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Boundary control of semilinear elliptic equations – existence of optimal solutions

by

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Abstract: A class of semilinear elliptic boundary control problems is considered. Owing to measurable and bounded controls we use the weak solution approach for the state equation. For the derivation of solvability results we present two methods. The first method works with the complete continuity of the state mapping, whereas in the second one theorems about the separation of convex sets and measurable selection are applied to overcome the complete continuity of the state mapping.

Keywords: boundary control, semilinear elliptic equations, existence theory

1. Introduction

Optimal control problems governed by nonlinear partial differential equations are a field of active research. Most of the work is done to derive necessary or sufficient optimality conditions of first and second order. The aim is to justify numerical algorithms for solving such problems. The question of existence of optimal controls has not been considered in such detail. Mostly, the assumptions on the control problem are chosen in such a way that a more or less known standard method can be applied to derive the existence of solutions. Meanwhile, there are some papers, for instance Papageorgiou (1991) in the case of optimal control problems for nonlinear elliptic equations, dealing especially with the problem of existence of solutions for optimal control problems. In Li and Yong (1995) a short overview of the history in this special field with some other references is given. They also show existence results for optimal control problems governed by evolution equations and elliptic variational inequalities. Papageorgiou as well as Li and Yong consider problems with distributed controls in the elliptic case.

In this paper we consider a boundary control problem for semilinear elliptic differential equations. We want to prove only the existence of a solution. For first order necessary optimality conditions we refer for instance to Casas (1993) and Bonnans and Casas (1991). Moreover, a new development in second order optimality conditions occurs (consult, for instance, Casas, Tröltzsch and Unger, 1996). The questions concerning optimality conditions requiring assumptions on differentiability will not be considered here.

As mentioned above, a standard method for the proof of existence is known. This method was applied, for instance, in Lions (1968), Zeidler (1990), and in Casas (1993). The main disadvantage of this method consists in the assumption of linearity of the partial differential equation with respect to the control. Nevertheless, we will recall it for convenience. Another method in this field is to assume compactness of the set of admissible controls (see, e.g., Zeidler, 1990). But this restriction is rather strong and is not satisfied in our concrete setting. Therefore, we will use a method based on measurable selection theory, which is known for control problems governed by ordinary differential equations from, for instance, Macki and Strauss (1982). In the case of parabolic control problems the method was applied in Eppler (1993).

Also in Papageorgiou (1991) results of measurable selection theory were used to derive existence of solutions for nonlinear elliptic control problems. But we want to consider boundary control problems in contrary to problems with distributed control as in Papageorgiou (1991). On the other hand we will allow in some sense more general cost functionals.

2. Preliminaries

In this paper we want to investigate the following control problem:

$$\Phi(w,u) = G(w) + H(u) = \min! \tag{1}$$

subject to

$$\begin{array}{rcll} (Aw)(x) & = & f(x) & \text{in } \Omega, \\ \partial_{\nu_A} w(x) & = & b(w(x),u(x)) & \text{on } \Gamma, \end{array} \tag{2}$$

$$u \in \mathcal{U}^{ad}. \tag{3}$$

We make the following assumptions:

(A1) $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a simply connected domain with a boundary Γ of class C^1.

(A2) $\mathcal{U}^{ad} := \{u(\cdot) \in L_\infty(\Gamma) \mid u(x) \in [a,d] \text{ a. e. on } \Gamma\}$, where $a < d$ are fixed real numbers. $\mathcal{U}^{ad}$ is a convex, bounded and closed subset of $L_\infty(\Gamma)$.

(A3) $G(\cdot)$ is a continuous functional on $C(\overline{\Omega})$ and $H(\cdot) : L_q(\Gamma) \mapsto \mathbb{R}$ is convex and continuous for a certain fixed $q > n-1$.

(A4) The mapping A given by

$$(Aw)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij}(x) \frac{\partial w}{\partial x_i}(x) + a_0(x)w(x) \tag{4}$$

generates a continuous and coercitive bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to$ IR with

$$a(w,v) = \int\limits_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial w}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x) + a_0(x) w(x) v(x) \right) dx,$$

where $a_{ij} \in C^{0,1}(\overline{\Omega})$ and $a_0 \in L_\infty(\Omega)$ such that $a_0(x) \geq m > 0$ a.e. in Ω.

$$\partial_{\nu_A} w(x) = a_{ij}(x) \frac{\partial w}{\partial x_i}(x) n_j(x),$$

where n_j denotes the jth component of the unit outward normal on Γ in x.

(A5) The function $b : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is uniformly Lipschitz-continuous in both arguments and monotone decreasing with respect to the first variable, i. e., for all $w_i, u_i \in \mathbb{R}$, $i = 1, 2$ we have:

$$|b(w_1,u_1) - b(w_2,u_2)| \leq C(|w_1 - w_2| + |u_1 - u_2|, \tag{5}$$

$$(b(w_1,u) - b(w_2,u))(w_1 - w_2) \leq 0. \tag{6}$$

(A6) $f(\cdot) \in L_p(\Omega)$ with $p > n/2$.

We use the following notations in the paper:
$H^1(\Omega)$ is the usual Sobolev space. Moreover, we denote by "$\rightharpoonup$" and "$\rightarrow$" weak and strong convergence of elements, respectively. The underlying Banach space, such as "L_p", "C" or "C^α", is only shortly indicated at the top of "rightharpoonup" and "rightarrow", respectively, because the correct space becomes clear from the context.
Furthermore, we introduce
$\mathcal{W}^{ad} = \{w(u) |\ w(\cdot)$ is the (weak) solution of the boundary value problem (BVP) (2) for a given $u \in \mathcal{U}^{ad}\}$ as the set of admissible states and
$\mathcal{B}^{ad} = \{b(w(u), u) |\ u \in \mathcal{U}^{ad},\ w(u) \in \mathcal{W}^{ad}\}$ as the set of all "admissible right-hand sides" of the boundary condition in (2).
Besides, the notion of *complete continuity* is used as follows:
We call a mapping $P : X \mapsto Y$, $(X, Y$ Banach spaces) *completely continuous*, if the weak convergence of a sequence $\{x_n\} \subset X$ implies the strong convergence of the images with respect to Y, i. e.: $x_n \rightharpoonup_X x_0 \in X \Rightarrow P(x_n) \rightarrow_Y P(x_0) \in Y$. Note that this notion may be used in a different way in other papers.

In the next section we will discuss some properties of the boundary value problem (2), which is called the state equation.

3. The weak solution of the state equation

In view of our assumptions on the control and the right hand side of the boundary value problem, we use the weak solution approach for the state equation. The weak formulation of the boundary value problem (2) is given by

$$\begin{aligned} a(w,v) &= \int\limits_{\Gamma} b(w(x), u(x)) v(x) dS_x + \int\limits_{\Omega} f(x) v(x) dx \\ &= B(w,u,v) + F(v) \end{aligned} \tag{7}$$

for all $v \in H^1(\Omega)$.

The assumed properties of the bilinear form a ensure the estimates

$$|a(w,v)| \leq M\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

and

$$a(w,w) \geq m\|w\|^2_{H^1(\Omega)}.$$

For every $u \in \mathcal{U}^{ad}$ the assumptions (A5) and (A6) on b and f, respectively, together with these estimates imply the existence of a unique solution $w \in H^1(\Omega)$. See for instance Kinderlehrer and Stampacchia (1983) or Zeidler (1990). Note that in our setting we can not expect coercitivity or monotonicity of a with respect to norms of Sobolev–Slobodetskiĭ spaces $W_p^1(\Omega)$ with $p > 2$. On the other hand, we have supposed that G is a functional on $C(\overline{\Omega})$. Therefore, existence theory for weak solutions of elliptic boundary value problems is not sufficient. We need additional statements from regularity theory. The results presented here are going back to Kinderlehrer and Stampacchia (1983) and are summarized in Li and Yong (1995).

Before we start the consideration of the nonlinear problem, a statement with respect to linear boundary value problems should be in order. This statement will allow some useful estimations in relation to the nonlinear problem. Let the linear problem be given by

$$\begin{aligned} (Aw)(x) = &\quad f(x) &\quad \text{in } \Omega, \\ \partial_{\nu_A} w(x) + \beta_1(x)w(x) = &\quad \beta_2(x) &\quad \text{on } \Gamma. \end{aligned} \tag{8}$$

Then it holds:

LEMMA 3.1 *Let the assumptions on Ω, A and f be fulfilled. Suppose that $\beta_1 \in L_\infty(\Gamma)$ with $\beta_1(x) \geq 0$ almost everywhere on Γ and $\beta_2 \in L_q(\Gamma)$. Then there is a constant $\alpha \in (0,1)$ such that the unique weak solution w of (8) belongs to $H^1(\Omega) \cap C^\alpha(\overline{\Omega})$. Moreover, we find a constant $C > 0$ not depending on f and β_2 with*

$$\|w\|_{H^1(\Omega)} + \|w\|_{C^\alpha(\overline{\Omega})} \leq C(\|f\|_{L_p(\Omega)} + \|\beta_2\|_{L_q(\Gamma)}). \tag{9}$$

We return to the nonlinear problem. As mentioned above we need a regularity result which ensures that the weak solution of the boundary value problem (2) belongs to the space $C(\overline{\Omega})$. The last and the following statement are taken from Li and Yong (1995):

THEOREM 3.1 *Let the assumptions (A1)–(A6) be fulfilled. Then there exists a constant $\alpha \in (0,1)$ such that for every control $u \in \mathcal{U}^{ad}$ the corresponding weak solution of the problem (2) belongs to $H^1(\Omega) \cap C^\alpha(\overline{\Omega})$.*

Now we are able to introduce an operator describing the state mapping:

Definition 3.1 *We define the operator* $K : \mathcal{U}^{ad} \mapsto H^1(\Omega) \cap C^\alpha(\overline{\Omega})$ *(the state mapping) by* $w = K(u)$, *where* w *is the solution of equation (7) corresponding to the control* u. *A pair* $(u_0, w_0) \in \mathcal{U}^{ad} \times H^1(\Omega) \cap C^\alpha(\overline{\Omega})$ *is called an admissible pair, iff* $u_0 \in \mathcal{U}^{ad}$ *and* $w_0 = K(u_0)$.

It would be useful to have some continuity properties of the mapping K. The assertion of the next theorem appears to be natural:

Theorem 3.2 *Let the assumptions of Theorem 3.1 be satisfied. Then there is a constant* $C > 0$ *such that for all admissible pairs* (w_1, u_1) *and* (w_2, u_2) *the estimate*

$$\|w_1 - w_2\|_{H^1(\Omega)} + \|w_1 - w_2\|_{C^\alpha(\overline{\Omega})} \leq C\|u_1 - u_2\|_{L_q(\Gamma)} \tag{10}$$

holds.

Proof: In view of Lemma 3.1 the statement seems to be obvious. Nevertheless, the proof using only this lemma is lengthy and technical. We will give here only the main ideas.

The difference $w_1 - w_2$ satisfies the equation

$$a(w_1 - w_2, v) = B(w_1, u_1, v) - B(w_2, u_2, v)$$

for all $v \in H^1(\Omega)$. Looking upon the right hand side as given, we see that this is a linear problem. The properties of a and b and Lemma 3.1 in connection with the Sobolev embedding theorem imply the estimates

$$\begin{aligned}
\|w_1 - w_2\|_{L_t(\Gamma)} &\leq c\|u_1 - u_2\|_{L_2(\Gamma)} \\
\|w_1 - w_2\|_{L_t(\Gamma)} &\leq c(\|u_1 - u_2\|_{L_2(\Gamma)} + \|w_1 - w_2\|_{L_2(\Gamma)}) \\
\|w_1 - w_2\|_{L_\infty(\Gamma)} &\leq c(\|u_1 - u_2\|_{L_q(\Gamma)} + \|w_1 - w_2\|_{L_q(\Gamma)})
\end{aligned}$$

with generic constant c and t depending on boundary embedding of $H^1(\Omega)$.

The interpolation property of L_s spaces (see, e.g., Triebel, 1983) ensures

$$\|w_1 - w_2\|_{L_{t'}(\Gamma)} \leq c(\|u_1 - u_2\|_{L_s(\Gamma)} + \|w_1 - w_2\|_{L_s(\Gamma)}),$$

where $s, t' \in [2, \infty)$ with $t' - s \geq s' > 0$.

Now starting with the estimate from Lemma 3.1 the assertion can be proved by a bootstrapping argument. □

Remark 1: As a consequence of estimate (10), the set $\mathcal{W}^{ad}$ of all admissible states is uniformly bounded: There exists a $C_o > 0$ with

$$\sup_{u \in \mathcal{U}^{ad}} \sup_{x \in \Omega} |w(u; x)| \leq C_o, \tag{11}$$

which implies the uniform a.e. boundedness of the set $\mathcal{B}^{ad}$, too. In addition we have the estimate

$$\|w_1 - w_2\|_{C^\alpha(\overline{\Omega})} \leq c\|b(w_1, u_1) - b(w_2, u_2)\|_{L_q(\Gamma)}. \tag{12}$$

Therefore, for all sequences $\{b_n(\cdot)\}_{n=1}^{\infty} \subset \mathcal{B}^{ad}$ with $b_n \rightarrow_{L_q} \bar{b} \in L_q(\Gamma)$ there is $w_n \rightarrow_{C^\alpha} \bar{w}$, where w_n, $\bar{w}$ are the associated solutions of (7) to b_n and $\bar{b}$, respectively. Furthermore, due to the linearity of the mapping, the same is true with respect to weak convergence, i. e., $b_n \rightharpoonup_{L_q} \bar{b}$ implies $w_n \rightharpoonup_{C^\alpha} \bar{w}$.

4. Existence of optimal solutions

Although the continuity of the state mapping K is not sufficient to guarantee existence of optimal solutions, the underlying investigations are very useful for showing the closedness of $\mathcal{W}^{ad}$ in $C(\overline{\Omega})$. Moreover, with an additional assumption, we get the complete continuity of the state mapping from $L_q(\Gamma)$ to $C(\overline{\Omega})$. The next result will be proved by means of separation techniques for convex sets and measurable selection theorems. This method was at first developed for control problems governed by ordinary differential equations (see, for instance, Macki and Strauss, 1982). Later, similar techniques were used by Eppler (1993) for parabolic boundary control problems. This works well even in the case of elliptic control problems.

LEMMA 4.1 *The assumptions (A1)-(A6) ensure the compactness of $\mathcal{W}^{ad}$ in $C(\overline{\Omega})$.*

Proof: At first we remark that $\mathcal{W}^{ad}$ is a relatively compact subset of $C(\overline{\Omega})$. This follows from the estimate (10), the boundedness of $\mathcal{U}^{ad}$ and the compact embedding of $C^\alpha(\overline{\Omega})$ in $C(\overline{\Omega})$.
Now we take a sequence $\{w_n\}_{n=1}^{\infty} \subset \mathcal{W}^{ad}$ $(w_n = w(u_n)$ for some $u_n \in \mathcal{U}^{ad})$ and assume without loss of generality

$$w_n \rightarrow_C \bar{w}. \tag{13}$$

For $\bar{w}$ we have $\|\bar{w}\|_{C(\overline{\Omega})} \leq C_o$ (cf. (11)). Moreover, for the associated "right hand sides" $b_n = b(w_n, u_n)$ of the boundary condition (2) the relation $b_n \rightharpoonup_{L_q} \bar{b}$ is fulfilled, where $\bar{w}$ is the solution of equation (7) associated to the right hand side $\bar{b}$ in the boundary condition. This can be shown as follows:
Every subsequence $\{b_{n'}\} \subset \{b_n\}$ contains a sub-subsequence $\{b_{n"}\} \subset \{b_{n'}\}$ with $b_{n"} \rightharpoonup_{L_p} b"$, because $\mathcal{B}^{ad}$ is bounded in $L_q(\Gamma)$. This implies $w_{n"} \rightarrow_C w"$ ($w"$ is the solution of (7) associated to $b"$) and from (13) we have $w" = \bar{w} \Rightarrow b" = \bar{b}$.

If we were able to show the existence of a control $u_0 \in \mathcal{U}^{ad}$ such that $\bar{b} = b(\bar{w}, u_0)$ (in detail: $\bar{b}(x) = b(\bar{w}(x), u_0(x))$ a.e. on Γ), then, altogether we have $\bar{w} = w(u_0)$ and the theorem will be proved.
To this aim we introduce the sets

$$M(\bar{w}(x)) \;=\; \{b(\bar{w}(x), y)|\; y \in [a, d]\} \;\subset \mathbb{R},$$

and the functions $\underline{m}(\cdot)$ and $\overline{m}(\cdot)$, defined by

$$\underline{m}(x) = \min\{z|\; z \in M(\bar{w}(x))\} = \min\{b(\bar{w}(x), y)|\; y \in [a, d]\},$$

$$\overline{m}(x) = \max\{z|\ z \in M(\bar{w}(x))\} = \max\{b(\bar{w}(x), y)|\ y \in [a, d]\},$$

for all $x \in \Gamma$. The continuity of $\bar{w}(\cdot)$ on Γ and the uniform continuity of $b(\cdot,\cdot)$ on the compact set $[-C_o, C_o] \times [a, d]$ guarantee the continuity, hence Lebesgue measurability of $\underline{m}(\cdot)$ and $\overline{m}(\cdot)$. Moreover, we have

$$M(\bar{w}(x)) \ = \ [\underline{m}(x), \overline{m}(x)],$$

i. e., all sets $M(\bar{w}(x))$ are convex subsets of R. The measurability of the set M_1, defined by

$$M_1 \ = \ \{x \in \Gamma| \quad \bar{b}(x) \notin M(\bar{w}(x))\},$$

is an immediate consequence.
In the next step we prove that $\mathrm{mes}(M_1) > 0$ is a contradiction to $b_n \rightharpoonup_{L_q} \bar{b}$. In doing so we rewrite M_1 as $M_1 = \underline{M}_1 \cup \overline{M}_1$ with

$$\underline{M}_1 \ = \ \{x|\ \bar{b}(x) < \underline{m}(x)\}, \quad \overline{M}_1 \ = \ \{x|\ \bar{b}(x) > \overline{m}(x)\},$$

and assume without loss of generality $\mathrm{mes}(\overline{M}_1) > 0$. ¿From this we get the existence of a constant $\delta > 0$ and of a subset $\overline{M}_1^\delta \subset \overline{M}_1$ with $\mathrm{mes}(\overline{M}_1^\delta) > 0$, such that

$$\bar{b}(x) \geq \delta + \overline{m}(x) \geq \delta + \sup_{n\geq 1} b(\bar{w}(x), u_n(x)), \qquad \forall x \in \overline{M}_1^\delta$$

(the last inequality follows by the definition of $\overline{m}$). The uniform continuity of b and $w(u_n) \rightarrow_C \bar{w}$ ensure

$$|b(w_n(x), u_n(x)) \ - \ b(\bar{w}(x), u_n(x))| \leq \delta/2,$$

for all $x \in \Gamma$, $n \geq N_0(\delta)$, and therefore

$$\bar{b}(x) \geq \delta/2 + \sup_{n\geq N_0(\delta)} b(w_n(x), u_n(x)), \qquad \forall x \in \overline{M}_1^\delta.$$

Integration over $\overline{M}_1^\delta$ yields

$$\int_{\overline{M}_1^\delta} \bar{b}(x)\, dS_x \geq \ \mathrm{mes}(\overline{M}_1^\delta)\delta/2 \ + \int_{\overline{M}_1^\delta} b_n(x)\, dS_x, \quad \forall n \geq N_0.$$

This is a contradiction to $b_n \rightharpoonup_{L_q} \bar{b}$, because $\chi_{\overline{M}_1^\delta} \in L_{q'}(\Gamma)$ for the characteristic function $\chi(\cdot)$ of the set $\overline{M}_1^\delta$ $(1/q + 1/q' = 1)$. Consequently, we get

$$\bar{b}(x) \in \{b(\bar{w}(x), y)|\ y \in [a, b]\} \ \text{ a.e. on } \Gamma.$$

That means that there exists at least one selection $\bar{u}(x) \in [a, d]$ with

$$\bar{b}(x) \ = \ b(\bar{w}(x), \bar{u}(x)) \ \text{ a.e. on } \Gamma.$$

The application of the Fillipov lemma (Macki and Strauss, 1982) guarantees a *measurable* selection $u_0(\cdot)$, by setting

$$u_0(x) = \begin{cases} a\,, & x \in M_1 \\ \min\{y \in [a,d] | \bar{b}(x) = b(\bar{w}(x), y)\,, & x \notin M_1. \end{cases} \quad \square$$

Remark 2: Obviously, the set $\mathcal{W}^{ad}$ is also weakly closed in $C^{\alpha}(\overline{\Omega})$ and the set $\mathcal{B}^{ad}$ is weakly closed in $L_q(\Gamma)$, too (but both sets are not necessarily convex).

In order to prove the complete continuity of the state mapping, we need an additional structural assumption on the function b. This well known idea was realized, for instance, by Sperber (1983) for optimal control problems governed by semilinear parabolic equations, and was further developed and applied by Eppler (1988), (1993).

LEMMA 4.2 *In addition to (A1) - (A6) we suppose that*

$$b(w, u) \;=\; b_1(w) \cdot u + b_2(w), \tag{14}$$

with Lipschitz-continuous functions $b_1(\cdot)$ and $b_2(\cdot)$ (i. e., the function b is affine-linear with respect to u).
Then, the state mapping K is completely continuous on $\mathcal{U}^{ad}$ from $L_q(\Gamma)$ to $C(\bar{\Omega})$. More precisely: The condition $u_n \rightharpoonup_{L_q} u_0$, $u_n \in \mathcal{U}^{ad}$, $n = 0, 1, \ldots$ implies $w(u_n) \rightarrow_C w(u_0)$ for the associated states.

Proof: We take a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{U}^{ad}$ with $u_n \rightharpoonup_{L_p} u_0 \in \mathcal{U}^{ad}$. Similar to the discussion at the beginning of the proof of Theorem 4.1 we get for every subsequence $\{u_{n'}\} \subset \{u_n\}$ a sub-subsequence $\{u_{n''}\} \subset \{u_{n'}\}$ with $b_{n''} = b(w(u_{n''}), u_{n''}) \rightharpoonup_{L_q} b''$ and $w(u_{n''}) \rightarrow_C w''$ (w'' is the solution of (7) associated to b'').

If we have in addition the "strong-weak" continuity of the Nemytskiĭ-Operator defined by the function $b(\cdot,\cdot)$, more precisely, if we have:
the conditions $u_n \rightharpoonup_{L_q} u_0$ and $w(u_n) \rightarrow_C \bar{w}$ imply the weak convergence $b(w(u_n), u_n) \rightharpoonup_{L_p} b(\bar{w}, u_0)$, then $b'' = b(w'', u_0)$ and therefore $w'' = w(u_0)$, which concludes the proof.

In order to show this, we use the special structure of the function b and the concrete duality product of L_q. Although this is well known, we want to outline the proof once more:
We fix an arbitrary $f(\cdot) \in L_{q'}(\Gamma)$ $(1/q + 1/q' = 1)$, and estimate as follows:

$$\int_{\Gamma} f(x) \cdot [b_1(w(u_n; x),)u_n(x) + b_2(w(u_n; x)) - b_1(\bar{w}(x))u_0(x) - b_2(\bar{w}(x))]dS_x|$$

$$\leq |I_n^1| + |I_n^2| + |I_n^3|,$$

with

$$I_n^1 = \int\limits_{\Gamma} f(x)u_n(x)[b_1(w(u_n;x)) - b_1(\bar{w}(x))]dS_x,$$

$$I_n^2 = \int\limits_{\Gamma} f(x)b_1(\bar{w}(x))(u_n(x) - u_0(x))dS_x,$$

and

$$I_n^3 = \int\limits_{\Gamma} f(x)[b_2(w(u_n;x)) - b_2(\bar{w}(x))]dS_x.$$

Because of $w(u_n) \to_C \bar{w}$ and the continuity of $b_2(\cdot)$ we have $b_2(w(u_n;\cdot)) \to_C b_2(\bar{w}(\cdot))$, and therefore $I_n^3 \to 0$ for $n \to \infty$. Analogously, we deduce from the uniform boundedness of all u_n the relation $I_n^1 \to 0$. Finally, we get $I_n^2 \to 0$ from the weak convergence of the controls u_n ($f(\cdot)b_1(\bar{w}(\cdot)) \in L_q(\Gamma)$ is a consequence of $b_1(\bar{w}(\cdot)) \in C(\Gamma) \subset L_\infty(\Gamma)$). Hence, we have the announced "strong-weak" continuity of the Nemytskiĭ-Operator and the lemma is proved. □

Remark 3: In fact we have weak continuity for K from $L_q(\Gamma)$ to $C^\alpha(\bar{\Omega}$, i. e., $u_n \rightharpoonup_{L_q} u_0$ implies $w(u_n) \rightharpoonup_{C^\alpha} w(u_0)$). Moreover, compactness of $\mathcal{W}^{ad}$ (in $C(\bar{\Omega})$) and weak closedness of $\mathcal{B}^{ad}$ (in $L_p(\Gamma)$) now follow immediately from the properties of the underlying mappings.

A well known difficulty for the existence of optimal controls is the noncompactness of the set of admissible controls, if it is defined as above. However, $\mathcal{U}_{ad}$ is closed, bounded and convex in $L_q(\Gamma)$, hence weakly compact. Therefore, one way to overcome this difficulty consists in using the complete continuity of the state mapping.

THEOREM 4.1 *Under the assumptions stated in Lemma 4.2 the optimal control problem (1) - (3) is solvable.*

Proof: With the help of our investigations of Section 3 and Lemma 4.2 this is a simple conclusion from the Weierstrass Theorem, applied to the weak topology on $L_q(\Gamma)$. Note that continuity and convexity of $H(\cdot)$ are sufficient for the weak lower semicontinuity. □

Remark 4: Obviously, every weak accumulation point $\bar{u} \in \mathcal{U}^{ad}$ of a minimizing sequence $\{u_n\}$ is an optimal control of the control problem.

Although this is a standard result, the assumption (14) seems to be rather strong. Therefore, we present another result, based on Lemma 4.1.

THEOREM 4.2 *If the assumptions (A1)-(A6) are satisfied and if the objective of the control problem does not explicitly depend on the control u (i. e., if the functional $H(\cdot)$ is vanishing), then problem (1) - (3) has at least one solution.*

Proof: Because the part $H(\cdot)$ is not present, the optimal control problem can be formally reformulated as

$$G(w) = \min! \quad \text{subject to} \quad w \in \mathcal{W}^{ad}.$$

Now Lemma 4.1 and assumption (A3) ensure that the application of the Weierstrass-Theorem is also possible in this case. □

Remark 5: In contrary to the theorem above, on the one hand a weak accumulation point of the sequence $\{u_n\}$ is generally not an optimal control. On the other hand the optimal control u_0 need not to be a weak accumulation point of the sequence $\{u_n\}$, whereas the state $w(u_0)$ is an accumulation point of $\{w(u_n)\}$.

5. Concluding remarks

Similarly to Papageorgiou (1991) an existence result is possible for an objective of integral type explicitly depending on the control. To this aim some appropriate convexity hypotheses with respect to sets, concerning the integrand of the objective together with the right hand side of the boundary condition, have to be satisfied. In our situation this reduces to the convexity of the sets $M(w(\cdot))$ (compare the proof of Lemma 4.1), which is obvious and therefore no additional assumption is required. Moreover, the resulting closedness of $\mathcal{W}^{ad}$ is of interest in its own right.

The use of classical results on $C^{\alpha}(\overline{\Omega})$-regularity for the state may be substituted by $W_1^{q+1}(\Omega)$-regularity, because some compact embedding in $C(\bar{\Omega})$ is essential for our investigations of Section 4. However, the assumption known to the authors, guaranteeing such a property, describes some relations between the ellipticity of the operator A and the domain Ω. This is rather technical and should be avoided. A more useful condition seems to be an open problem.

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