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# Trade-off between energy and entropy of information

by

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**Abstract:** Two ways of speed stabilization of the d-c motor are considered. One way consists in the use of additional kinetic energy accumulated in a wheel with a great inertial moment $J$. The other way consists in the use of additional information supplied by the feedback loop with gain $K$. In both cases the motor is under influence of the same white Gaussian noise. These two ways of stabilization are compared under the assumption of the same value of the speed error in the steady state.

**Keywords:** entropy of information, energy, stabilization, d-c motor, white noise

## 1. Introduction

In the paper the relation between energy and entropy of information is found. This serious problem was considered in Kolmogorov (1956;1968), McMillan (1953), Rissanen (1978), Verdugo Laro and Rathie (1978), Żurek (1989;1990), Jaynes (1957), Brillouin (1962), Bennet (1987), Axzel and Daroczy (1973), Chaitin (1987), Gray (1990), Kaczorek (1981). This relation seems to be very important from theoretical point of view and, besides, it has some practical implications for the design of electrical and hydraulical systems. The references give an idea as to where this and related problems have been considered

The speed stabilization of the direct current motor with external excitation is considered. Two ways of speed stabilization are analyzed. One way is by using a wheel with a great inertial moment $J$. The second is by the use of feedback with a static controller of gain $K$. The motor is supplied by the constant voltage $u$ and is under the influence of the white Gaussian disturbance, Kwakernaak, Sivan (1972).

The same accuracy of these two ways of stabilization is assumed.

The relation between the kinetic energy of the inertial wheel and the amount of information in the feedback is found.

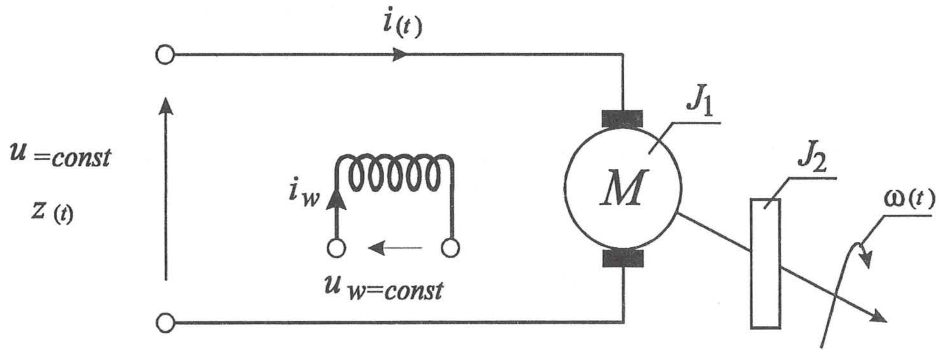


Figure 1. The scheme of the speed stabilization of the motor using inertial wheel

## 2. The two methods of stabilization

### 2.1. The first method of stabilization (using the inertial wheel)

In Fig. 1, the scheme of motor stabilization is presented. It is assumed that the current in the excitation circuit has attained its steady state.

For the sake of simplicity the nonlinear effects such as saturation, histeresis and the reaction of the winding are neglected.

The electrical moment of the motor is described by the relation

$$M_e(t) = k_1 i_w i(t) \tag{1}$$

where
$i_w$ denotes the steady state current in the excitation circuit
$i(t)$ the current in the main circuit
$k_1$ constant coefficient.

The equation of the voltage balance is

$$u + z(t) = Ri(t) + L\frac{di(t)}{dt} + e(t) \tag{2}$$

where
the voltage $u$ represents the set-value of the speed
$z(t)$ represents the white Gaussian noise; we assume farther that $z(t) = Zw(t)$, where $Z = 1$ represents intensity of this noise.
$R$ the resistance of the main circuit
$L$ the inductance of the main circuit

$$e(t) = k_2\omega(t) \tag{3}$$

voltage induced by the rotation
$\omega(t)$ angular velocity of the motor
$k_2$ constant coefficient

The equation of the moments is as follows:

$$M_{st} + (J_1 + J_2)\frac{d\omega}{dt} = k_1 i_w i(t) \tag{4}$$

where
$M_{st}$ static moment
$J_1$ the inertial moment of the moving parts of the motor
$J_2$ the inertial moment of the wheel

In the equilibrium state, if $z(t) = 0$, the angular acceleration

$$\frac{d\omega\,(t)}{dt} = 0 \tag{5}$$

and

$$\frac{di\,(t)}{dt} = 0 \tag{6}$$

The nominal value $\omega_n$ of the speed after taking into account (3) and (6) in the equation (2) is

$$\omega_n = \frac{u - i_n R}{k_2} \tag{7}$$

where
$i_n$ is the nominal value of the current in the main circuit.

After denoting

$$\Delta i(t) = i(t) - i_n \tag{8}$$

$$\Delta\omega(t) = \omega(t) - \omega_n \tag{9}$$

$\frac{L}{R} = T_0$ the electromagnetic time constant of the main circuit
$\frac{J_1 R}{i_w k_1 k_2} = T_1$ the electromechanical time constant of the motor
$\frac{J_2 R}{i_w k_1 k_2} = T_2$ the time constant of the wheel
equations (2) and (4) can be written down as follows:

$$\begin{bmatrix} \frac{d\Delta\omega(t)}{dt} \\ \frac{d\Delta i(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{R}{k_2}\frac{1}{T_1+T_2} \\ -\frac{1}{T_0}\frac{k_2}{R} & -\frac{1}{T_0} \end{bmatrix} \begin{bmatrix} \Delta\omega(t) \\ \Delta i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{T_0 R} \end{bmatrix} z(t) \tag{10}$$

The total kinetic energy with inertial wheel is equal to:

$$E_2 = \frac{1}{2}(J_1 + J_2)\omega^2(t) \tag{11}$$

The kinetic energy without inertial wheel is equal to:

$$E_1 = \frac{1}{2}J_1\omega^2(t) \tag{12}$$

The additional energy of the inertial wheel supplied to the system is equal to

$$\Delta E = E_2 - E_1 = \frac{1}{2} J_2 \omega^2(t) \tag{13}$$

The increment of this energy $\Delta E$ caused by the increment of the speed $\Delta\omega$ according to relation (9) is equal to

$$\delta(\Delta E) = J_2 \omega_n \Delta\omega(t) + \frac{1}{2} J_2 \left[\Delta\omega(t)\right]^2 \tag{14}$$

or in the differential form

$$d(\Delta E) = J_2 \omega_n d\omega \tag{15}$$

Returning to the equation (10) we denote

$$A = \begin{bmatrix} 0 & \frac{R}{k_2} \frac{1}{T_1+T_2} \\ -\frac{1}{T_0} \frac{k_2}{R} & -\frac{1}{T_0} \end{bmatrix} \tag{16}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{RT_0} \end{bmatrix} \tag{17}$$

The steady state error is equal to

$$I_1 = \lim_{T \to \infty} \frac{1}{T} \left\{ \int_0^T x^T(t) W x(t) dt \right\} = tr\left(B^T P B\right) \tag{18}$$

where

$$x(t) = \begin{bmatrix} \Delta\omega(t) \\ \Delta i(t) \end{bmatrix} \tag{19}$$

$W$ is a symmetric nonnegative definite matrix

$$W = \begin{bmatrix} 1 & 0 \\ 0 & \frac{R^2}{k_2^2} \end{bmatrix} \tag{20}$$

and $P$ is a constant matrix which satisfies the matrix algebraic equation

$$0 = A^T P + PA + W \tag{21}$$

The assumption that the matrix $A$ is asymptotically stable guarantees that this algebraic equation has a unique solution.

The substitution of (16) and (20) into the equation (21) gives the solution for matrix $P$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \tag{22}$$

where

$$P_{11} = \frac{1}{2}T_0 + T_1 + T_2 \tag{23}$$

$$P_{12} = \frac{1}{2}\frac{R}{k_2}T_0 \tag{24}$$

$$P_{22} = \frac{1}{2}\frac{R^2}{k_2^2}T_0\left(1 + \frac{T_0}{T_1 + T_2}\right) \tag{25}$$

The steady state error according to (18) after the substitution of (17) and (23)-(25) is equal to

$$I_1 = \frac{1}{2k_2^2}\left(\frac{1}{T_0} + \frac{1}{T_1 + T_2}\right) \tag{26}$$

## 2.2. The second method of stabilization

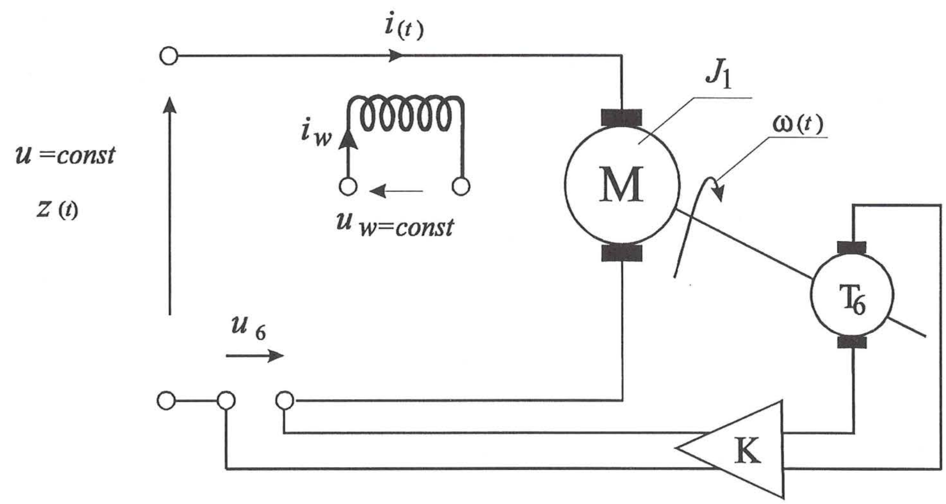


Figure 2. The scheme of speed stabilization of the motor using feedback loop

In Fig. 2 the scheme of stabilization of the speed of the motor using a tachometer $T_G$ and the gain $K$ in the feedback loop is presented.

The equations of motion are the following:

$$M_e(t) = k_1 i_w i(t)$$

$$u + z(t) = Ri(t) + L\frac{di(t)}{dt} + e(t) + u_G(t) \tag{27}$$

where

$u_G(t)$ the voltage from tachometer

$$u_G(t) = K\Delta\omega(t) \tag{28}$$

The equation of moments

$$M_{st} + J_1 \frac{d\omega(t)}{dt} = k_1 i_w i(t) \tag{29}$$

Following the same way as in the first method of stabilization we obtain a matrix equation corresponding to the matrix equation (10):

$$\begin{bmatrix} \frac{d\Delta\omega(t)}{dt} \\ \frac{d\Delta i(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{T_1}\frac{R}{k_2} \\ -\frac{k_2+K}{RT_0} & -\frac{1}{T_0} \end{bmatrix} \begin{bmatrix} \Delta\omega(t) \\ \Delta i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{T_0 R} \end{bmatrix} z \tag{30}$$

The solution of the corresponding Lyapunov equation gives the steady state error equal to

$$I_2 = \frac{1}{2k_2^2}\left(\frac{1}{T_0} + \frac{k_2}{(k_2+K)\,T_1}\right) \tag{31}$$

These two systems of stabilization are equivalent if the steady state errors are the same

$$I_1 = I_2 \tag{32}$$

Comparing (26) with (31) gives

$$T_2 = \frac{K}{k_2} T_1 \tag{33}$$

or

$$J_2 = \frac{K}{k_2} J_1 \tag{34}$$

The substitution of the relation (33) into (14) yields

$$KJ_1 = \frac{k_2\delta(\Delta E)}{\omega_n \Delta\omega + \frac{1}{2}\left[\Delta\omega\right]^2} \tag{35}$$

or in a differential form

$$KJ_1 = \frac{k_2}{\omega_n}\frac{d\Delta E}{d\omega} \tag{36}$$

The increment of entropy of information in the system with the second method of stabilization is, Kolmogorov (1968):

$$\Delta H = H_2 - H_1 \tag{37}$$

where

$H_2$ denotes the entropy of information of the system with feedback
$H_1$ denotes the entropy of information of the system without feedback

The entropy of information for a continuous system is

$$H(p) = -\int_{-\infty}^{\infty} p(x)\ln[p(x)]dx \tag{38}$$

where $p(x)$ is the density of the probability distribution.

In this case

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{39}$$

where $m$ is the expected value, and $\sigma$ the dispersion or standard deviation.

The substitution of the expression (39) into (38) gives

$$H = \left[\frac{\ln\sigma\sqrt{2\pi}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}dx + \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{(x-m)^2}{2\sigma^2}e^{-\frac{(x-m)^2}{2\sigma^2}}dx\right] \tag{40}$$

Putting $\frac{x-m}{\sigma\sqrt{2}} = t$ in (40) gives

$$H = \left[\frac{\ln\sigma\sqrt{2\pi}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2 e^{-t^2}dt\right] \tag{41}$$

But

$$\left.\begin{array}{l} \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} e^{-t^2}dt = 1 \\ \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} t^2 e^{-t^2}dt = \frac{1}{2} \end{array}\right\} \tag{42}$$

as the integrals of Euler-Poisson.

Finally

$$H = \ln\sigma\sqrt{2\pi} + \frac{1}{2} \tag{43}$$

Returning to (37) gives

$$\Delta H = \ln\sigma_2\sqrt{2\pi} - \ln\sigma_1\sqrt{2\pi} = \ln\frac{\sigma_2}{\sigma_1} \tag{44}$$

The steady state variance matrix $Q$ is the solution of the following algebraic equation

$$AQ + QA^T + BzB^T = 0 \tag{45}$$

The solution of this equation in the case of the state equation (30) is

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \tag{46}$$

where

$$q_{11} = \frac{1}{2k_2T_1(k_2+K)} = \sigma_2^2 \tag{47}$$

$$q_{22} = \frac{1}{2R^2T_0} = \sigma_0^2 \tag{48}$$

In the case of the system without feedback loop ($K = 0$) we have

$$q_{11} = \frac{1}{2k_2^2T_1} = \sigma_1^2 \tag{49}$$

$$q_{22} = \frac{1}{2R^2T_0} = \sigma_0^2 \tag{50}$$

From the equation (44) after using (47) and (48) we find that

$$\Delta H_1 = \ln\frac{\sigma_2}{\sigma_1} = -\ln\sqrt{1+\frac{K}{k_2}} \tag{51}$$

$$\Delta H_2 = \ln\frac{\sigma_0}{\sigma_0} = 0 \tag{52}$$

We can rewrite the equation (51) in the form

$$1+\frac{K}{k_2} = e^{-2\Delta H_1} \tag{53}$$

The substitution of the relation (53) into (35) gives

$$\left(e^{-2\Delta H_1}-1\right) J_1 = \frac{\delta(\Delta E)}{\omega_n\Delta\omega+\frac{1}{2}\left(\Delta\omega\right)^2} \tag{54}$$

After denoting the kinetic energy of the system without any stabilization in the nominal state by

$$E_1 = \frac{1}{2}J_1\omega_n^2 \tag{55}$$

and after introducing relative variables

$$\Delta E_r = \frac{\Delta E}{E_1} \tag{56}$$

$$\Delta\omega_r = \frac{\Delta\omega}{\omega_n} \tag{57}$$

the relation (54) takes the form

$$e^{-2\Delta H_1} = 1+\frac{1}{2}\frac{\delta(\Delta E_r)}{\Delta\omega_r+\frac{1}{2}\left[\Delta\omega_r\right]^2} \tag{58}$$

or in the differential form

$$e^{-2\Delta H_1} = 1+\frac{1}{2}\frac{d\Delta E_r}{d\omega_r} \tag{59}$$

which is the basic relation.

## 3. Conclusion

The relation (59) leads to the following fundamental statement:

> The increment of information or decrement of entropy of information in the system is proportional to the logarithm of the derivative of the additional energy supplied to the system with respect to its carrier. Observe that if the $\frac{d\Delta E_r}{d\omega_r} = 0$ then $\Delta H$ is also equal to zero, and vice-versa, if $\Delta H = 0$ then $\frac{d\Delta E_r}{d\omega_r} = 0$.

**Remark** Generalization for multivariable systems, Kaczorek (1981).

The formal generalization is straightforward, Kwakernaak, Sivan (1972).

The analogon of equation (10), the equation of motion, is the stochastic vector differential equation.

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bz(t) \\ &x(t_0) = x_0 \end{aligned} \tag{60}$$

where

$A$ and $B$ are constant matrices.

$z(t)$ is a vector of white noise Gaussian stochastic process with constant intensity $Z$.

Then if $A$ is asymptotically stable and $t_0 \to -\infty$ or $t \to \infty$ the variance matrix of $x(t)$ tends to the constant nonnegative-definite matrix.

The analogy to relations (18), (26) and (31) is as follows.

Vector of steady-state error is equal

$$\lim_{T\to\infty} \frac{1}{T} \left\{ \int_0^T x^T(t) R x(t) dt \right\} = tr\left(B^T P B\right) \tag{61}$$

where

$R$ is a symmetric nonnegative-definite constant matrix

$\bar{P}$ is a constant matrix, which is a unique solution of the Lyapunov equation

$$A^T \bar{P} + \bar{P} A + R = 0 \tag{62}$$

The vector probability density function can be written in the analogy to relation (39) as

$$\begin{aligned} &p_x\left(x_1, x, ..., x_n\right) = \\ &= \frac{1}{\left[\sqrt{(2\pi)^n \det C_x}\right]^{1/2}} \exp\left\{ \frac{1}{2} \frac{\det \begin{bmatrix} C_x & x - m_x \\ (x-m_x)^T & 0 \end{bmatrix}}{\det C_x} \right\} \end{aligned} \tag{63}$$

where $C_x$ is the compound covariance matrix

$$C_x = \left[\begin{array}{cc|c} C_{x_1x_1} & C_{x_1x_2} & C_{x_1x_n} \\ C_{x_2x_1} & C_{x_2x_2} & C_{x_2x_n} \\ \hline C_{x_nx_1} & C_{x_nx_2} & C_{x_nx_n} \end{array}\right] \tag{64}$$

$$x - m_x = \begin{bmatrix} x_1 - mx_1 \\ x_2 - mx_2 \\ \dots \\ x_n - mx_n \end{bmatrix} \tag{65}$$

If the components $x_1, \dots x_n$ of the stochastic vector $x$ are not correlated, then the vector probability density function is equal

$$p_x(x_1, x_2, \dots, x_n) = p_{x_1}(x_1)p_{x_2}(x_2)...p_{x_n}(x_n) \tag{66}$$

and

$$p_{x_i}(x_i) = \frac{1}{\sqrt{2\pi C_{x_ix_i}}} \exp\left[-\frac{1}{2}\frac{(x_i - m_{x_i})^2}{C_{x_ix_i}}\right] \tag{67}$$

The subsequent calculations follow the same way as for scalar case.

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