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# Optimal controller for discrete systems with indeterminate perturbations

by

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**Abstract:** The paper concerns a new approach to construction of optimal feedback control in discrete systems. Indeterminate perturbations are supposed to influence the system during control. The algorithm taking into account the acting perturbations is offered. It is an adaptation of the dual methods of linear programming to optimization of dynamic systems.

**Keywords:** discrete systems, feedback control, synthesis problem, support control, controller

## 1. Introduction

Optimal control of dynamic systems in real time is currently a very important problem for engineering. There are a lot of approaches to this problem in the modern optimal control theory (see Bellman, 1963, Feldbaum, 1963, Krasovskii, 1977, Leondes, 1976, Moroz, 1987). However, until now, satisfactory results have not been obtained for multidimensional problems with direct restrictions on control.

The approach presented in this paper is based on adaptation of the dual methods of linear programming to optimization of dynamic systems (see Gabasov, Kirillova, 1977,1980). The main idea of this approach consists in embedding the optimal control problem in a one-parameter family of problems along a realizing trajectory. Real time is the parameter of the family.

At each particular moment our controller constructs an optimal control for the current state where the system transfers from the previous one under the influence of the control and the perturbation realized at this moment. This allows us to save enormous volume of memory at the expense of relatively little computational work during the control process. The advantage in the amount of current calculations is achieved because the optimal control is not constructed anew for each current state, but results from correction of the old optimal control in response to continuous perturbations affecting the system trajectory.

It is known from linear programming that the dual method is an extremely effective one for correction of optimal feasible solutions for slight changes of problem parameters. That is why the proposed approach is based on special implementation of the dual method for solving optimal control problems.

Under optimal system synthesis the controller algorithm depends on the accessible information about perturbations acting on the system. In Gabasov, Kirillova, Prischepova (1991) the controller counteracting perturbations which are measured during the control process has been described. Now we consider another formulation of the synthesis problem. We shall assume that during control design the perturbation is indeterminate and only the domain of its possible values is known. Since the quantization period is assumed to be small and is defined by the controller speed the variation of the system states over this period will be not large. In this context it is interesting to establish a relation between the results obtained in the present paper and in Gabasov, Kirillova, Prischepova (1991).

## 2. Statement of the problem

Consider the terminal problem of optimal control

$$f(u) = h'x(t^*) \to \max, \tag{1}$$

$$x(t+h) = Ax(t) + bu(t), x(0) = x_0, \tag{2}$$

$$h_i'x(t^*) \geq g_i, i = \overline{1,m}; \tag{3}$$

$$u_*(t) \leq u(t) \leq u^*(t), \tag{4}$$

$$t \in T(0) = \{0, h, \ldots, t^* - h\}.$$

About the $n \times n$-matrix $A$ and the $n$-vector $b$ we shall assume that they have been obtained either by a result of discretization of the continuous system $\dot{x} = \bar{A}x + \bar{b}u$ or after application of impulse controls. In the first case the simplest Euler method leads to $A = E + h\bar{A}$, $b = h\bar{b}$. The second one gives $A = \exp \bar{A}h$, $b = |\int_0^h e^{\bar{A}(h-\tau)}\bar{b}|d\tau$. From the preceding, the assumption $detA \neq 0$ is not an essential restriction.

Without loss of generality the system (2) is supposed to be controllable over restrictions (3).

As usual a control $u(t)$, $t \in T(0)$, (4), will be called admissible for the position $\{0, x(0)\}$ if the corresponding trajectory $x(t)$, $t \geq 0$, of system (2) at $t^*$ satisfies (3). The admissible control $u(t)$, $t \in T(0)$, is said to be an optimal (program) one $(u^0(t), t \in T(0))$, if the quality criterion (1) attains the maximal value.

Here we shall distinguish between two types of solution to problem (1)-(4): program and feedback.

A finite effective algorithm for constructing the optimal (program) control $u^0(t|0, x_0)$, $t \in T(0)$, for a fixed initial position $\{0, x_0\}$ is presented in the monograph of Gabasov, Kirillova et al. (1984-1991).

If perturbations do not affect the system (2) then the constructed program control solves the synthesis problem completely since for an arbitrary moment $\tau \in T(0)$ system (2) is in the state $x_0(\tau)$ and the optimal control value for the position $\{\tau, x_0(\tau)\}$ is equal to $u^0(\tau|0, x_0\}$.

We understand an optimal control of the classical feedback type to be a piecewise constant function $v^0(x, t)$, $x \in R^n$, $t \in T(0)$, which for every moment $\tau \in T(0)$ and every initial state $x_0$ from the set of controllability generates the trajectory of the system

$$x(t+h) = Ax(t) + bv^0(x(t), t), x(\tau) = x_0$$

coinciding with the optimal trajectory $x_0(t)$, $t \in T(0)$, of problem (1)-(4).

The main reason why engineers prefer control of the feedback type to program control is that real movements of dynamic systems are described not by the equation (2) but by

$$x(t+h) = Ax(t) + bu(t) + y(t), x(0) = x_0, \tag{5}$$

where $y(t)$, $t \in T(0)$, is an unknown $n$-vector function. Under the conditions (5), program control cannot provide even admissibility of trajectories. Feedback control can cope successfully with a great number of perturbations.

We define our way of control design of feedback type.

We shall assume that the controller begins to operate at the moment $t = 0$ from the state $x(0) = x_0$ and its work is based on a special program solution of problem (1)-(4) (see Gabasov, Kirillova et al., 1984-1991). Let the controller have operated during

$$t = 0, h, \ldots, \tau - h,$$

and the system (2) having transferred from the state $x(\tau - h)$ not to the state

$$\hat{x}(\tau) = Ax(\tau - h) + bu(\tau - h)$$

but to

$$x(\tau) = \hat{x}(\tau) + y(\tau)$$

under the action of controls $u(0), \dots, u(\tau - h)$, and the perturbations $y(0), \dots,$ $y(\tau - h)$. We suppose that at the moment $\tau$ the value of perturbation $y(\tau)$ that will act on the state $x(\tau)$ is not known. It is only known that

$$y(\tau) \in Y = \{z \in R^n : Gz = q, d_* \le z \le d^*\}.$$

Using the principle of obtaining the guaranteed result (see Gabasov, Kirillova, 1991) we shall call controller synthesis (or, in other words, the design of the control feedback type) the calculation of the control

$$u^0(t|\tau, x(\tau)), t \in T(\tau) = \{\tau, \tau + h, \dots, t^* - h\},$$

for which all the trajectories of the system (2) with initial conditions

$$x(\tau) + y(\tau), y(\tau) \in Y,$$

will satisfy the terminal restrictions (3) and the guaranteed (the worst) value of the quality criterion

$$f(u(\cdot)) = \min_{y(\tau) \in Y} h' x(t^*)$$

will be maximal ($f(u^0(\cdot)) = \max_u f(u(\cdot))$).

## 3. Optimal control under uncertainty conditions

From linearity of the problem (1)-(4) one can show that the result $u^0(t|\tau, x(\tau))$, $t \in T(\tau)$, of the synthesis at the moment $\tau$ can be calculated as a solution of the following extremal problem:

$$\begin{aligned}
& f(u) = h_0' x(t^*) + \gamma_0^\tau \to \max, \\
& x(t+h) = Ax(t) + bu(t), x(\tau) = x_\tau, \\
& h_i' x(t^*) \ge g_i^\tau, \; i = \overline{1,m}; \\
& u_*(t) \le u(t) \le u^*(t), \; t \in T(\tau),
\end{aligned} \tag{6}$$

where $x_\tau$ is the measured state $x(\tau)$ at the moment $\tau$; and $g_i^\tau = g_i - \gamma_i^\tau$, $\gamma_i^\tau, i = \overline{0,m}$, are the estimates of the following linear programming problems

$$\begin{aligned}
\gamma_0^\tau &= \min h_i' A^{(t^*-\tau)/h_z}, \\
Gz &= q, d_* \le z \le d^*,
\end{aligned}$$

which can be computed in a parallel way.

According to the investigations of Gabasov, Kirillova et al. (1984-1991) solution of the terminal control problem (6) is the set

$$\{u^0(\cdot|\tau, x(\tau)), S_{\sup}(\tau)\},$$

where

$$S_{\text{sup}}(\tau) = \{I_{\text{sup}}(\tau), T_{\text{sup}}(\tau)\}, I_{\text{sup}}(\tau) \subseteq I = \{1, 2, \ldots, m\},$$

$$T_{\text{sup}}(\tau) = \{\tau_1, \ldots, \tau_l\}, \tau \leq \tau_1(\tau) < \tau_2(\tau) < \ldots < \tau_l(\tau)t^* - h.$$

The set $S_{\text{sup}}(\tau)$ is called a support.

The relations

$$|I_{\text{sup}}(\tau)| = |T_{\text{sup}}(\tau)| = l, \; 0 \leq l \leq m, det P(\tau) \neq 0,$$

$$P(\tau) = \left[ \begin{array}{c} h_i' A^{(t^*-t)/h-1} b, \; t \in T_{\text{sup}}(\tau) \\ i \in I_{\text{sup}}(\tau) \end{array} \right].$$

are fulfilled.

The vector of potentials

$$\nu' = \nu'(\tau) = c_{\text{sup}}' Q(\tau),$$

$$c_{\text{sup}} = (c(t), \; t \in T_{\text{sup}}(\tau)), \; c(t) = h_0' A^{(t^*-t)/h-1} b, \; \; t \in T(\tau),$$

$$Q(\tau) = P^{-1}(\tau)$$

corresponds to the support $S_{\text{sup}}(\tau)$.

With the help of the above the accompanying co-trajectory

$$\psi(t) = \psi(t|\tau), \; t \in T(\tau),$$

is constructed as a solution to the conjugate system

$$\psi'(t-h) = \psi'(t)A, \psi'(t^*-h) = h_0' - \nu'(I_{\text{sup}})H(I_{\text{sup}}, J),$$

$$H(I, J) = \left[ \begin{array}{c} h_i'(J) \\ i \in I \end{array} \right], \; J = \{1, 2, \ldots, n\}.$$

The co-trajectory generates the co-control

$$\Delta(t) = \Delta(t|\tau), \; t \in T(\tau) : \; \Delta(t) = -\psi'(t)b. \tag{7}$$

The optimal control $u^0(t|\tau, x(\tau))$, $t \in T_N(\tau) = T(\tau) \in T_{\text{sup}}(\tau)$, at the non-support moments of time is calculated by

$$u^0(t|\tau, x(\tau)) \begin{cases} = u_*(t) \text{ when } \Delta(t) > 0; \\ = u_*(t) \text{ when } \Delta(t) < 0; \\ \in [u_*(t), u^*(t)] \text{ when } \Delta(t) = 0, \; t \in T_N(\tau). \end{cases} \tag{8}$$

Without loss of generality we may suppose that

$$h_i' x(t^*) = g_i^T, \; i \in I_{\text{sup}}(\tau)(\nu(i) \leq 0, i \in I_{\text{sup}}(\tau)).$$

The set of values $u^0_{\text{sup}} = (u^0(t),\ t \in T_{\text{sup}}(\tau))$ of optimal control at the support moments is found by

$$u^0_{\text{sup}} = Q(\tau)g(\tau),$$

where

$$\begin{aligned} g(\tau) &= \begin{bmatrix} g_i(\tau) \\ i \in I_{\text{sup}}(\tau) \end{bmatrix},\ g_i(\tau) = g^\tau - \\ &- \sum_{t \in T_N(\tau)} h_i' A^{(t^*-t)/h-1} bu(t) - h_i' A^{(t^*-\tau)/h} x_\tau . \end{aligned}$$

According to (8) the form of optimal control is defined by the time moments in which the co-control $\Delta(t),\ t \in T(\tau)$, (7), changes its sign. Because of this, the algorithm proposed below is essentially based on observing the movement of these moments at the interval $T(\tau)$. We shall consider for simplicity that except the support moments $T_{\text{sup}}(\tau)$ the sign of co-control may be changed only at one non-support moment $\tau_N \in T_N(\tau)$.

We construct the auxiliary sets $T_{N^+}(\tau)$, $T_{N^-}(\tau)$ connected with alternation of the co-control signs according to relations

$$\begin{aligned} T_{N^+}(\tau) &= \{t \in T_N(\tau) : \Delta(t) > 0, \Delta(t-h) < 0\} \cup \\ &\cup\ \{t \in T_N(\tau) : \Delta(t) > 0, t-h \in T_{\text{sup}}(\tau)\}, \\ T_{N^-}(\tau) &= \{t \in T_N(\tau) : \Delta(t) < 0, \Delta(t-h) > 0\} \cup \\ &\cup\ \{t \in T_N(\tau) : \Delta(t) < 0, t-h \in T_{\text{sup}}(\tau)\}, \\ |T_{N^+}(\tau)| + |T_{N^-}(\tau)| &= m+1, (\tau_N \in T_{N^+}(\tau) \cup T_{N^-}(\tau), \tau_N - h \notin T_{\text{sup}}(\tau)). \end{aligned}$$

Let the information array be known

$$V_\tau(t) = A^{(t^*-t)/h-1} b,\ t \in T_{\text{sup}}(\tau) \cup \tau_N \cup (t^* - h).$$

## 4. Optimal controller synthesis

Let us pass to the description of the algorithm for the optimal controller for an arbitrary position $\{\tau, x(\tau)\}$. In this context we shall assume that the system state does not leave the domain of controllability while the controller acts.

The set

$$\begin{aligned} &C^k(\tau) = \{u^{(k)}(t), t \in T(\tau); W^k; S^k; T^k_{N+}; T^k_{N-}; \\ &\Delta g^k; V^k(t), t \in T^k_{\text{sup}} \cup \tau_N \cup (t^*-h);\ \Delta^k(t-h), \Delta^k(t+h), t \in T^k_{\text{sup}}, \\ &\Delta(t), t = \tau_N, \tau_N - h, t^* - h; Q^k; \nu^k\} \end{aligned}$$

will be called the current state of the algorithm at the moment $\tau$. As an initial state $C^0(\tau)$ we shall choose the set with the following components:

$$u^{(0)}(t) \ = \ u^0(t|\tau - h),\ t \in T(\tau); W^0 = Hx^0(t^*) - g^\tau;$$

$$
\begin{aligned}
S^0_{\text{sup}} &= S_{\text{sup}}(\tau);\ T^0_{N+} = T_{N+}(\tau-h);\ T^0_{N-} = T_{N-}(\tau-h); \quad (9)\\
\Delta g^0 &= \gamma^{\tau-h} - \gamma^t + H(Ax_{\tau-h} + bu^0(\tau-h|\tau-h, x(\tau-h)) - x_\tau);\\
V^0(t) &= V_\tau(t-h), t \in T^0_{\text{sup}} \cup \tau_N \cup (t^*-h);\\
\Delta^0(t-h) &= \Delta(t-h|\tau-h),\ \Delta^0(t+h) = \Delta(t+h|\tau-h), t \in T^0_{\text{sup}};\\
\Delta^0(\tau_N) &= \Delta(\tau_N|\tau-h),\ \Delta^0(\tau_N-h) = \Delta(\tau_N-h|\tau-h),\\
\Delta^0(t^*-h) &= \Delta(t^*-h|\tau-h);\ Q^0 = Q(\tau-h);\ \nu^0 = \nu(\tau-h).
\end{aligned}
$$

A starting state $C^0(\tau = h)$ is constructed by (9) with help of the optimal program solution $u^0(t|0, x_0)$, $t \in T(h)$ (see Gabasov, Kirillova et al., 1984-1991).

The algorithm's iteration $C^k(\tau) \to C^{k+1}(\tau)(C^k(\tau) \to C^0(\tau+h))$ is the implementation of the dual method of linear programming (see Gabasov, Kirillova, 1977-1980) for the situation arising in the discrete control problem and consists of the following steps.

Step 1. If $l = 0$, then pass to Step 2. Let $l \geq 1$. Compare $\tau - h$ with $\tau_1$. If $\tau - h < \tau_1$ then pass to the following Step. At $\tau - h = \tau_1$ pass to Step 8.

Step 2. Calculate the change directions for the control and the vector of inequality restrictions

$$
\begin{aligned}
\Delta u^k(T^k_{\text{sup}}) &= (\Delta u^k(t), t \in T^k_{\text{sup}}) = Q^k \Delta g^k(I^k_{\text{sup}});\\
\Delta u^k(T^k) &= 0, T^k_N = T(\tau) \setminus T^k_{\text{sup}};\\
\Delta W^k(I^k_N) &= \sum_{t \in T^k_{\text{sup}}} H(I^k_N, J) V^k(t) \Delta u^k(t),\\
\Delta W^k(I^k_{\text{sup}}) &= 0,\ I^k_N = I \setminus I^k_{\text{sup}}.
\end{aligned}
$$

Step 3. Calculate the maximal feasible steps $\alpha^k$, $\beta^k$, $\Theta^k$ along the directions from Step 2

$$\alpha^k = \alpha(\tau_s) = \min \alpha(t),\ t \in T^k_{\text{sup}},$$

$$
\alpha(t) = \begin{cases} \frac{u_*(t) - u^{(k)}(t)}{\Delta u^k(t)}, & \text{when } \Delta u^k(t) < 0,\\ \frac{u^*(t) - u^{(k)}(t)}{\Delta u^k(t)}, & \text{when } \Delta u^k(t) > 0,\\ \infty, & \text{when } \Delta u^k(t) = 0,\ t \in T^k_{\text{sup}}; \end{cases}
$$

$$\beta^k = \beta(i_0) = \min \beta(i),\ i \in I^k_N,$$

$$
\beta(i) = \begin{cases} -\frac{W^k_i}{\Delta W^k_i - \Delta g^k_i}, & \text{when } \Delta W^k_i - \Delta g^k_i < 0,\\ \infty, & \text{when } \Delta W^k_i - \Delta^k_i \geq 0. \end{cases}
$$

Let $\Theta^k = \min\{1, \alpha^k, \beta^k\}$. If $\Theta^k = 1$, then pass to Step 4. At $\Theta^k < 1$ we pass to Step 5.

Step 4. Compute $u^0(\tau|\tau, x(\tau)) = u^{(k)}(\tau) + \Delta u^k(\tau)$. If $\tau = t^* - lh$ then the algorithm completes the work:

$$u^0(\tau + ih|\tau + ih, x(\tau + ih)) = u^{(k)}(\tau + ih) + \Delta u^k(\tau + ih), i = \overline{1, l-1}.$$

If $\tau < t^* - lh$ we construct the initial state $C^0(\tau + h)$ for the moment $\tau + h$

with the following components:

$$\begin{aligned}
u^{(0)}(t) &= u^{(k)}(t) + \Delta u^k(t), t \in T(\tau+h); W^0 = W^k + \Delta W^k;\\
S^0 &= S^k_{\text{sup}}; T^0_{N+} = T^k_{N+}; T^0_{N-} = T^k_{N-};\\
\Delta g^0 &= \gamma^\tau - \gamma^{\tau+h} + H(Ax_\tau + bu^0(\tau|\tau, x(\tau)) - x_{\tau+h});\\
\Delta^0(t-h) &= \Delta^k(t-h), \Delta^0(t+h) = \Delta^k(t+h), t \in T^0_{\text{sup}};\\
\Delta^0(\tau_N) &= \Delta^k(\tau_N), \Delta^0(\tau_N - h) = \Delta^k(\tau_N - h), \Delta^0(t^* - h) =\\
&\quad \Delta^k(t^* - h);\\
V^0(t) &= V^k(t), t \in T^0_{\text{sup}} \cup \tau_N \cup (t^* - h); Q^0 = Q^k; \nu^0 = \nu^k.
\end{aligned}$$

Pass to Step 1.

Step 5. If $\Theta^k = \beta^k = \beta(i_0)$ calculate the dual directions for the vector of potentials and the co-control

$$\begin{aligned}
\mu^k(I^k_{\text{sup}}) &= [h'_{i_0} V^k(t), t \in T^k_{\text{sup}}]Q^k,\\
\delta^k(t+h) &= [h'_{i_0} - \mu^{k'}(I^k_{\text{sup}})H(I^k_{\text{sup}}, J)]A^{-1}V^k(t), t \in T^k_{\text{sup}};\\
\delta^k(t-h) &= [h'_{i_0} - \mu^{k'}(I^k_{\text{sup}})H(I^k_{\text{sup}}, J)]AV^k(t), t \in T^k_{\text{sup}} \cup \tau_N;\\
\delta^k(t) &= [h'_{i_0} - \mu^{k'}(I^k_{\text{sup}})H(I^k_{\text{sup}}, J)]V^k(t), t \in \{\tau_N, t^* - h\}.
\end{aligned}$$

If $\Theta^k \doteq \alpha^k = \alpha(\tau_s)$ calculate these directions by formula

$$\begin{aligned}
\mu^k(I^k_{\text{sup}}) &= \rho Q^k(\tau_s, I^k_{\text{sup}}),\\
\delta^k(t+h) &= \rho Q^k(\tau_s, I^k_{\text{sup}})H(I^k_{\text{sup}}, J)]A^{-1}V^k(t), t \in T^k_{\text{sup}};\\
\delta^k(t-h) &= \rho Q^k(\tau_s, I^k_{\text{sup}})H(I^k_{\text{sup}}, J)]AV^k(t), t \in T^k_{\text{sup}} u \tau_N;\\
\delta^k(t) &= \rho Q^k(\tau_s, I^k_{\text{sup}})H(I^k_{\text{sup}}, J)]V^k(t), t \in \{\tau_N, t^* - h\};\\
\rho &= \text{sign}\Delta u(\tau_s) \text{ if } u^{(k)}(\tau_s) > 0,\\
\rho &= -\text{sign}\Delta u(\tau_s) \text{ if } u^{(k)}(\tau_s) < 0.
\end{aligned}$$

Pass to Step 6.

Step 6. Calculate the maximal feasible dual steps along directions from Step 5

$$\begin{aligned}
\sigma^k &= \min\{\sigma(t-h), \sigma(t+h), t \in T^k_{\text{sup}};\\
&\quad \sigma(t), t = \tau_N, \tau_N - h, t^* - h;\\
&\quad \omega(i), i \in I^k_{\text{sup}}\}; s(t), t \in T^k_{N+} \cup T^k_{N-} :\\
\sigma(t-h) &= -\frac{\Delta^k(t-h)}{\delta^k(t-h)}, s(t) = h,\\
&\quad \text{when } t - h \in T^k_{N+}, \delta^k(t-h) < 0\\
&\quad \text{or } t - h \in T^k_{N-}, \delta^k(t)0;\\
\sigma(t+h) &= -\frac{\Delta^k(t+h)}{\delta^k(t+h)}, s(t) = -h,\\
&\quad \text{when } t + h \in T^k_{N+}, \delta^k(t-h)\\
&\quad \text{or } t + h \in T^k_{N-}, \delta^k(t-h) > 0, t \in T^k_{\text{sup}};
\end{aligned}$$

$$\begin{aligned}
\sigma(t) &= -\frac{\Delta^k(t)}{\delta^k(t)}, \\
&\quad \text{when } \Delta^k(t)\delta^k(t) < 0, t = \tau_N, \tau_N - h, t^* - h, s(t) = 0; \\
&\sigma(t-h) = \sigma(t+h) = \sigma(t) = \infty \text{ in other cases.} \\
\omega(i) &= -\frac{\nu^k(i)}{\mu^k(i)}, \\
&\quad \text{when } \nu^k(i)\mu^k(i) < 0 \\
&\quad \text{or } \nu^k(i) = 0, \mu^k(i) > 0, \omega(i) = \infty \text{ in other cases,} \\
&i \in I^k_{\text{sup}}.
\end{aligned}$$

Pass to Step 7.

Step 7. Transform the support set $S^k_{\text{sup}} = \{I^k_{\text{sup}}, T^k_{\text{sup}}\}$ and the support matrix $Q^k$.

1. Let $\Theta^k = \beta(i_0) < 1$, $\sigma^k = \omega(i_*)$. Then

$$\begin{aligned}
I^{k+1}_{\text{sup}} &= (I^k_{\text{sup}}\ i_*) \cup i_0, T^{k+1}_{\text{sup}} = T^k_{\text{sup}}, \\
Q^{k+1}(\tau_j, i) &= Q^k(\tau_j, i) + Q^k(\tau_j, i_*)r^k(i)/r^k(i_*), i = i_*; \\
Q^{k+1}(\tau_j, i_*) &= Q^k(\tau_j, i_*)/r^k(i_*), j = \overline{1,l}, i = \overline{1,l},
\end{aligned}$$

where $r^k = (r^k(i), ieI^k_{\text{sup}}) = [h'_{i_0}V^k(t), t \in T^k_{\text{sup}}]Q^k(T^k_{\text{sup}}, I^k_{\text{sup}})$.

2. Let $\Theta^k = \beta(i_0) < 1$, $\sigma^k = \sigma(t_q - s(t_q))$. Then

$$\begin{aligned}
I^{k+1}_{\text{sup}} &= I^k_{\text{sup}} \cup i_0,\ T^{k+1}_{\text{sup}} = T^k_{\text{sup}} \cup (t_q - s(t_q)) \\
Q^{k+1}(\tau_j, i) &= Q^k(\tau_j, i) + r^k_1(\tau_j)r^k_2(i)/\rho\delta(t_q - s(t_q)), \\
Q^{k+1}(t_q - s(t_q), i) &= -r^k_2(i)/\rho\delta(t_q - s(t_q)), \\
Q^{k+1}(\tau_j, i_0) &= -r^k_1(\tau_j)/\rho\delta(t_q - s(t_q)), \\
Q^{k+1}(t_q - s(t_q), i_0) &= -1/\rho\delta(t_q - s(t_q)), j = \overline{1,l},\ i = \overline{1,l}, \\
r^k_1 &= (r^k_1(\tau_j), j = \overline{1,l}) = \\
&\quad Q^k H(I^k_{\text{sup}}, J)V^k(t_q - s(t_q))), \\
r^k_2 &= (r^k_2(i), i = \overline{1,l}) = \\
&\quad [h'_{i_0}V^k(t), t \in T^k_{\text{sup}}]Q^k.
\end{aligned}$$

3. Let $\Theta^k = \alpha^k = \alpha(\tau_s) < 1$, $\sigma^k = \omega(i_*)$. Then

$$\begin{aligned}
I^{k+1}_{\text{sup}} &= I^k_{\text{sup}} \setminus i_*,\ T^{k+1}_{\text{sup}} = T^k_{\text{sup}} \setminus \tau_s, \\
Q^{k+1}(T^{k+1}_{\text{sup}}, I^{k+1}_{\text{sup}}) &= Q^k(T^k_{\text{sup}} \setminus \tau_s, I^k_{\text{sup}1*}) - \\
&- Q^k(T^k_{\text{sup}} \setminus \tau_s, i_*)Q^k(\tau_s, I^k_{\text{sup}1*})/\rho\mu(i_*).
\end{aligned}$$

4. Let $\Theta^k = \alpha^k = \alpha(\tau_s) < 1$, $\sigma^k = \sigma(t_q - s(t_q))$. Then

$$\begin{aligned}
I^{k+1}_{\text{sup}} &= I^k_{\text{sup}}, T^{k+1}_{\text{sup}} = (T^k_{\text{sup}} \setminus \tau_s) \cup (t_q - s(t_q)), \\
Q^{k+1}(\tau_j, i) &= Q^k(\tau_j, i) - Q^k(\tau_s, i)r^k(\tau_j)/r^k(\tau_s), i = i_*;
\end{aligned}$$

$$\begin{aligned} Q^{k+1}(\tau_s, i) &= Q^k(\tau_s, i)/r^k(\tau_s), j = \overline{1,l},\ i = \overline{1,l}, \\ r^k &= (r^k(\tau_j), j = \overline{1,l}) = \\ & \quad Q^k H(I^k_{\text{sup}}, J) V^k(t_q - s(t_q)). \end{aligned}$$

The vector $V^k(t_q - s(t_q))$ is calculated easily with respect to the vector $V^k(t_q)$.

Let

$$\begin{aligned} u^{(k+1)}(t) &= u^{(k)}(t) + \Theta^k \Delta u^k(t), t \in T(\tau); \\ \Delta g^{k+1} &= (1 - \Theta^k)\Delta g^k; \Delta^{k+1}(t) = \Delta^k(t) + \sigma^k \delta^k(t), \\ & \quad t \in T^k_{N+} \cup T^k_{N-} \cup (t^* - h); \\ \nu^{k+1} &= \nu^k + \sigma^k \mu^k; W^{k+1} = W^k + \Theta^k \Delta W^k. \end{aligned}$$

Let in Cases 1), 3) $T^{k+1}_{N+} \cup T^{k+1}_{N-} = (T^k_{N+} \cup T^k_{N-})$, in Case 2) $(T^{k+1}_{N+} \cup T^{k+1}_{N-}) = (T^k_{N+} \cup T^k_{N-}) \cup (t_q - s(t_q) + h)$, in Case 4) $(T^{k+1}_{N+} \cup T^{k+1}_{N-}) = (T^k_{N+} \cup T^k_{N-}) \setminus (\tau_s + h) \cup (t_q - s(t_q) + h)$.

Pass to Step 2.

Step 8. Let $s = 1$, $\Theta^0 = \alpha^0 = \alpha(\tau_1) = 0$, $\Delta u^0(\tau_1) Q^k(\tau_1, I^k_{\text{sup}}) \Delta g(I^k_{\text{sup}})$, and pass to Step 5.

**Remarks:**

1. The situation $\sigma^k = \infty$ testifies that the arising perturbations do not allow us to satisfy the constraints (3) at the expense of choice of control on the segment $T(\tau)$. In this case the algorithm stops abnormally.
2. The method for constructing the optimal controller described above presupposes certain properties of co-control. Cases where the sets $T_{N+}(\tau) \cup T_{N-}(\tau)$ contain essentially more elements than the set $T_{\text{sup}}(\tau)$, and where in the process of operation of the controller there appear new elements in $T_{\text{sup}}(\tau)$ which differ from those described above, can be investigated according to the scheme outlined, but now the controller becomes more complicated.
3. It is not difficult to verify that the synthesis algorithm can be extended to include the case of nonstationary domain of perturbations when the set $Y = Y(x_\tau, \tau)$ depends both on time and values of the current state.
4. The algorithm can be obviously generalized for the case when it is known that the perturbation will act not only at the moment $\tau$ but also at the moments $\tau + h, \ldots, \tau + rh \le t^* - h$.

## 5. Example

Let us illustrate the results obtained by the simple example of optimization of the mechanical motion.

The material point, starting the motion in the rectilinear way from some neighbourhood of the given point and being acted upon by indeterminate perturbations, is required to be moved into the given domain at the given moment and to acquire velocity the guaranteed value of which is maximal.

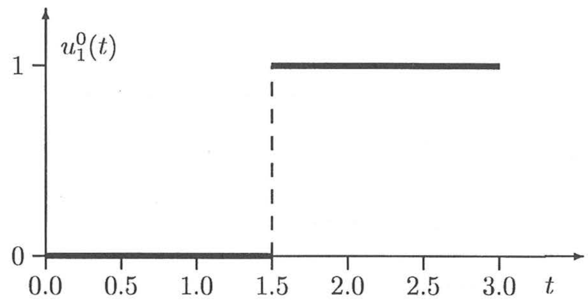


Figure 1.

The mathematical model of the problem is

$$x_2(3) \to \max, \; x_1(t+h) = x_1(t) + hx_2(t),$$

$$x_2(t+h) = x_2(t) + hu(t), \; x_1(3) \leq 2, \; x_1(0) = x_2(0) = 0,$$

$$0 \leq u(t) \leq 1, \; -1 \leq y_1(t) \leq 1,$$

$$y_2(t) = 0, \; t = 0, \; h, \ldots, 3, \; h = 0.5.$$

Let us present the results of the optimal controller's operation for the case when the perturbations

$$y_1(0) = 1/2, \; y_1(0.5) = 1/4, \; y_1(1) = -1/2,$$

$$y_1(1.5) = -1/4, \; y_1(2) = 1/4, \; y_1(2.5) = 0,$$

have been actually realized but the values of this noise are unknown for the controller at the corresponding moments.

During the operation described above the controller has designed the control presented in Fig. 1.

The efficiency of this control is equal to $f(u_1^0(\cdot)) = 3/2$.

If the information about the perturbations were entered during the process the controller would design the control presented in Fig. 2, with the value of the criterion for the designed control equal $f(u_2^0(\cdot)) = 5/3$.

Let the perturbation be known before the beginning of the process. Then the optimal control has the configuration presented in Fig. 3. The effectiveness of the control is now equal $f(u_3^0(\cdot)) = 9/4$.

The controller from Gabasov, Kirillova, Prischepova (1991) produces the control for the nearest period after the perturbation measurement which is made at already after the control has been adopted. Assume that the perturbation

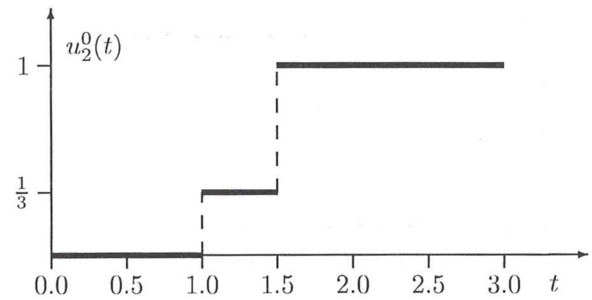


Figure 2.

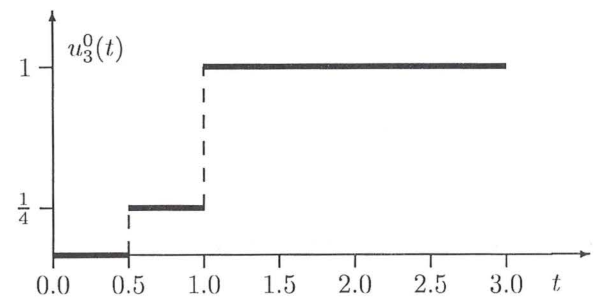


Figure 3.

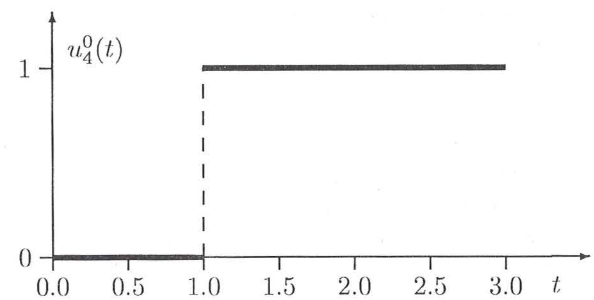


Figure 4.

measurement at each period is made before the control selection. In this case the controller produces the control presented in Fig. 4 ($f(u_4^0(\cdot)) = 2$).

From the above example the dependence between the control efficiency and information conditions is seen. In relation to full information the loss of efficiency is equal to $f(u_3^0(\cdot)) - f(u_1^0(\cdot)) = 3/4$, in relation to the partial information it is equal to $f(u_2^0(\cdot)) - f(u_1^0(\cdot)) = 1/6$.

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