**Control and Cybernetics**

vol. **26** (1997) No. 2

# A new approach to strict positive realness of interval transfer functions[1]

by

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**Abstract:** In this paper, we first present some intrinsic pointwise property for the real part of interval transfer function family evaluated along the imaginary axis, and then provide a simple direct proof of the eight-vertex result on strict positive realness of the interval transfer function family. One salient feature of our approach is that all the results are obtained directly from Kharitonov's Theorem on robust stability of interval polynomials. Some illustrative examples are also provided. Based on our discussion, a Kharitonov-like result can be established for the stronger version of strict positive realness of the interval transfer function family.

## 1. Introduction

The concept of strict positive realness of a transfer function plays an important role in absolute stability and adaptive control. The strict positive realness of *a family of transfer functions*, i.e. each transfer function in this family is strictly positive real, is very useful in robustness analysis of uncertain systems. This problem was first treated in Dasgupta (1987), and in Dasgupta and Bhagwat (1987), where it was shown that the strict positive realness of an interval transfer function family can be guaranteed by checking only *sixteen* prescribed vertex transfer functions. Chapellat et al. improved this result and proved that the strict positive realness of an interval transfer function family can be ascertained by the same property of only *eight* prescribed vertex transfer functions in this family, Chapellat (1991). This result is remarkable since it reduces the verification of the strict positive realness condition over infinitely many transfer

[1]Supported by Alexander von Humboldt Foundation of Germany. Dedicated to Professor J. Ackermann on the occasion of his sixtieth birthday.

functions in the interval family to the verification of the same property over only *eight* transfer functions, irrespective of the order of the transfer function family. In what follows, we first present some intrinsic pointwise property for the real part of the interval family evaluated along the imaginary axis, and then give a simple direct proof of the eight-vertex result. One distinguishing feature of this approach is that all the results are obtained directly from Kharitonov's Theorem on the robust stability of interval polynomials, Kharitonov (1978).

## 2. Main results

A polynomial $p(s)$ is said to be stable, which is denoted by $p(s) \in H$, if all its roots lie within the open left half of the complex plane.

A proper transfer function $\frac{p(s)}{q(s)}$ is said to be strictly positive real, which is denoted by $\frac{p(s)}{q(s)} \in SPR$, if

$$\begin{array}{l} 1)\ q(s) \in H \\ 2)\ \Re\frac{p(j\omega)}{q(j\omega)} > 0\ ,\ \forall \omega \in R \end{array} \tag{1}$$

Consider the $n$-th order real interval polynomial family

$$\Gamma = \left\{ p(s) \mid p(s) = \sum_{i=0}^{n} q_i s^i\ ,\ q_i \in [q_i^-\ ,\ q_i^+]\ ,\ i = 0, 1, \ldots, n \right\} \tag{2}$$

and define the four real Kharitonov polynomials as

$$K_1(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots \tag{3}$$

$$K_2(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots \tag{4}$$

$$K_3(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \cdots \tag{5}$$

$$K_4(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots \tag{6}$$

LEMMA 2.1 *(Kharitonov's Theorem for the Real Interval Polynomials)*

$$p(s) \in H\ ,\ \forall p(s) \in \Gamma \iff K_1(s)\ ,\ K_2(s)\ ,\ K_3(s)\ ,\ K_4(s) \in H \tag{7}$$

Now consider the first order complex interval polynomial family

$$\begin{array}{rl} \Gamma^c = & \{p(s) \mid p(s) = (r_0 + jt_0) + (r_1 + jt_1)s\ , \\ & r_i \in [r_i^-\ ,\ r_i^+]\ ,\ t_i \in [t_i^-\ ,\ t_i^+]\ ,\ i = 0, 1\} \end{array} \tag{8}$$

and define the eight complex Kharitonov polynomials as

$$K_1^+(s) = (r_0^- + jt_0^-) + (r_1^- + jt_1^+)s \tag{9}$$

$$K_2^+(s) = (r_0^+ + jt_0^+) + (r_1^+ + jt_1^-)s \tag{10}$$

$$K_3^+(s) = (r_0^+ + jt_0^-) + (r_1^- + jt_1^-)s \tag{11}$$

$$K_4^+(s) = (r_0^- + jt_0^+) + (r_1^+ + jt_1^+)s \tag{12}$$

$$K_1^-(s) = (r_0^- + jt_0^-) + (r_1^+ + jt_1^-)s \tag{13}$$

$$K_2^-(s) = (r_0^+ + jt_0^+) + (r_1^- + jt_1^+)s \tag{14}$$

$$K_3^-(s) = (r_0^+ + jt_0^-) + (r_1^+ + jt_1^+)s \tag{15}$$

$$K_4^-(s) = (r_0^- + jt_0^+) + (r_1^- + jt_1^-)s \tag{16}$$

LEMMA 2.2 *(Kharitonov's Theorem for the First Order Complex Interval Polynomials)*

$$p(s) \in H \ , \ \forall p(s) \in \Gamma^c \Longleftrightarrow K_i^+(s) \ , \ K_i^-(s) \in H \ , \ i = 1,2,3,4. \tag{17}$$

LEMMA 2.3 *Consider the transfer function* $\frac{p(s)}{q(s)}$. *For any fixed* $\omega \in R$, *suppose* $q(j\omega) \neq 0$, *then*

$$\Re\frac{p(j\omega)}{q(j\omega)} > 0 \Longleftrightarrow q(j\omega)s + p(j\omega) \in H \tag{18}$$

Proof: For any fixed $\omega \in R$, since $q(j\omega) \neq 0$, the polynomial $q(j\omega)s + p(j\omega)$ has a fixed order. Namely, $q(j\omega)s + p(j\omega)$ is a first-order polynomial with complex coefficients. Its unique root is $-\frac{p(j\omega)}{q(j\omega)}$. Hence

$$\begin{aligned} q(j\omega)s + p(j\omega) \in H \quad &\Longleftrightarrow \Re\left(-\tfrac{p(j\omega)}{q(j\omega)}\right) < 0 \\ &\Longleftrightarrow \Re\tfrac{p(j\omega)}{q(j\omega)} > 0 \end{aligned} \tag{19}$$

This completes the proof. ■

Note that Lemma 2.3 is important in its own right. It provides a connection between the strict positive realness condition and Hurwitz stability of a first-order complex polynomial. A similar result holds for $H_\infty$-norm of the strictly proper, stable transfer function $\frac{p(s)}{q(s)}$, i.e., the small gain condition $\left\|\frac{p(s)}{q(s)}\right\|_\infty < 1$ is satisfied if and only if, for any fixed $\omega \in R$, the first-order complex polynomial $q(j\omega)z + p(j\omega)$ is *Schur* stable.

Now consider the interval transfer function family

$$T = \left\{ \frac{p_u(s)}{p_v(s)} \mid p_u(s) \in \Gamma_u \ , \ p_v(s) \in \Gamma_v \right\} \tag{20}$$

where $\Gamma_u$ , $\Gamma_v$ are $n_u$-th, $n_v$-th order real interval polynomial families respectively.

Denote their Kharitonov polynomials as $K_i^u(s)$, $i = 1,2,3,4$ and $K_i^v(s)$, $i = 1,2,3,4$ respectively. We have

THEOREM 2.1 *For any fixed $\omega \in R$, suppose $p_v(j\omega) \neq 0$ , $\forall p_v(s) \in \Gamma_v$ . Then*

$$\begin{aligned}
&\Re\frac{p_u(j\omega)}{p_v(j\omega)} > 0 \ , \ \forall p_u(s) \in \Gamma_u \ , \ \forall p_v(s) \in \Gamma_v \\
&\Longleftrightarrow \Re\frac{K_1^u(j\omega)}{K_4^v(j\omega)} > 0 \ , \ \Re\frac{K_2^u(j\omega)}{K_3^v(j\omega)} > 0 \ , \\
&\Re\frac{K_3^u(j\omega)}{K_1^v(j\omega)} > 0 \ , \ \Re\frac{K_4^u(j\omega)}{K_2^v(j\omega)} > 0 \ , \\
&\Re\frac{K_1^u(j\omega)}{K_3^v(j\omega)} > 0 \ , \ \Re\frac{K_2^u(j\omega)}{K_4^v(j\omega)} > 0 \ , \\
&\Re\frac{K_3^u(j\omega)}{K_2^v(j\omega)} > 0 \ , \ \Re\frac{K_4^u(j\omega)}{K_1^v(j\omega)} > 0 \ . 
\end{aligned} \tag{21}$$

Proof: First note that, for any fixed $\omega \geq 0$ and $p_u(s) \in \Gamma_u$, we have

$$\Re K_1^u(j\omega) = \Re K_4^u(j\omega) \leq \Re p_u(j\omega) \leq \Re K_2^u(j\omega) = \Re K_3^u(j\omega) \tag{22}$$

$$\Im K_1^u(j\omega) = \Im K_3^u(j\omega) \leq \Im p_u(j\omega) \leq \Im K_2^u(j\omega) = \Im K_4^u(j\omega) \tag{23}$$

Thus, the set $\{p_u(j\omega) \mid p_u(s) \in \Gamma_u\}$ is an interval complex number (level rectangle) on the complex plane. Similarly, the set $\{p_v(j\omega) \mid p_v(s) \in \Gamma_v\}$ is also an interval complex number (level rectangle) on the complex plane.

Now by Lemma 2.3, we have

$$\begin{aligned}
&\Re\frac{p_u(j\omega)}{p_v(j\omega)} > 0 \ , \ \forall p_u(s) \in \Gamma_u \ , \ \forall p_v(s) \in \Gamma_v \\
&\Longleftrightarrow p_u(j\omega) + p_v(j\omega)s \in H \ , \ \forall p_u(s) \in \Gamma_u \ , \ \forall p_v(s) \in \Gamma_v
\end{aligned} \tag{24}$$

Let

$$r_0^- = \Re K_1^u(j\omega) = \Re K_4^u(j\omega) \tag{25}$$

$$r_0^+ = \Re K_2^u(j\omega) = \Re K_3^u(j\omega) \tag{26}$$

$$t_0^- = \Im K_1^u(j\omega) = \Im K_3^u(j\omega) \tag{27}$$

$$t_0^+ = \Im K_2^u(j\omega) = \Im K_4^u(j\omega) \tag{28}$$

$$r_1^- = \Re K_1^v(j\omega) = \Re K_4^v(j\omega) \tag{29}$$

$$r_1^+ = \Re K_2^v(j\omega) = \Re K_3^v(j\omega) \tag{30}$$

$$t_1^- = \Im K_1^v(j\omega) = \Im K_3^v(j\omega) \tag{31}$$

$$t_1^+ = \Im K_2^v(j\omega) = \Im K_4^v(j\omega) \tag{32}$$

Then by Lemma 2.2, we have

$$\begin{aligned}
& p_u(j\omega) + p_v(j\omega)s \in H \,, \; \forall p_u(s) \in \Gamma_u \,, \; \forall p_v(s) \in \Gamma_v \\
& \Longleftrightarrow K_1^u(j\omega) + K_4^v(j\omega)s \,, \; K_2^u(j\omega) + K_3^v(j\omega)s \,, \\
& K_3^u(j\omega) + K_1^v(j\omega)s \,, \; K_4^u(j\omega) + K_2^v(j\omega)s \,, \\
& K_1^u(j\omega) + K_3^v(j\omega)s \,, \; K_2^u(j\omega) + K_4^v(j\omega)s \,, \\
& K_3^u(j\omega) + K_2^v(j\omega)s \,, \; K_4^u(j\omega) + K_1^v(j\omega)s \; \in H
\end{aligned} \tag{33}$$

Again by Lemma 2.3, we get the desired result.

When $\omega < 0$, note that $p_u(s)$ , $p_v(s)$ are real polynomials, thus $p_u(j\omega)$ (or $p_v(j\omega)$) and $p_u(-j\omega)$ (or $p_v(-j\omega)$) are mirror images with respect to the real axis. By similar arguments we can reach the same conclusion. ■

EXAMPLE 1 Suppose $a \in [a^-, a^+]$, $b \in [b^-, b^+]$, $c \in [c^-, c^+]$, $d \in [d^-, d^+]$, and $(0,0) \notin [c^-, c^+] \times [d^-, d^+]$. For all possible $a, b, c, d$, how can we guarantee the following inequality

$$ac + bd > 0 \tag{34}$$

(which can be regarded as one of the conditions in the corresponding Hermite-Biehler root-interlacing theorem for stability of complex polynomials.) From the theory of interval analysis, we know that we need to check all the sixteen vertices $\{a^-, a^+\} \times \{b^-, b^+\} \times \{c^-, c^+\} \times \{d^-, d^+\}$. In order to use the result of Theorem 2.1, we first notice the following equivalence

$$ac + bd > 0 \Longleftrightarrow \Re\, \frac{a + jb}{c + jd} > 0 \tag{35}$$

Hence, by Theorem 2.1, we only need to check the following *eight* inequalities

$$a^-c^- + b^-d^+ > 0 \,, \;\; a^+c^+ + b^+d^- > 0 \tag{36}$$

$$a^+c^- + b^-d^- > 0 \,, \;\; a^-c^+ + b^+d^+ > 0 \tag{37}$$

$$a^-c^+ + b^-d^- > 0 \,, \;\; a^+c^- + b^+d^+ > 0 \tag{38}$$

$$a^+c^+ + b^-d^+ > 0 \,, \;\; a^-c^- + b^+d^- > 0 \tag{39}$$

THEOREM 2.2 *(*Chapellat et al.*)*

$$\begin{aligned}
& \frac{p_u(s)}{p_v(s)} \in SPR \,, \; \forall p_u(s) \in \Gamma_u \,, \; \forall p_v(s) \in \Gamma_v \\
& \Longleftrightarrow \frac{K_1^u(s)}{K_4^v(s)} \,, \; \frac{K_2^u(s)}{K_3^v(s)} \,, \; \frac{K_3^u(s)}{K_1^v(s)} \,, \; \frac{K_4^u(s)}{K_2^v(s)} \,, \\
& \frac{K_1^u(s)}{K_3^v(s)} \,, \; \frac{K_2^u(s)}{K_4^v(s)} \,, \; \frac{K_3^u(s)}{K_2^v(s)} \,, \; \frac{K_4^u(s)}{K_1^v(s)} \; \in SPR
\end{aligned} \tag{40}$$

Proof: Necessity: Obvious.

Sufficiency: Since $K_i^v(s) \in H$ , $i = 1, 2, 3, 4$, by Lemma 2.1, we have $p_v(s) \in H$ , $\forall p_v(s) \in \Gamma_v$. This also guarantees that $p_v(j\omega) \neq 0$ , $\forall \omega \in R$ , $\forall p_v(s) \in \Gamma_v$. Since the eight vertex transfer functions are strictly positive real, for any fixed $\omega \in R$, the real parts of these transfer functions evaluated at $j\omega$ are greater than zero. Thus, by Theorem 2.1, for any fixed $\omega \in R$, we have

$$\Re \frac{p_u(j\omega)}{p_v(j\omega)} > 0 \ , \ \forall p_u(s) \in \Gamma_u \ , \ \forall p_v(s) \in \Gamma_v \tag{41}$$

Therefore, we have

$$\frac{p_u(s)}{p_v(s)} \in SPR \ , \ \forall p_u(s) \in \Gamma_u \ , \ \forall p_v(s) \in \Gamma_v \tag{42}$$

This completes the proof. ∎

REMARK 2.1 *From the proof of Theorems 2.1 and 2.2, we see that the eight critical vertices in Chapellat's theorem come from the eight* first order *complex Kharitonov polynomials.*

REMARK 2.2 *Theorem 2.1 reveals a pointwise property for the interval transfer functions evaluated along the imaginary axis (the quotient of two complex interval numbers).*

REMARK 2.3 *In the proof of Theorem 2.1, a very special case of Kharitonov's Theorem, i.e. the case of the first order complex interval polynomials, has been resorted to. If Kharitonov's Theorem can be improved for low-order complex polynomials (in the real coefficients case, Kharitonov's Theorem can be simplified for low-order polynomials, see Anderson, 1987), then Theorems 2.1 and 2.2 can further be improved.*

EXAMPLE 2 Consider the interval transfer function family

$$\frac{[3,5]s + [-7,9]}{[5,8]s + [1,2]} \tag{43}$$

Suppose $\omega = 1$. It is easy to verify that all the eight critical vertices have positive real parts at this frequency. Hence, by Theorem 2.1, we know that every transfer function in this family evaluated at $\omega = 1$ has a positive real part. Note that Theorem 2.2 does not apply in this case since some transfer functions in this family, like $\frac{4s-6}{7s+2}$, are not strictly positive real. Similar results can be shown for the following transfer function families:

$$\frac{[-2,5]s^2 + [3,5]s + [-2,7]}{[2,4]s^5 + [3,4]s + [1,2]} \tag{44}$$

$$\frac{[2,3]s^9 + [1,2]s + [-7,9]}{[-3,-2]s^3 + [3,5]s + [1,2]} \tag{45}$$

REMARK 2.4 *Motivated from network theory, a slightly stronger definition for strictly positive real transfer function is as follows, Ioannou and Tao (1987):*

*A transfer function $G(s)$ of the complex variable $s = \sigma + j\omega$ is positive real, if*

$$\begin{aligned} &1)\quad G(s) \text{ is real for real } s \\ &2)\quad \Re G(s) \geq 0 \text{ for all } \Re[s] > 0 \end{aligned} \tag{46}$$

*Assume that $G(s)$ is not identically zero for all $s$. Then $G(s)$ is strictly positive real, if and only if $G(s-\epsilon)$ is positive real for some $\epsilon > 0$.*

*An equivalent characterization of the stronger strictly positive real transfer function is as follows:*

*Assume that $G(s)$ is not identically zero for all $s$. Then $G(s)$ is strictly positive real, if and only if*

$$\begin{aligned} &1)\quad G(s) \text{ is analytic in } \Re[s] \geq 0 \\ &2)\quad \Re G(j\omega) > 0,\ \forall \omega \in (-\infty, \infty) \\ &3)\quad \begin{array}{ll} \lim_{\omega^2 \to \infty} \omega^2 \Re G(j\omega) > 0, & n^* = 1 \\ \lim_{\omega^2 \to \infty} \Re G(j\omega) > 0 \text{ and } \lim_{|\omega| \to \infty} \frac{G(j\omega)}{j\omega} > 0, & n^* = -1 \end{array} \end{aligned} \tag{47}$$

*where $n^*$ is the relative degree of $G(s)$.*

*Assume that $G(s)$ is of the following form*

$$G(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots\cdots + a_1 s + a_0}{b_{n+1} s^{n+1} + b_n s^n + \cdots\cdots + b_1 s + b_0} \tag{48}$$

*where $a_n \neq 0$, $b_{n+1} \neq 0$. Then, it is easy to verify that condition 3) is equivalent to*

$$a_n b_n > a_{n-1} b_{n+1} \tag{49}$$

*Alternatively, assume that $G(s)$ is of the following form*

$$G(s) = \frac{a_{n+1} s^{n+1} + a_n s^n + \cdots\cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots\cdots + b_1 s + b_0} \tag{50}$$

*where $a_{n+1} \neq 0$, $b_n \neq 0$. Then, it is easy to verify that condition 3) is equivalent to*

$$a_n b_n > a_{n+1} b_{n-1} \quad \text{and} \quad a_{n+1} b_n > 0 \tag{51}$$

By similar arguments, we can establish strong Kharitonov-like results for the stronger version of strict positive realness of interval transfer function family.

## Acknowledgement

The author is grateful to the reviewers for their constructive comments.

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