omega_p} \Gamma_{\alpha\beta}^\lambda \mu_\lambda - \int_{\omega_p} [(b_\alpha^\lambda)_{|\lambda} v_\beta + (b_\lambda^\alpha)_{|\lambda} v_\alpha] \right] \end{aligned} \quad (32)$$

where $\nu = \nu_\alpha \tilde{a}^\alpha$ is the outward normal along the boundary of ω_p (see Destuynder, 1990, for details concerning the Stokes formula applied to shell models).

First of all let us emphasise that $l(v)$ contains boundary terms and surface ones (for instance $\int_{\partial\omega} v_\alpha \nu_\beta + v_\beta \nu_\alpha$ and $-2 \int_{\omega_p} \Gamma_{\alpha\beta}^\lambda v_\lambda$). Hence $l(\cdot)$ can be reduced to the combination of line-forces (or moments) and surface-forces.

Then, we consider particular cases of piezo-electric material and shell-geometry. Let us deal first with unioriented piezo-devices such that:

$$h_{\alpha\beta} = H \delta_{l\beta} \delta_{l\alpha} \quad (\delta_{\alpha\beta} \text{ is Kronecker symbol}).$$

We obtain:

$$\begin{aligned} l(v) = & -\frac{HV}{c} \left[\int_{\partial\omega_p} v_1 \nu_1 - \int_{\omega_p} \Gamma_{11}^\lambda v_\lambda - \int_{\omega_p} b_{11} v_3 \right] \\ & \left[\int_{\partial\omega_p} \mu_1 \nu_1 + \int_{\partial\omega_p} b_1^\lambda v_1 \nu_1 - \int_{\omega_p} b_1^\lambda b_{1\lambda} v_3 \right. \\ & \left. - \int_{\omega+p} \Gamma_{11}^\lambda \mu_\lambda - \int_{\omega_p} \frac{1}{|g|^{\frac{1}{2}}} (b_1^\lambda)_{|\lambda} v_1 \right] \end{aligned} \quad (33)$$

a) Case of a cylinder If the shell is a cylinder as shown in Fig. 5, the previous expression can be simplified depending on the orientation of the material with respect to the curvature:

i) ξ^1 is parallel to the axis of the cylinder (ν is the unit outward normal to $\partial\omega_p$)

$$l(v) = -\frac{HV}{c} \left[\int_{\partial\omega_p} v_1 \nu_1 \right] - \frac{(\epsilon + a)HV}{c} \left[\int_{\partial\omega_p} \mu_1 \nu_1 \right] \quad (34)$$

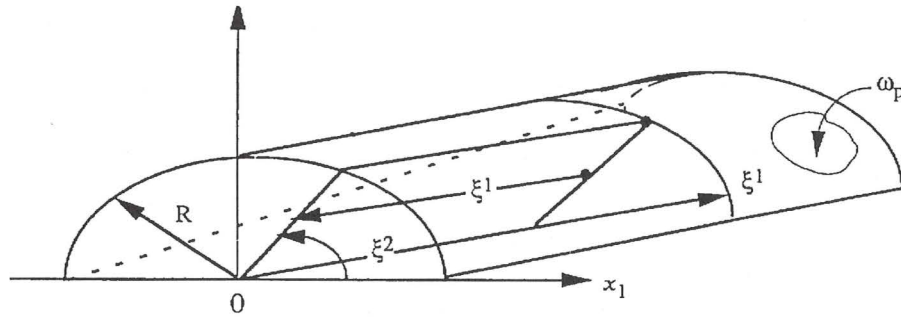


Figure 5. A cylinder shell with a piezo-device

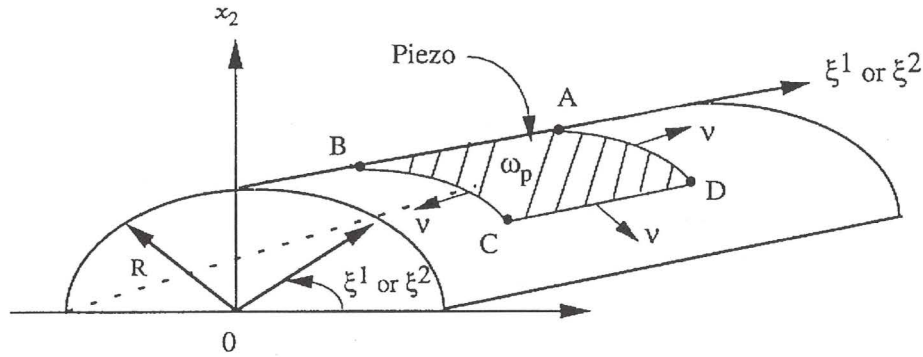


Figure 6. A particular wafer on a cylindrical shell

ii) ξ^2 is parallel to the axis of the cylinder

$$l(v) = -\frac{HV}{c} \left[\int_{\partial\omega_p} v_1\nu_1 + R \int_{\omega_p} v_3 \right] - \frac{(\epsilon + a)HV}{c} \left[\int_{\partial\omega_p} \mu_1\nu_1 + \int_{\partial\omega_p} \frac{1}{R} v_1\nu_1 - \int_{\omega_p} v_3 \right] \quad (35)$$

It is particularly interesting to make a little bit more explicit these last formulae when ω_p is a "rectangular" wafer as shown in Fig. 6.

In the first configuration the piezo-device has a double effect: one is a localized force along the lines AD and BC (see Fig. 6). The other is a localized bending moment along the same lines. In the second configuration, these localized forces and bending-moments are applied on the lines BA and CD (see Fig. 6). But additionally there is a transverse pressure

applied on ω_p , which is given by:

$$\frac{HV}{c}[R - (\epsilon + a)], -\frac{HV}{c}(R - \epsilon), -\frac{HVR}{c} \quad (36)$$

(because $a \ll \epsilon \gg R$!).

From this remark we can state that the second configuration is well adapted to control the bending vibrations of a shell. But the surface ω_p should be chosen in order that $\int_{\omega_p} u_3 \neq 0$ for the movement to be controlled.

- b) Case of a plate** In a such a configuration all the Christoffel symbols and the curvature are zero. Then we obtain the following expression for the right hand side $l(v)$ due to the piezo-electric effect for a unioriented material:

$$l(v) = -\frac{HV}{c} \int_{\partial\omega_p} v_1 \nu_1 + \frac{(\epsilon + a)HV}{c} \int_{\partial\omega_p} v_{3,1} \nu_1 \quad (37)$$

or for an isotropic piezo-material ($h_{\alpha\beta} = H\delta_{\alpha\beta}$):

$$l(v) = -\frac{HV}{c} \int_{\partial\omega_p} v_\alpha \nu_\alpha + \frac{(\epsilon + a)HV}{c} \int_{\partial\omega_p} \frac{\partial v_3}{\partial \nu}. \quad (38)$$

In a pure bending deformation (no membrane effect), we obtain the simplified expressions:

- i) **unioriented material**

$$l(v) = (\epsilon + a) \frac{HV}{c} \int_{\partial\omega_p} v_{3,1} \nu_1 \quad (39)$$

- ii) **isotropic material**

$$l(v) = (\epsilon + a) \frac{HV}{c} \int_{\partial\omega_p} \frac{\partial v_3}{\partial \nu} \quad (40)$$

The corresponding forces are symbolically represented in Fig. 7.

Remark. It is possible to split bending and stretching effect in an arbitrary shell for the control by piezo-electric devices. This can be easily obtained by setting two identical piezo-wafers on respectively the upper and the lower faces of the shell. Then, if the voltage V has an opposite sign, the membrane effect is cancelled in the expression of $l(\cdot)$. If V is the same on both wafers, then the bending effect is eliminated. But in many application it is not easy to manufacture such a coupled structure. For instance, if the outside of the shell is submitted to wind loading, or any surrounding conditions, it is quite impossible to equip the outside of the shell structure with piezo-devices.

2.2.4. Existence and uniqueness of a solution to the coupled model

We suggest in this section to apply the general Theorem 2 just by checking the assumptions. As a matter of fact, this verification is restricted to the right hand side. Obviously the variational formulation is not useful because the solution and the test-functions are not in the right space which would enable one to assign sense to the terms written. Therefore the equation of the coupled model

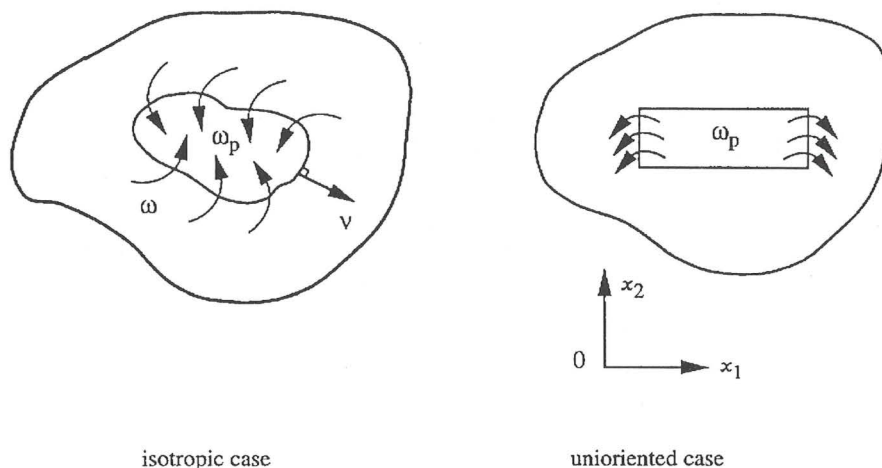


Figure 7. Forces due to a piezo-electric wafer on a bending plate

should be understood as in Theorem 2. In order to state the existence and uniqueness theorem, we just have to determine in which functional space is the right hand side $l(v)$. Let us first recall that it is defined by (31). Clearly $l(v)$ is a linear form with respect to v . The element v being in the space V , one has from Schwarz inequality:

$$|l(v)| \leq K|V| \left[\sum_{\alpha=1,2} (\|v_\alpha\|_{1,\omega} + \|\mu_\alpha\|_{1,\omega}) + \|v_3\|_{0,\omega} \right]$$

where K is a constant and V the electric potential which is a function of time. Then from:

$$\mu_\alpha = -b_\alpha^\lambda v_\lambda - v_{3,\alpha}$$

we deduce that:

$$|l(v)| \leq K'|V| \left[\sum_{\alpha=1,2} (\|v_\alpha\|_{1,\omega}) + \|v_3\|_{2,\omega} \right].$$

Therefore for almost every time t , $l(v)$ is a linear and continuous form defined on the functional space V introduced at (25). In other words, it means that l is an element of V' (dual space of V). Hence, we can state the following theorem.

THEOREM 2.3 *Let $V(t)$ be a function of the space $L^\infty(OT)$ and (u^0, u^1) two functions lying in the space V . Then there exists a unique solution $u(x, t)$ to*

the coupled model.

$$\begin{cases} M\ddot{u} + Au = l \\ u(x, 0) = u^0 \quad \dot{u}(x, 0) = u^1 \\ u(x, t) \in H^1(OT; V') \cap L^2(OT; D_0(\omega)) \end{cases}$$

where l is an element of $L^2(OT; V')$ defined at (32) and A is the operator introduced in Theorem 2.

The proof of the previous result is straightforward from Theorem 2. But let us point out that the assumed regularity concerning the initial conditions is not necessary but only convenient for the formulation and corresponds to most of the physical cases.

Let us now try to make explicit the basic feature of a solution to the coupled model. Let us therefore consider an infinite beam, modelled by the following equation:

$$\begin{cases} \ddot{u} + \gamma^4 \frac{\partial^4 u}{\partial x^4} = V\delta'_0(x)Y(t) \\ u(x, 0) = 0 \quad \dot{u}(x, 0) = 0. \end{cases} \quad \begin{array}{l} (Y(t) \text{ being the Heavyside func-} \\ \text{tion and } \delta'_0 \text{ the derivative of the} \\ \text{Dirac function).} \end{array} \quad (41)$$

The solution can be explicitated using Fourier transform with respect to the coordinate x :

$$\begin{cases} \hat{u} + k^4 \gamma^4 \hat{u} = iVkY(t) \\ \hat{y}(k, 0) = 0 \quad \dot{\hat{u}}(k, 0) = 0 \\ \hat{u}(k, t) = \int_{-\infty}^{+\infty} e^{-ikx} u(x, t) dx \end{cases} \quad (42)$$

(k is the wave number up to an ad hoc normalization).

$$\begin{cases} \hat{u}(k, t) = 0 \text{ for } t \leq 0 \\ \hat{u}(k, t) = \frac{iV}{k^3 \gamma^4} [1 - \cos(k^2 \gamma^2 t)] \text{ for } t \geq 0 \end{cases} \quad (43)$$

This function has been represented in Fig. 8 for various values of time t . With another respect the solution $u(x, t)$ is plotted in Fig. 9 (a numerical computation using an integration scheme has been used).

3. Controllability results for the coupled model

3.1. A criterion

First of all, let us recall the definition of the exact controllability of a second order system (with respect to time). For an arbitrary large time delay T , we define $u(x, t)$ as the solution of the model:

$$\begin{cases} M\ddot{u} + Au = l \quad \forall (x, t) \in \omega \times]0, T[, \\ u(x, 0) = u^0(x) \in D_0(\omega), \\ \dot{u}(x, 0) = u^1(x) \in V' \end{cases} \quad (44)$$

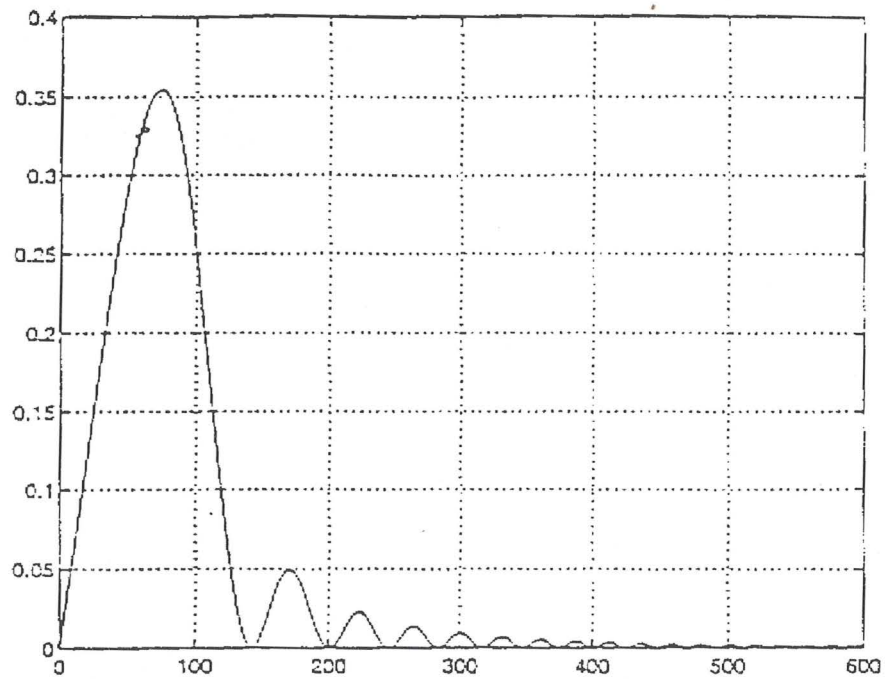


Figure 8. The variation of the space Fourier transform for varying time (k is the wave number) with the beam equation

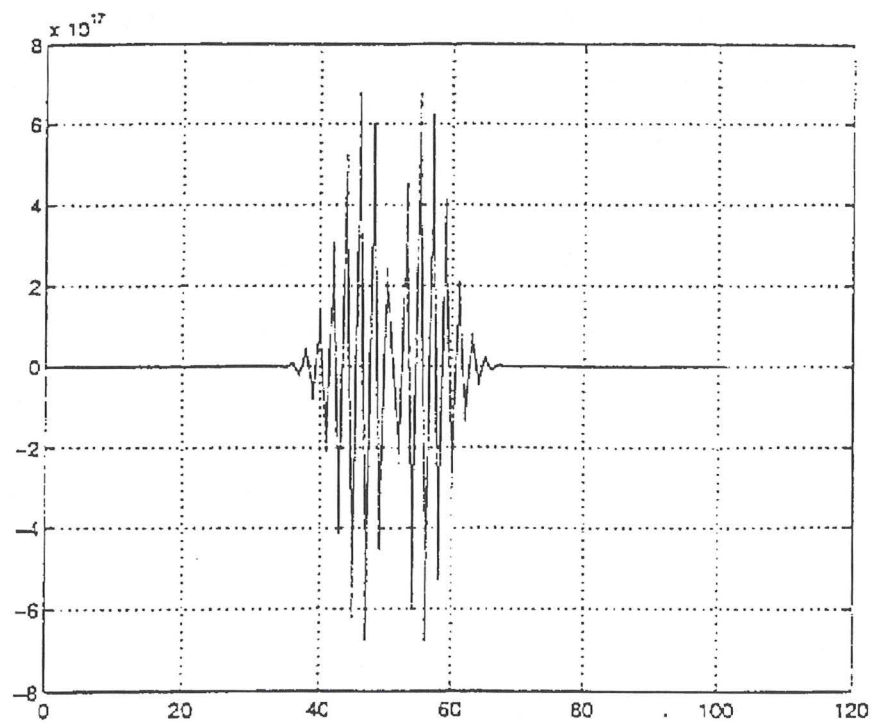


Figure 9. The solution with respect to the coordinate x (beam equation)

where the operators M and A have been defined previously. Furthermore we assume that the right hand side satisfies:

$$l \in L^2(OT; V').$$

Therefore, from Theorem 3 we deduce that:

$$u \in L^2(OT; V'). \quad (45)$$

Then we define a convex, closed and bounded (or not) subset of $L^2(OT; V')$ by C , assuming that the origin is an interior point of C (the topology is the one induced by the Hilbert space $L^2(OT; V')$).

We say that the system (44) can be exactly controlled if and only if for any initial condition – say (u^0, u^1) – lying in the space $D_o(\omega) \times V'$, there exists an element $l \in C$ such that:

$$u(x, t) = 0 \quad \forall t \geq T_c \quad (\forall x \in \omega)$$

where T_c is a finite time delay (assuming that T is sufficiently large so that $T_c \leq T$).

In order to formulate a practical criterion which could enable one to check the controllability property, let us reverse the time t and let us consider the following problem.

$$\begin{cases} M\ddot{u} + Au = l \quad \forall (x, t) \in \omega \times]0, T[, \\ u(x, 0) = 0, \\ \dot{u}(x, 0) = 0 \\ l \in C \end{cases} \quad (46)$$

Then we set

$$\begin{aligned} A(C) &= \{(u(x, t), \dot{u}(x, t)), \\ &\quad u \text{ being solution to the above equations for arbitrary } l \text{ in } C\} \end{aligned} \quad (47)$$

Due to the linearity of the model (46), $A(C)$ is also a convex set of $D_o(\omega) \times V'$. The system can be exactly controlled if and only if $A(C)$ is equal to $D_o(\omega) \times V'$. In other words, it means that the mapping: $l \rightarrow (u(x, t), \dot{u}(x, t))$ is onto, or else that $A(C)$ is dense and closed. Therefore if the system can be exactly controlled, then **a necessary condition is that** for any element $(u^0, u^1) \in D_o(\omega) \times V'$, satisfying:

$$\forall l \in C, \langle u^0, u(x, T) \rangle_{D_o(\omega)} + \langle u^1, \dot{u}(x, T) \rangle_{V'} = 0, \quad (48)$$

one has:

$$(u^0, u^1) = 0;$$

(let us point out that in the previous equation u is solution to (46)). The symbol \langle, \rangle denotes the scalar products in $D_o(\omega)$ or V' defined, respectively, as follows:

$$\forall \chi, y \in V', \chi = \sum_{i \geq 1} \chi_i W^i, y = \sum_{i \geq 1} y_i W^i,$$

then:

$$\langle \chi, y \rangle_{V'} = \sum_{i \geq 1} \frac{1}{\lambda^i} \chi_i y_i,$$

and

$$\langle \chi, y \rangle_{D_o(\omega)} = \sum_{i \geq 1} \chi_i y_i.$$

Let us write the necessary condition (48) more explicitly. First of all we set (u being solution to (46)):

$$u(x, t) = \sum_{n \geq 1} \alpha_n(t) W^n(x)$$

where:

$$\alpha_n(t) = \frac{1}{\sqrt{\lambda^n}} \int_0^t \langle l, W^n \rangle_{D_o(\omega)}(s) \sin \left[\sqrt{\lambda^n}(t-s) \right] ds.$$

Then the relation (48) can be written:

$$\forall l \in C(\subset L^2(OT; V')); \quad (49)$$

$$\begin{aligned} & \sum_{n \geq 1} \frac{\langle u^0, W^n \rangle_{D_o(\omega)}}{\sqrt{\lambda^n}} \int_0^t \langle l, W^n \rangle_{D_o(\omega)}(s) \sin \left(\sqrt{\lambda^n}(t-s) \right) ds \\ & + \langle u^1, W^n \rangle_{V'} \int_0^t \langle l, W^n \rangle_{D_o(\omega)}(s) \cos \left(\sqrt{\lambda^n}(t-s) \right) ds = 0 \end{aligned}$$

and setting:

$$\xi^n = \langle u^0, W^n \rangle_{D_o(\omega)}, \eta^n = \langle u^1, W^n \rangle_{V'}, \quad (50)$$

the condition (48) is equivalent to $\xi^n = \eta^n = 0$.

Remark. The relation (48) does not depend on the magnitude of l . But it will appear in the following that there is a connexion between the time delay T and the upper bound for l .

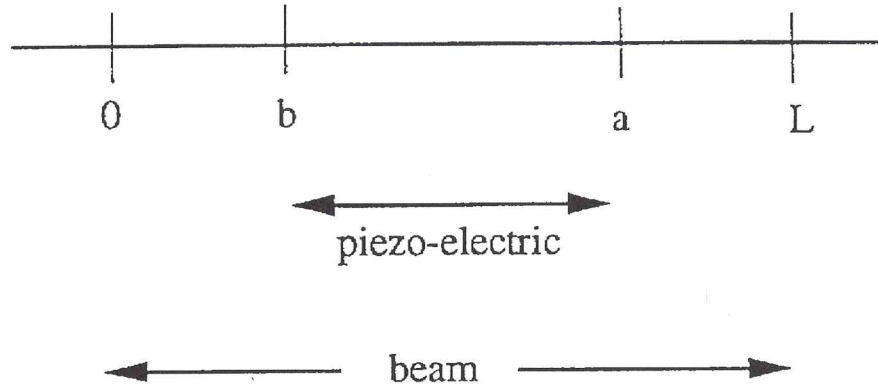


Figure 10. The beam model

3.2. A simple example in one dimension: the bending of a beam

Let us consider the case of a bending beam. The coupled model can be specified as follows:

$$\begin{cases} \text{find } u(x, t) \text{ (see Fig. 10), solution to:} \\ \rho^S \ddot{u} + D \frac{\partial^4 u}{\partial x^4} = KV[\delta'_a - \delta'_b] \end{cases} \quad (51)$$

where ρ^S and D are two mechanical constants, K is a piezo-coefficient and a, b are the abscissae of the two extremities of the piezo-electric wafer (see Fig. 10).

For the sake of brevity (see section 2.3), we assume that the beam is "simply supported" at both extremities:

$$u(0, t) = u(L, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = \frac{\partial^2 u}{\partial x^2}(L, t) = 0. \quad (52)$$

Then a simple and classical exercise leads to the following expression for the eigenmodes:

$$\begin{cases} \lambda^n = \frac{D}{\rho^S} \frac{n^4 \Pi^4}{L^4}, \\ W^n(\chi) = \frac{2}{\rho^S L} \sin\left(\frac{n \Pi \chi}{L}\right) \end{cases} \quad (53)$$

and the relation (49) becomes:

$$\begin{aligned} & \forall V(t) \in L^2(0, T), |V(t)| \leq V_{\max}, \\ & \sum_{n \geq 1} \frac{L \xi^n}{n \Pi} \sqrt{\frac{\rho^S}{D}} \left(\cos \frac{n \Pi a}{L} - \cos \frac{n \Pi b}{L} \right) \int_0^T V(s) \sin \left(\sqrt{\frac{D}{\rho^S}} \frac{n^2 \Pi^2}{L^2} (T - s) \right) ds \\ & - \eta^n \frac{n \Pi}{L} \left(\cos \frac{n \Pi a}{L} - \cos \frac{n \Pi b}{L} \right) \int_0^T V(s) \cos \left(\sqrt{\frac{D}{\rho^S}} \frac{n^2 \Pi^2}{L^2} (T - s) \right) ds = 0 \end{aligned} \quad (54)$$

Let us now consider the characteristic function $\chi_{T_0}(t)$ defined on $[0, T]$ by:

$$\chi_{T_0}(t) = \begin{cases} 0 & \text{if } t \in [0, T_0], \\ 1 & \text{otherwise} \end{cases} \quad (55)$$

The time T_0 is chosen so that:

$$\sqrt{\frac{D}{\rho^S}} \frac{\Pi^2}{L^2} (T - T_0) = 2\Pi \text{ or else } T_0 = T - \frac{2}{\Pi} \sqrt{\frac{\rho^S}{D}} L^2.$$

Furthermore it is always possible, by adjusting the characteristic function, to assume that the time delay T is such that:

$$T = \frac{2}{\Pi} \sqrt{\frac{\rho^S}{D}} L^2 k \text{ where } k \text{ is a positive integer.}$$

(Obviously it is necessary to assume that T is sufficiently large). Then, observing by a simple calculus, that if $n \neq m$, one has:

$$\begin{cases} \int_0^T \chi_{T_0}(t) \sin \sqrt{\lambda^n}(T-s) \sin \sqrt{\lambda^m}(T-s) ds = 0, \\ \int_0^T \chi_{T_0}(t) \sin \sqrt{\lambda^n}(T-s) \cos \sqrt{\lambda^m}(T-s) ds = 0, \\ \int_0^T \chi_{T_0}(t) \sin \sqrt{\lambda^n}(T-s) \cos \sqrt{\lambda^m}(T-s) ds = 0. \end{cases}$$

we can simplify the relation (49) by setting:

$$\begin{aligned} V(t) &= K_{\chi_{T_0}}(t) V(t) = K_{\chi_{T_0}}(t) \\ &\quad \left[\sum_{n \geq 1} \left(\frac{n\Pi\xi^n}{L} \sqrt{\frac{D}{\rho^S}} \sin \left(\sqrt{\frac{D}{\rho^S}} \frac{n^2\Pi^2}{L^2} (T-s) \right) \right) \right. \\ &\quad \left. + \frac{L}{n^5\Pi} \eta^n \cos \left(\sqrt{\frac{D}{\rho^S}} \frac{n^2\Pi^2}{L^2} (T-s) \right) \right] \left(\cos \frac{n\Pi a}{L} \cos n\Pi b L \right) \end{aligned}$$

K being a coefficient chosen so that $|V(t)| \leq V_{\max}$.

Let us point out that the previous series is convergent because $\lambda^n \cong n^4$. Then one obtains:

$$\sum_{n \geq 1} \left[(\xi^n)^2 + \frac{(\eta^n)^2}{n^4} \right] \left(\cos \left(\frac{n\Pi a}{L} \right) - \cos \left(\frac{n\Pi b}{L} \right) \right)^2 = 0 \quad (56)$$

which implies:

$$\forall n \geq 1 \left\{ \begin{array}{l} \xi^n = \eta^n = 0 \text{ or} \\ \cos \frac{n\Pi a}{L} = \cos \frac{n\Pi b}{L} \end{array} \right. \quad (57)$$

The last condition is equivalent to:

$$a \pm b = 2 \frac{k}{n} L \quad k \in BfN^*. \quad (58)$$

Therefore the controllability criterion is satisfied as soon as $\frac{a \pm b}{L}$ is an irrational number (for instance a/L can be rational and b/L would be irrational).

As a matter of fact it could be proven on this simple example that the previous necessary condition is almost sufficient when T is large enough (see Tucsnak, 1994, for another formulation based on the so-called H.U.M. method). But it is clear that this condition leads to negative conclusions, because it cannot be satisfied practically. This basic point is at the origin of the finite dimensional strategy that we suggest in the following. But before, let us make few remarks concerning the number of piezo-wafers which are used in the control process.

First of all let us assume that the abscissae of the two extremities of the wafer are such that:

$$a = \frac{q_1}{p_1}L \quad b = \frac{q_2}{p_2}L \quad (b > a!)$$

where $q_1, p_1, q_2, p_2 \in \mathbb{N}^*$. Then there are an infinite number of eigenvalues which correspond to uncontrolled eigenmodes. They correspond to the index n such that:

$$\cos \frac{n\pi a}{L} - \cos \frac{n\pi b}{L} = 0$$

or else:

$$n = \frac{2kp_1p_2}{q_2p_1 \pm q_1p_2} \text{ for any } k \in \mathbb{N}^*. \quad (59)$$

This very simple example can be generalized to further boundary conditions. The method is the same as the one we develop in the next section but for a more interesting situation. Let us just point out that the difficulty (for other boundary conditions), is due to the fact that the distribution of the eigen-frequencies is no more harmonic. Therefore the orthogonality between the sine functions with respect to time, is lost.

Let us now consider that there are two piezo-electric wafers. The extremities of each of them are denoted by (a, b) and (a', b') and are such that:

$$\begin{cases} a = \frac{q_1}{p_1}L, \quad b = \frac{q_2}{p_2}L \text{ for the first one, and} \\ a' = \frac{q'_1}{p'_1}L, \quad b' = \frac{q'_2}{p'_2}L \text{ for the second one.} \end{cases}$$

The indices “ n ” of the uncontrolled eigenmodes for the first wafer are given by:

$$n = \frac{2kp_1p_2}{p_1q_2 \pm q_1p_2}, \quad \forall k \in \mathbb{N}^*,$$

and for the second wafer:

$$n' = \frac{2k'p'_1p'_2}{p'_1q'_2 \pm q'_1p'_2}, \quad \forall k' \in \mathbb{N}^*.$$

Unfortunately, one can always find integers k and k' such that $n = m'$. These values are solutions of the following relation:

$$kp_1p_2(p'_1q'_2 \pm q'_1p'_2) = k'p'_1p'_2(p_1q_2 \pm q_2p_2). \quad (60)$$

It is clear that there is an infinite number of possibilities, for instance for any $m \in \mathbf{N}^*$:

$$\begin{cases} k = m[p'_1p'_2(p_1q_2 \pm q_1p_2)], \\ k' = m[p_1p_2(p'_1q'_2 \pm q'_1p'_2)]. \end{cases} \quad (61)$$

Therefore the exact control of a “simply attached beam” is impossible. This example is sufficient for suggesting to stop the investigations concerning the addition of further piezo-electric wafers.

3.3. A cylindrical shell in axisymmetrical bending

Let us consider a cylindrical shell such that the piezo-electric wafer is unoriented and positioned all around the cylinder as shown in Fig. 7. Then one has (see (35)):

$$l(v) = -V \frac{H}{c} [R - a - \epsilon] \int_{\omega_p} v_3, \quad (62)$$

and the coupled model for a pure bending shell is in this particular case:

$$\rho^S \ddot{u} + D \left[\gamma^4 u_3 + \frac{\partial^4 u_3}{(\partial \xi^2)^4} \right] = KV(t) \chi_{[a,b]}(\xi^2) \quad (63)$$

where γ is the so-called Batdorf coefficient $\left(\gamma^4 = \frac{3(1-\nu^2)}{\epsilon^2 R^2} \right) + \frac{1}{R^4}$, for the Koiter model (see Destuynder, 1990), and K is the piezo-electric constant. The characteristic function $\chi_{[a,b]}$ is equal to 1 on $[a, b]$ and 0 everywhere else. Finally, $V(t)$ is the voltage applied to the wafer and u_3 is the axisymmetrical component of the deflection (see Fig. 11).

In order to simplify our analysis we restrict the previous example to a simply supported shell at both extremities:

$$u_3(0, t) = u_3(L, t) = \frac{\partial^2 u_3}{\partial \xi^2}(0, t) = \frac{\partial^2 u_3}{(\partial \xi^2)^4}(L, t) = 0. \quad (64)$$

Then the eigenmodes and the eigenvectors of the model are:

$$\begin{cases} \lambda^n = \frac{D}{\rho^S} \left[\gamma^4 + \frac{n^4 \Pi^4}{L^4} \right], \\ W^n = \frac{2\rho^S}{L} \sin \left(\frac{n\Pi\xi}{L} \right). \end{cases} \quad (65)$$

Hence this structure is not harmonic because the frequencies $\left(\frac{1}{2\Pi} \sqrt{\lambda^n} \right)$ are not obtained from the fundamental one by a multiplication by an integer (excepted

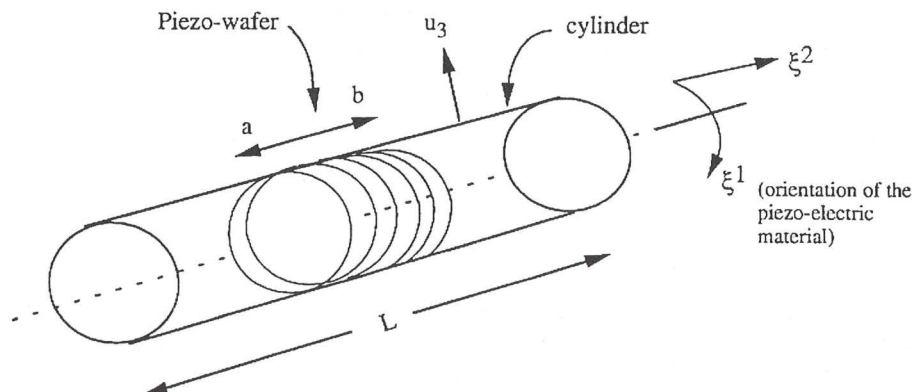


Figure 11. A cylindrical shell with a circular piezo-device

if the Batdorf's γ is zero). Similar situations can be met, for instance, with clamped boundary conditions for a bending beam. Let us point out that the basic trick which enables our theory to work is that the spectrum of the structure is asymptotically harmonic (i.e. it is almost harmonic when $\lambda^n \rightarrow \infty$). Unfortunately it is not easy to check this property for most realistic structures.

Let us set:

$$\sqrt{\lambda^n} = \sqrt{\lambda_0^n}(1 + \epsilon(n)), \text{ and } \sqrt{\lambda_0^n} = \frac{n^2 \Pi^2}{L^2} \sqrt{\frac{D}{\rho^S}}$$

$$\text{with: } \epsilon(n) = \sqrt{1 + \frac{\gamma^4 L^4}{n^4 \pi^4}} - 1. \quad (66)$$

We can observe (for instance with the sine functions) that the quantity:

$$A^{nm} = \int_0^T \sin \sqrt{\lambda^n}(T-s) \sin \sqrt{\lambda^m}(T-s) \chi_{T_0}(s) ds,$$

(χ_{T_0} and T_0 being the same as defined in the previous section at (55)), is equal to:

$$\frac{1}{2} \left[\frac{\sin \left(\sqrt{\lambda^n} - \sqrt{\lambda^m} \right) t}{\sqrt{\lambda^n} - \sqrt{\lambda^m}} - \frac{\sin \left(\sqrt{\lambda^n} + \sqrt{\lambda^m} \right) t}{\sqrt{\lambda^n} + \sqrt{\lambda^m}} \right]_{t=T-T_0}^{t=T}$$

Then using formulae (65) and (66), one deduces that for $n \neq m$ (k is defined at (55)):

$$A^{nm} = \frac{L^2}{2\Pi^2} \sqrt{\frac{\rho^S}{D}} \left[\frac{\sin(2\Pi j(n^2\epsilon(n) - m^2\epsilon(m)))}{n^2(1 + \epsilon(n)) - m^2(1 + \epsilon(m))} \right] \quad (67)$$

$$- \left. \frac{\sin(2\Pi j(n^2\epsilon(n) + m^2\epsilon(m)))}{n^2(1 + \epsilon(n)) + m^2(1 + \epsilon(m))} \right]_{j=k-1}^{j=k}$$

Therefore, assuming here again that the abscissae a and b are such that (in order to avoid that the scalar product between the eigen functions and the characteristic function of the support of the control is different from zero):

$$\frac{a \pm b}{L} \text{ is an irrational number"}$$

we can deduce from (54) that an additional **necessary** condition for the controllability (we shorten the details) can be formulated as follows:

$$\left| \begin{array}{l} \text{If } \{\xi^n\} \text{ satisfies:} \\ 1) \text{ the series } \{\xi^n\} \text{ is in } l^2, \text{ i.e. the series } \xi^{n^2} \text{ is convergent,} \\ 2) \sum_{n \geq 1} A^{nm} \xi^n = 0 \quad \forall m \in \mathbf{N}^* \\ \text{then one must have: } \xi^n = 0 \quad \forall n \in \mathbf{N}^*. \end{array} \right. \quad (68)$$

The same thing can be formulated for the cosine functions (i.e. the series: $\{\eta^n\}$).

But one can prove easily that:

$$|A^{nm}| \leq \frac{T_0 C(\gamma)}{(n^2 + m^2)} \text{ for } n \neq m$$

and where $C(\gamma)$ is a positive constant which depends of the Batdorf's γ , such that $C(0) = 0$ and T_0 is just set for convenience in the following.

Furthermore for $n = m$ and because of the definition (55) of T and T_0 , we can found a strictly positive constant α such that:

$$A^{nn} = \frac{T_0}{2} \left[1 - \frac{\sin 2\Pi n^2 \epsilon(n)}{2\Pi n^2 (1 + \epsilon(n))} \right] \geq \alpha T_0$$

Hence if ξ is solution to (68), one has also:

$$\alpha |\xi^m| \leq \sum_{n \neq m} C(\gamma) \frac{|\xi^n|}{n^2 + m^2}, \quad \forall m \geq 1$$

Because $C(0) = 0$, it can be proved from a continuity property, that for γ small enough one has $\xi = 0$. A similar conclusion can then be derived for the series η^n . But because the negative aspect of the conclusions (practically unuseful), we do not go further in this kind of results which could be easily extended to other examples.

3.4. Reduced controllability of the coupled model

Due to the difficulty of proving that the piezo-electric wafers can control the vibrations of the structure, we use now a weak formulation based on a simplified definition of the controllability. Let us define by V^N the subspace of V' (defined earlier) and such that:

$$V^N = \left\{ v = \sum_{i=1,N} \alpha_i w^i, \alpha = (\alpha_i) \in \mathbf{R}^N \right\}.$$

Obviously dimension of V^N is equal to N . Then we consider an approximation of the coupled model (shell equipped with the piezo-electric system). We set:

$$\left\{ \begin{array}{l} \text{find } u^N \in L^2(OT; V^N), \dot{u}^N \in L^2(OT; V^N) \text{ such that:} \\ \forall v \in V^N, m(\ddot{u}^N, v) + a(u^N, v) = l(v). \end{array} \right. \quad (69)$$

The bilinear forms $m(\cdot, \cdot)$ and $a(\cdot, \cdot)$ and the linear one $l(\cdot)$, have been defined earlier in Section 1. Furthermore, u^N must satisfy initial conditions:

$$\left\{ \begin{array}{l} u^N(x, 0) = u^0(x) \in V^N, \\ \dot{u}^N(x, 0) = u^1(x) \in V^N, \end{array} \right.$$

u^0 and u^1 being two given functions.

The system (69) is a finite dimensional differential equation (second order). We set:

$$\left\{ \begin{array}{l} M_{ij} = m(w^i, w^j) = \delta_{ij} \text{ (Kronecker)} \\ A_{ij} = a(w^i, w^j) = \lambda^i \delta_{ij} \\ L_i = l(w^i) \end{array} \right.$$

so that (69) is equivalent to:

$$\left\{ \begin{array}{l} \text{find } \alpha(t) \in \mathbf{R}^N, \alpha_i \in L^2(OT), \forall i, \\ \ddot{\alpha}_i + \lambda^i \alpha_i = L_i, \\ \alpha_i(0) = m(u^0, w^i), \\ \dot{\alpha}_i(0) = m(u^1, w^i). \end{array} \right.$$

Or else in a matrix form with self-explanatory notations:

$$\ddot{\alpha} + \Lambda \alpha = L \quad \alpha(0) = \alpha^0, \quad \dot{\alpha}(0) = \alpha^1 \quad (70)$$

As a matter of fact, the term L can be made explicit with respect to the voltage of each piezo-electric wafer.

If we denote by $V^p(t)$ the electric potential of each wafer one can write (see the expression of $l(\cdot)$ given in the first section):

$$L = \sum_{p=1,P} V^p(t) B^p$$

where P is the number of wafers and B^p the vector of \mathbf{R}^N corresponding to the wafer number p . For instance, if we consider a plate one deduces from the general expressions for $l(\cdot)$ (see (39)):

$$B_i^p = (\epsilon + a) \frac{H}{c} \int_{\partial\omega_p} w_{3,1}^i \nu_1.$$

This previous formula corresponds to a unioriented material (in the direction χ_1). In order to apply the classical Bellman theorem for controllability, we set the model (70) into a more usual form, with the following notations:

$$X = \begin{pmatrix} \alpha \\ \dot{\alpha} \end{pmatrix} \in \mathbf{R}^{2N}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B^1 & |B^2| & & B^p \end{pmatrix} \in \mathbf{R}^{2N} \times \mathbf{R}^p$$

$$S = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \mathbf{R}^{2N} \times \mathbf{R}^{2N}$$

$$X^0 = \begin{pmatrix} \alpha^0 \\ \alpha^1 \end{pmatrix}, \quad V = \begin{pmatrix} V^1 \\ - \\ V^p \end{pmatrix} \in \mathbf{R}^P.$$

Then (70) is equivalent to:

$$\begin{cases} \dot{X} = SX + CV \\ X(0) = X^0 \end{cases}, \quad (71)$$

and the controllability criterion can be stated as follows. We set:

$$\mathcal{C} = [C | SC | \dots | S^{2N-1}C] \in \mathbf{R}^{2N} \times \mathbf{R}^{2N \times P}$$

and the system (70) can be exactly controlled if the rank of matrix \mathcal{C} is equal to $2N$ (exactly controlled means that for any initial conditions, there exists a control law (i.e. voltages) such that the solution $X(t)$ of (71) or $(\alpha, \dot{\alpha})(t)$ solution to (70) is exactly zero after a finite time T_c).

Because of the particular structure of S and \mathcal{C} , it is possible make more explicit the previous rank condition. Assuming that there is only one piezo-device, one obtains:

$$\det \mathcal{C} = \det \begin{bmatrix} \frac{O}{B} & \left| \frac{B}{O} \right| & \frac{O}{-\Lambda B} & \left| \frac{-\Lambda B}{O} \right| \text{etc.} \end{bmatrix}$$

and (Λ is the diagonal matrix: $\{\lambda^i\}$):

$$\det \mathcal{C} \neq 0 \Leftrightarrow \det [B | \Lambda B | \Lambda^2 B \dots \Lambda^{N-1} B] \neq 0.$$

The above Vandermonde matrix is invertible as soon as:

- i) $\lambda^i \neq \lambda^j \quad \forall j \neq i$
and

ii) $B_i \neq 0 \ \forall i$ (where B_i are the components of the vector B).

The first condition ensures that all the eigenvalues λ^i are single; the second one that the piezo-wafer is able to detect all the eigenmodes. For instance in a beam, we obtain the same condition as the one already mentioned in section 2.2. For a plate (formula (39)) equipped with a unioriented piezo-electric material one has the following condition:

$$\forall i = 1, N, \int_{\partial\omega_p} w_{3,1}^i \nu_1 \neq 0$$

w_3^i being the third component of the eigenvector. In mechanical terms, it means that the boundary of the piezo-device should "avoid" the lines such that $w_{3,1}^i \nu_1$ is zero. But this is just an indication. The reader will find further comments for multiple eigenvalues in (Destuynder, Legrain, Castel, Richard, 1990).

4. Regulation laws based on quadratic criteria

Because of the conclusions of the previous section ("it is practically impossible to control exactly in a finite time a shell-like structure using piezo-wafers"), we strongly suggest to use a stabilization procedure based on the minimization of a quadratic criterion. The method is very classical and we just recall the main features of this kind of regulation law. Several numerical results are then given. But first of all, let us underline that only a finite number of eigenmodes are controlled in the method described hereafter.

4.1. The quadratic regulator

Let us consider the finite dimensional system introduced in section 2.3, but with an additional term F :

$$\begin{cases} \dot{X} = SX + CV + F, \ F = \begin{pmatrix} O \\ G \end{pmatrix} \in \mathbf{R}^{2N}, \\ X(0) = X^0. \end{cases}$$

The term F is a time dependent vector which represents external perturbations assumed to be smooth enough ($G_i(t) \in L^2(\mathbf{R})$ for instance). With another respect X^0 is the initial perturbation and N the number of eigenmodes which are controlled. For a given time control – say T – we set:

$$\begin{aligned} J(V) &= \frac{1}{2} \int_0^T (DX(t), X(t)) dt + \frac{1}{2} (EX(T), X(T)) \\ &+ \frac{1}{2} \int_0^T (NV(t), V(t)) dt \end{aligned} \quad (72)$$

where D , E and N are positively definite matrices on respectively \mathbf{R}^{2N} , \mathbf{R}^{2N} and \mathbf{R}^P . In the applications the best choice seems to be the one corresponding

to the mechanical energy for D and E and a scalar matrix for N . Then we set the following optimization problem:

$$\begin{cases} \text{minimize } J(V) \\ V = (V_p) \in L^2(OT; \mathbf{R}^P) \\ |V_p| \leq V_{\max}. \end{cases} \quad (73)$$

It is well known that (73) has a unique solution – say V – such that:

$$\begin{cases} \dot{X} = SX + CV + F, X(0) = X^0 \\ \dot{\xi} = -{}^tS\xi - DX, \xi(T) = EX(T) \\ ({}^tC\xi + NV, U - V) \geq \forall |U| \leq V_{\max}, \quad \forall t \in]O, T[\end{cases} \quad (74)$$

The solution of (74) is quite simple using for instance a gradient algorithm with respect to V . Obviously a projection is used. The state equation and its adjoint one (the second one) can be solved efficiently by Newmark algorithm. Some improvements can be done by modifications of the gradient direction in order to take into account the admissible set for the control V (see Saidi, 1997, for further indications).

But from a practical point of view (i.e. real time implementation) it is worth introducing a slightly different formulation based on the Riccati equations. First of all we consider the solution of (74) for which the control is not restricted (i.e. $V_{\max} = \infty$). Let us denote by X^∞ , ξ^∞ and V^∞ such a solution. Therefore:

$$\begin{cases} \dot{X}^\infty = SX^\infty + CV^\infty + F, X^\infty(0) = X^0, \\ \dot{\xi}^\infty = -{}^tS\xi^\infty - DX^\infty, \xi^\infty(T) = EX^\infty(T), \\ {}^tC\xi^\infty + NV^\infty = 0, \quad \forall t \in]O, T[. \end{cases}$$

The solution to this linear system is the sum of the one corresponding to F (the right hand side) and the one due to X^0 . Hence we set:

$$X^\infty = X_F + X_I, \quad \xi^\infty = \xi_F + \xi_I, \quad V^\infty = V^F + V_I$$

where:

$$\begin{cases} \dot{X}_F = SX_F + CV_F + F, X(0) = 0 \\ \dot{\xi}_F = -{}^tS\xi_F - DX_F, \xi_F(T) = EX_F(T) \\ {}^tC\xi_F + NF_F = 0, \end{cases} \quad (75)$$

and

$$\begin{cases} \dot{X}_I = SX_I + CV_I, X_I(0) = X^0 \\ \dot{\xi}_I = -{}^tS\xi_I - DX_I, \xi_I(T) = EX_I(T) \\ {}^tC\xi_I + NV_I = 0. \end{cases} \quad (76)$$

Assuming that F is a well-known perturbation the system (75) can be solved once for all so that we can construct a linear operator which transforms $F(t)$ into $V_F(t)$. For instance if $F(t)$ is decomposed on a basis, the same linear

combination which gives F , will give the regulation law V_F , but in the basis constructed by solving each elementary system with a right hand side equal to a basis vector for F .

The second system (76) is more difficult to solve as far as X^0 is random. A classical strategy is to use the Riccati equation. The basic idea is the following. We look for solutions of (76) such that:

$$\xi = RX \quad (77)$$

where R is $2N \times 2N$ symmetrical matrix which is time dependent. Then a simple calculus leads to:

$$\begin{cases} \dot{R}X_I + RSX_I - RCN^{-1}CRX_I + {}^tSRX_I + DX_I = 0 \\ R(T)X_I(T) = EX_I(T). \end{cases}$$

Therefore a solution is obtained by choosing R being the solution of the Riccati equation:

$$\begin{cases} \dot{R} + RS + {}^tSR - RCN^{-1}{}^tCR + D = 0, \\ R(T) = E. \end{cases} \quad (78)$$

One classical possibility is to use Runge-Kutta algorithm (order 4). But several meaningful improvements can be obtained if we use a semi-implicit scheme (partially implicit) as it has been proved by Dubois and Saidi (1995). Such improvements are mostly valuable for real time implementations. Then the optimal control V which should be applied is defined as follows:

$$V^\infty(t) = V_F(t) - N^{-1}{}^tCR(t)X(t)$$

but V^∞ is not necessarily acceptable. Hence we replace the previous formula by the following one:

$$V_{\text{opt}}(t) = P_{0,V_{\text{max}}}[V^\infty(t)]. \quad (79)$$

Another very convenient and frequently used strategy consists in choosing T quite large in order to use the stationary Riccati solution (which does not depend any more on time).

If we prefer to choose T small (which is not very satisfying from the mechanical point of view), it is better to consider the inverse of R which is solution of:

$$\dot{K} - SK - K^tS - (DK, K) + CN^{-1}{}^tC = 0$$

and

$$K(T) = E^{-1}$$

This system is linear if $D = 0$ (or neglected for T small). Then the solution is faster and more precise.

5. Optimum design of the wafers

The efficiency of a regulation law for the voltage of the piezo-electric wafers depends strongly on the shape and the position of the piezo-devices. Hence, it is worth to define a method which enables one to improve the efficiency of the control loop and based on an adjustment of the piezo-wafer geometry. Therefore we define the following problem:

$$\begin{aligned} & \text{minimize } \{ \mathcal{J}(\omega_p) = \min J(V) \} \\ & \omega_p \in \omega_{ad} \quad |V(t)| \leq V_{\max} \\ & V(t) \in L^2(OT) \end{aligned} \quad (80)$$

where $J(\cdot)$ is the quadratic criterion defined at (72) and $\mathcal{J}(\omega_p)$ is its optimum with respect to the voltage-function: $V(t)$. The set ω_{ad} denotes an admissible set for the wafer-geometries. As a matter of fact they will be characterized by a mapping with a suitable smoothness. Let us point out that $\mathcal{J}(\omega_p)$ is well defined because the minimum of \mathcal{J} is unique. But the minimisation of \mathcal{J} is not well posed.

One can prove that there are continuity and derivability of \mathcal{J} as soon as the mapping which defines ω_p is correctly chosen. The existence of at least one solution to (80) rests upon the compactness of the set ω_{ad} . In the following we define an expression for the first order derivative of $\mathcal{J}(\omega_p)$. It will be convenient for mechanical interpretation but also for practical use in order to compute the solution to (80). Let us point out that for sake of brevity we restrict our analysis to plates.

5.1. A brief recall on optimum design using parametric representation of open sets

The method we are using is the so-called Lagrangian technique. It consists of representing varying domains on a reference open set, through a mapping. The required smoothness of this mapping is governed by the order of the partial derivative operator that we consider on the varying domains. In the case of shells this order is four. Then the domain must be represented by a mapping which is an element of the space $W^{2,\infty}(\omega)$.

5.1.1. The domain representation by a mapping (Murat, Simon, 1976)

Let us consider a vector function – say θ – the components of which are denoted by θ_α ($\alpha = 1, 2$). They are defined over the plane open set ω (we only consider plates as said previously). Each function θ_α belongs to the space $W^{2,\infty}(\omega)$. We assume that it satisfies the following properties:

- i) θ_α has a support strictly included in ω ,
- ii) we assume that there is only one wafer (just for sake of simplicity) and that the support of θ_α is contained in a neighbourhood of this wafer.

Let us now define a family of open sets ω^η which are all globally equal to ω and such that:

$$\omega^\eta = F^\eta(\omega),$$

where:

$$\forall x \in \omega, x = (x_\alpha), F^\eta(x) = x^\eta = x + \eta\theta(x), \quad (81)$$

η denoting a small enough real and positive parameter. It is easy to check that F^η is an internal transformation of ω which can allow translations and rotations of the wafer ω_p . But it also authorizes deformation of the boundary of ω_p .

Let, now, ϕ^η be a function defined on the domain ω^η . We associate $\phi(\eta)$ defined on ω by:

$$\phi(\eta)(x) = \phi^{\eta_0} F^\eta(x) \quad \forall x \in \omega. \quad (82)$$

Then several calculus rules can be derived from this relation. Let us summarize the most useful of them.

i) *Derivation*

$$\frac{\partial \phi^\eta}{\partial x_\alpha^\eta} = \frac{\partial \phi(\eta)}{\partial x_\beta} \frac{\partial x_\beta}{\partial x_\alpha^\eta}$$

(with the implicit summation convention over the repeated indices). From (81) and for η small enough, one can write:

$$\frac{\partial \phi^\eta}{\partial x_\alpha^\eta} = \left[\frac{\partial \phi(\eta)}{\partial x_\mu} (\delta_{\mu\alpha} - \eta \partial_\alpha \theta_\mu + \eta^2 \dots \text{etc.}) \right]$$

ii) *Integration*

$$\int_{\omega^\eta} \phi^\eta = \int_\omega \phi(\eta) \det \left(I + \eta \frac{\partial \theta}{\partial x} \right)$$

where

$$\left(\frac{\partial \theta}{\partial x} \right)_{\alpha\beta} = \partial_\beta \theta_\alpha = \frac{\partial \theta_\alpha}{\partial x_\beta}.$$

By iterating the previous derivation formula, one obtains:

$$\begin{aligned} \frac{\partial^2 \phi^\eta}{\partial x_\alpha^\eta \partial x_\beta^\eta} &= \frac{\partial}{\partial x_\alpha^\eta} \left[\frac{\partial \phi(\eta)}{\partial x_\mu} (\delta_{\mu\beta} - \eta \partial_\beta \theta_\mu + \dots) \right] \\ &= \frac{\partial}{\partial x_\lambda} \left[\frac{\partial \phi(\eta)}{\partial x_\beta} - \eta \frac{\partial \phi(\eta)}{\partial x_\mu} \partial_\beta \theta_\mu \right] (\delta_{\lambda\alpha} - \eta \partial_\alpha \theta_\lambda + \dots) \end{aligned}$$

and finally:

$$\begin{aligned} \frac{\partial^2 \phi^\eta}{\partial x_\alpha^\eta \partial x_\beta^\eta} &= \frac{\partial^2 \phi(\eta)}{\partial x_\alpha \partial x_\beta} - \eta \left[\frac{\partial^2 \phi(\eta)}{\partial x_\alpha \partial x_\mu} \partial_\beta \theta_\mu + \frac{\partial \phi(\eta)}{\partial x_\mu} \frac{\partial^2 \theta_\mu}{\partial x_\alpha \partial x_\beta} \right. \\ &\quad \left. + \frac{\partial^2 \phi(\eta)}{\partial x_\beta \partial x_\mu} \partial_\alpha \theta_\mu \right] + \eta^2 \dots \text{etc.} \end{aligned} \quad (83)$$

This formula will be very important in the following. Let us first point out that even if the previous calculus seems to be formal, it is not, as long as h is

small enough. The basic point is just to make sense of the following identity (see Destuynder, Djaoua, 1981):

$$\left[\frac{\partial F^\eta}{\partial x} \right]^{-1} = \left[I + \eta \frac{\partial \theta}{\partial x} \right]^{-1} = I - \eta \frac{\partial \theta}{\partial x} + \eta^2 \left(\frac{\partial \theta}{\partial x} \right)^2 \dots \text{etc.} \quad (84)$$

5.1.2. The Lagrangian formulation of the reduced coupled-model

In order to be able to use the variational formulation, we restrict the analysis done in this section, to the reduced coupled-model (i.e. the solution obtained by a truncation on the eigenmode series). Therefore, for each position ω_p^η of the piezo-wafer, we consider u^N solution of:

$$\begin{aligned} &\text{find } u^N \in \mathcal{C}^1(OT; V^N) \text{ such that:} \\ &\forall v \in V^N, m(\ddot{u}^N, v) + a(u^N, v) = l^\eta(v) \end{aligned} \quad (85)$$

with the initial conditions:

$$\begin{cases} u^N(x, 0) = u^0(x), \\ \dot{u}^N(x, 0) = u^1(x) \end{cases} \quad (86)$$

and V^N is the finite dimensional space span by the N first eigenvectors. The bilinear forms $m(\cdot, \cdot)$ and $a(\cdot, \cdot)$ are those defined in section 2. The right hand side $l^\eta(\cdot)$ is the linear (and continuous on V^N) form which takes into account the effect of the piezo-electric-device. The bilinear forms $m(\cdot, \cdot)$ and $a(\cdot, \cdot)$ do not depend on η (because we have neglected the stiffness contribution of the devices). But the change of variables we introduced in (82) depends on η . Let us recall that l^η is defined for a plate by:

$$\begin{aligned} l^\eta(v) = & \frac{-2(\epsilon + a)}{c} V(t) \int_{\omega_p} h_{\alpha\beta} \partial_{\alpha\beta} v \\ & + \eta \frac{2(\epsilon + a)}{c} V(t) \left[\int_{\omega_p} h_{\alpha\beta} (\partial_{\alpha\mu} v \partial_\beta \theta_\mu + \partial_{\beta\mu} v \partial_\alpha \theta_\mu + \partial_\mu v \partial_{\alpha\beta} \theta_\mu) \right. \\ & \left. - \int_{\omega_p} h_{\alpha\beta} \partial_{\alpha\beta} v \operatorname{div} \theta \right] + \eta^2 \dots \text{etc.} \end{aligned} \quad (87)$$

Obviously the partial derivatives which appear in the previous expression, are with respect to the coordinates χ_α (on the reference open set). Let us set:

$$u^N(\eta) = u^{0N} + \eta u^{1N} + \eta^2 \dots \text{etc.} \quad (88)$$

Then introducing (88) into (85), we obtain, by equating the terms with the same power with respect to η :

$$\begin{cases} u^{0N} = u^N \in \mathcal{C}^1(OT; V^N) \text{ such that:} \\ \forall v \in V^N, m(u^{0N}, v) + a(u^{0N}, v) = l^0(v), \end{cases}$$

with the initial conditions:

$$\begin{cases} u^{0N}(x, 0) = u^0(x), \\ \dot{u}^{0N}(x, 0) = u^1(x). \end{cases}$$

Then:

$$u^{1N} \in \mathcal{C}^1(OT; V^N) \text{ such that:} \quad (89)$$

$$\begin{aligned} \forall v \in V^N, \quad m(\ddot{u}^{1N}, v) + a(u^{1N}, v) = & -\frac{2(\epsilon + a)}{c} V(t) \left[\int_{\omega_p} h_{\alpha\beta} \partial_{\alpha\beta} v \operatorname{div} \theta \right. \\ & \left. + \int_{\omega_p} h_{\alpha\beta} (\partial_{\alpha\mu} v \partial_{\beta} \theta_{\mu} + \partial_{\beta\mu} v \partial_{\alpha} \theta_{\mu} + \partial_{\mu} v \partial_{\alpha\beta} \theta_{\mu}) \right] \\ & - \int_{\omega} \rho^S \ddot{u}^{0N} v \operatorname{div} \theta - \int_{\omega} R_{\alpha\beta\mu\lambda} \partial_{\alpha\beta} u^{0N} \partial_{\mu\lambda} v \div \theta \\ & + \int_{\omega} R_{\alpha\beta\mu\lambda} (\partial_{\alpha\beta} u^{0N} \partial_{\beta} \theta_{\mu} + \partial_{\beta\mu} u^{0N} \partial_{\alpha} \theta_{\mu} + \partial_{\mu} u^{0N} \partial_{\alpha\beta} \theta_{\mu}) \partial_{\xi\lambda} v \\ & + \int_{\omega} R_{\alpha\beta\xi\lambda} \partial_{\alpha\beta} u^{0N} (\partial_{\xi\mu} v \partial_{\lambda} \theta_{\mu} + \partial_{\lambda\mu} v \partial_{\xi} \theta_{\mu} + \partial_{\mu} v \partial_{\xi\lambda} \theta_{\mu}) \end{aligned}$$

where $R_{\alpha\beta\xi\lambda}$ is the bending stiffness tensor of the plate given by (E is the Young modulus and ν the Poisson coefficient):

$$R_{\alpha\beta\mu\lambda} = \frac{E\epsilon^3}{3(1-\nu^2)} \{ (1-\nu)(\delta_{\alpha\mu}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\mu}) + 2\nu\delta_{\alpha\beta}\delta_{\mu\lambda} \}.$$

The equation (89) defining u^{1N} is complemented with the initial conditions:

$$\begin{cases} u^{1N}(x, 0) = 0, \\ \dot{u}^{1N}(x, 0) = 0. \end{cases}$$

Let us notice that u^{1N} can also be defined by:

$$u^{1N} = \lim_{\eta \rightarrow 0} \frac{u^N(\eta) - u^{0N}}{\eta} = \lim_{\eta \rightarrow 0} \frac{u^N(\eta) - u^N(0)}{\eta}.$$

The limit must be considered in the space $\mathcal{C}^1(OT; V^N)$. The full justification is quite standard and rests upon the implicit function theorem. As a matter of fact it can be proved that $u^N(\eta)$ is analytical with respect to η (see Destuynder, 1976).

5.1.3. Calculus of the gradient of the criterion $\mathcal{J}(\omega_p)$

First of all let us make precise the derivative that we consider. Setting:

$$\begin{aligned}\mathcal{J}(\omega_p^\eta) &= f(\eta) = \frac{1}{2} \int_0^T [m^\eta(\dot{u}, \dot{u}) + a^\eta(u, u) + \xi V^2(t)] dt \\ &\quad + \frac{\chi}{2} [m^\eta(\dot{u}, \dot{u})(T) + a^\eta(u, u)(T)]\end{aligned}$$

where ξ and χ are two real constants and where we have set for arbitrary elements u, v in V^N :

$$\begin{aligned}m^\eta(u, v) &= \int_\omega \rho^S uv \det \left(I + \eta \frac{\partial \theta}{\partial x} \right) \\ a^\eta(u, v) &= \int_\omega R_{\alpha\beta\epsilon\delta} \partial_\mu \left(\partial_\lambda u \frac{\partial x_\lambda}{\partial x_\beta^\eta} \right) \frac{\partial x_\mu}{\partial x_\alpha^\eta} \partial_\gamma \left(\partial_\xi v \frac{\partial x_\xi}{\partial x_\epsilon^\eta} \right) \frac{\partial x_\gamma}{\partial x_\delta^\eta} \\ &\quad \det \left(I + \eta \frac{\partial \theta}{\partial x} \right)\end{aligned}$$

and for η small enough one has for instance:

$$\frac{\partial x_\lambda}{\partial x_\beta^\eta} = \delta_{\lambda\beta} - \eta \partial_\beta \theta_\lambda + \dots \text{etc.}$$

Then a simple (formal but rigorous!) calculus leads to the formula:

$$\begin{aligned}G(\omega_p^o) &= f'(0) = \lim_{\eta \rightarrow 0} \frac{f(\eta) - f(0)}{\eta} \\ &= \frac{1}{2} \int_0^T \left[\frac{\partial m^o}{\partial \eta}(\dot{u}^N, \dot{u}^N) + \frac{\partial a^o}{\partial \eta}(u^N, u^N) \right] \\ &\quad + \int_0^T [m^o(\dot{u}^N, \dot{u}^{1N}) + a^o(u^N, u^{1N})] \\ &\quad + \frac{\chi}{2} \left[\frac{\partial m^o}{\partial \eta}(\dot{u}^N, \dot{u}^N)(T) + \frac{\partial a^o}{\partial \eta}(u^N, u^N)(T) \right] \\ &\quad + \chi [m^o(\dot{u}^N, \dot{u}^{1N})(T) + a^o(u^N, u^{1N})(T)].\end{aligned}$$

Let us point that in the previous expression, the following equality has been used:

$$(\dot{u})^{1N} = \frac{\partial \dot{u}^N}{\partial \eta}(\eta = 0) = \frac{\partial}{\partial t}(u^{1N}) = \frac{\partial}{\partial t} \left(\frac{\partial u^N}{\partial \eta}(\eta = 0) \right) (= u^{1N}).$$

The proof is left to the reader.

We now introduce the adjoint state variable by:

$$\begin{cases} \text{find } p(x, t) \in \mathcal{C}(O, T; V^N) \text{ such that:} \\ \forall v \in V^N, m(\ddot{p}, v) + a(p, v) = -m(\ddot{u}, v) + a(u, v) \end{cases}$$

with the **final** conditions:

$$\begin{cases} \forall v \in V^N, m(p(T), v) = \chi m(\dot{u}^N, v)(T) \\ \forall v \in V^N, m(\dot{p}(T)) = -[m(\dot{u}^N(T), v) + \chi a(u(T), v)]. \end{cases}$$

Then one deduces that:

$$\begin{aligned} K &= \int_0^T [m(\dot{u}^N, \dot{u}^{1N}) + a(u^N, u^{1N})] + \\ &\quad \chi[m(\dot{u}^N, \dot{u}^{1N})(T) + a(\dot{u}^N, \dot{u}^{1N})(T)] \\ &= m(\dot{u}^N, u^{1N})(T) - \int_0^T m(\ddot{u}^N, u^{1N}) + \int_0^T a(u^N, u^{1N}) \\ &\quad + \chi[m(\dot{u}^N, \dot{u}^{1N}) + a(u^N, u^{1N})](T), \end{aligned}$$

and from the definition of the adjoint state function $p(x, t)$, we deduce that:

$$\begin{aligned} K &= \int_0^T [m(\ddot{p}, u^{1N}) + a(p, u^{1N})] + m(\dot{u}^N, u^{1N})(T) \\ &\quad + \chi[m(\dot{u}^N, \dot{u}^{1N}(T) + a(u^N, u^{1N})(T)] \\ &= m(\dot{p}^N, u^{1N})(T) - m(p, \dot{u}^{1N})(T) + \int_0^T p(\ddot{u}^{1N}, p) \\ &\quad + a(u^{1N}, p)] + m(\dot{u}^N, u^{1N})(T) \\ &\quad + \chi[m(\dot{u}^N, \dot{u}^{1N})(T) + a(u^N, u^{1N})(T)] \\ &= \int_0^T [m(\ddot{u}^N, p) + a(u^1, p)] \\ &= - \int_0^T \frac{\partial m^o}{\partial \eta}(\ddot{u}^N, p) - \int_0^T \frac{\partial a^o}{\partial \eta}(u^N, p) + \int_0^T \frac{\partial l^o}{\partial \eta}(p) \end{aligned}$$

the last equality being deduced from the characterization of u^{1N} .

Finally we can summarize the result as follows:

$$\begin{aligned} G(\omega_p^o) &= - \int_0^T \frac{\partial m^o}{\partial \eta}(\ddot{u}^N, p) - \int_0^T \frac{\partial a^o}{\partial \eta}(u^N, p) \\ &\quad + \frac{1}{2} \int_0^T \frac{\partial m^0}{\partial \eta}(\dot{u}^N, \dot{u}^N) + \int_0^T \frac{\partial l^o}{\partial \eta}(p) + \frac{1}{2} \int_0^T \frac{\partial a^o}{\partial \eta}(u^N, u^N) \\ &\quad + \frac{1}{2} \frac{\partial m}{\partial \eta}(\dot{u}^N, \dot{u}^N)(T) + \frac{1}{2} \frac{\partial a}{\partial \eta}(u^N, u^N)(T). \end{aligned} \tag{90}$$

This expression of the domain derivatives can also be transformed into another one which is only a function of θ along the boundary of ω_p . This can be obtained using Stokes formula. The calculus is a little bit long but without major difficulty. We obtain the following expression which takes into account

the continuity of u^N and p across $\partial\omega_p$ (because the eigenvectors W^n are smooth in the interior of ω).

$$g(\omega_p^o) = \frac{2(\epsilon + a)}{c} \left[\int_{\gamma_p} h_{\alpha\beta} [\partial_{\alpha\mu} p \theta_\mu \nu_\beta + \delta_{\mu p} \delta_\alpha \theta_\mu \nu_\beta - \partial_{\alpha\beta} p \theta_\lambda \nu_\lambda] \right] \quad (91)$$

where: $\gamma_p = \partial\omega_p$ and $\nu = (\nu_\alpha)$ is the unit normal; the jump $[\]$ of a function being evaluated in the direction of ν . But as $h_{\alpha\beta}$ is zero outside ω_p , $[X]$ represents the value of the quantity X from the inner side of ω_p and ν is oriented outwards ω_p .

Let us make more explicit the expression (91). If the piezo-electric material is isotropical one has:

$h_{\alpha\beta} = H\delta_{\alpha\beta}$ (Kronecker symbol), and therefore:

$$G(\omega_p^o) = -\frac{2(\epsilon + a)}{c} H \int_{\gamma_p} [\partial_{\alpha\mu} p \theta_\mu \nu_\alpha + \partial_\mu p \partial_\alpha \theta_\mu \nu_\alpha - \Delta p \theta_\lambda \nu_\lambda]$$

and if the boundary γ_p is piecewise linear, one has (assuming χ_1 is colinear to the boundary of ω_p and χ_2 being normal):

$$G(\omega_p^o) = \int_0^T \int_{\gamma_p} \left[g(\theta, \nu) + k \frac{\partial}{\partial \nu}(\theta, \nu) \right]$$

where:

$$\begin{cases} g = \frac{2(\epsilon+a)}{c} H(\partial_{11}p), \\ k = -\frac{2(\epsilon+a)}{c} H\partial_2 p, \end{cases}$$

(we assume that θ is colinear to ν for sake of simplicity). Hence if $\partial_{11}p$ is positive on γ_p it is suitable to increase the wafer (locally). The optimum is locally obtained where p is linear along γ_p . The coefficient $\frac{\partial}{\partial \nu}(\theta, \nu)$ is only different from zero if we deform the wafer. The sign of k defines how to deform ω_p . Let us notice that it is necessary to compute the adjoint state p in the optimum design problem. But the solution is very much simplified if we use the eigenmode decomposition.

6. Conclusions

We have suggested in this paper an analysis of the piezo-electric devices coupled with shell or plate models. The main results obtained are the following:

- i) It is quite impossible to perform an exact control of the vibrating structure. Hence only a finite number of eigenmodes can be considered in a realistic process.

- ii) The control law can be computed using classical L.Q.R algorithms (Least Square Regulator). The Riccati model can also be applied as far as the external perturbation is quite well known. Therefore real time procedure based on Riccati model seems to be a nice choice. Then a low frequency adjustment of the regulator can be used .
- iii) Because of the singular behavior of the piezo-effect, the regulation loops are very sensitive to the geometry of the piezo-wafer (see Saidi, 1997, for numerical examples). Hence an optimum design algorithm is suitable and the gradient of the L.Q.R. criterion can be numerically evaluated by simple formula. The expression has been obtained using domain derivatives strategy. Then an optimum design algorithm could be used in order to improve the efficiency of the piezo-devices.

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