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Common internal properties among power indices

by

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Abstract: On the basis of several common internal properties among the best known power indices, we consider a subset in the imputation set to which standard power indices belong. This notion originates an interval of power that assigns intervals of real numbers to the players. We generate power indices based on marginal contributions and consider a second notion of interval which contains Shapley-Shubik and Banzhaf indices for every game.
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1. Introduction

The problem of designing fair voting systems is central to Social Choice Theory. In this paper we are concerned with voting systems in which an alternative (such as a bill or an amendment) is pitted against the status quo (no change in the body of the law). Such systems are conveniently formalized by simple games.

Standard power indices, introduced by Shapley-Shubik, Banzhaf, Schmeidler, Johnston, Deegan-Packel and others, assign real numbers to the players in a simple game as a quantitative measure of their influence in the voting situation represented by the game. They are based on different axiomatic grounds and bargaining models. Nevertheless, of the various attempts to make the notion of power indices precise, none has yet emerged as being the most suitable.

We consider instead a subset in the imputation set to which the four power indices cited above belong. This notion is based on the existence of certain regularities among power indices and the idea of eliminating certain values as being clearly inappropriate, which leaves us with a range of possibilities. Furthermore, as point out by Taylor and Zwicker (1995) "there are situations for which our intuitions will never be sharp enough to pin down an exact power index that correctly measures the player's influence". The subset in consideration originates an interval notion of power that assigns intervals of real numbers to the players.

Using a characterization of semivalues by means of weighting coefficients, we generate distinct power indices (all based on marginal contributions) that measure different types of power. This class of solution satisfies strict monotonicity, which allows us to consider a second notion of interval of power.

2. Power indices

For our purposes, a *simple game* G is a structure $(N, \mathcal{W})$ in which N is the set $\{1, 2, \ldots, n\}$ and $\mathcal{W}$ is an arbitrary collection of subsets of N. The idea is that the bill being voted on passes if the set of those in favor belongs to $\mathcal{W}$. For this reason, elements of N are called *players*, sets in $\mathcal{W}$ are called *winning coalitions*, and subsets of N not in $\mathcal{W}$ are called *losing coalitions*. Within this paper we assume that all games are *monotonic* (every superset of a winning coalition is winning) and have the property that N is a winning coalition and $\emptyset$ is a losing coalition. We place no other restrictions on G. Note that a simple game is completely determined by a real-valued function v defined on N such that $v(S) = 1$ if $S \in \mathcal{W}$ and $v(S) = 0$ if $S \notin \mathcal{W}$. Then the structure (N, v) also defines G.

Interesting problems arise when attempting to design fair voting systems for circumstances in which it is appropriate for different players to have different amounts of influence. For example, suppose that each of the four shareholders in a certain company sends one representative to the board of directors. If the first and second one represent 33% each of the shares, the third 20% and the fourth 14%, how should the representatives' votes be counted? The naive solution is to weight the representatives' votes so that a "yes" by the first representative counts as .33, etc. and to declare that the proposal passes if the total weight in favor is .51 or more. The result is an example of a weighted game; more generally, $G = (N, \mathcal{W})$ is *weighted* (or weighted majority game) if there exists a function $w : N \to \mathbf{R}$ and a *quota* $q \in \mathbf{R}$ such that a coalition S is winning precisely if $\sum\{w(i) : i \in S\} \geq q$.

A particular choice of weights for a weighted game may be greatly disproportionate to the distribution of influence. In the above example it is easy to see that if we use the naive solution, the representative with the 14% of shares has no influence (that representative is a *null* – meaning that a change in his or her vote can never affect the outcome) while the other three have equal influence (any permutation among them leaves the set $\mathcal{W}$ fixed).

The amount of influence in a simple game is conventionally measured by means of an index. Formally, a power ψ is a function defined on $(N, \mathcal{W})$, representing a reasonable expectation of the percentage share of the decisional power among the various players, in relation to their strength in the game. We denote $\psi(v)(i)$ the quota of power that the index ψ grants to the ith player in the game v; such a quota is called *power index of the ith player*. Obviously, a power index is an *imputation*, i.e., a function

$$\psi : N \to [0,1] \quad \text{satisfying} \quad \sum_{i=1}^{n} \psi(i) = 1. \tag{1}$$

and this is the only requirement to define a power index. We will omit v when the game is known or a property is verified for all games.

Several authors have proposed various power indices on the basis of different axiomatic grounds and bargaining models. The most widely used of them are the Shapley-Shubik index (Shapley-Shubik, 1954), for its axiomatic derivations, see Dubey (1975), and the Banzhaf index (introduced in Banzhaf, 1965 and Coleman, 1971), for its axiomatic derivations, see Owen (1978). A court case has even led to the inclusion of the Banzhaf index (Banzhaf, 1965) in New York state law. Other indices introduced in the 1970s, can be found in Schmeidler (1969), Johnston and Deegan-Packel (1978).

Now we recall the two best known notions of power: the Shapley-Shubik and Banzhaf indices. Both notions are similar in the sense that are based on marginal contributions. We continue with the nucleolus (introduced by Schmeidler in 1969) for simple games, and the Johnston and Deegan Packel indices.

The *Shapley-Shubik index of i*, denoted here by SSI(i), is the number obtained as follows: $SSI(i) = \sum s!(n-s-1)!/n!$ where the sum is extended to all coalitions (of s members) for which the ith player is *crucial* (Gambarelli, 1983), that is, the coalition is winning with him or her and losing without him or her, (i.e., $S \cup \{i\} \in \mathcal{W}$ but $S \notin \mathcal{W}$). The *normalized Banzhaf index of i*, denoted here by $BI(i)$, assigns to each player the ratio between the number of coalitions for which he or she is crucial, the total Banzhaf power, and the sum of all these quantities for each player.

The excess functions which are implied in Schmeidler's nucleolus are the real-valued

$$e(x, S) = \begin{cases} 1 - x(S) & \text{if } S \in \mathcal{W} \\ -x(S) & \text{if } S \notin \mathcal{W} \end{cases}$$

defined for all imputations x and every coalition S. Having ordered these 2^n coordinates for every imputation x, in a vector $e(x)$ in a weakly decreasing order, then the *nucleolus* (denoted here by $SI(i)$, to refer to Schmeidler index) is defined by the imputation which, is the minimum in the lexicographic order. (i.e., $y \preceq_{lex} z$ if $y = z$ or there is an index $k \leq m$ such that $y_i = z_i$ for $i < k$ and $y_k < z_k$).

The total *Johnston power* of i, denoted here by $TJP(i)$, is the number obtained as follows:

Suppose $S_1, \cdots, S_p$ are the winning coalitions for which i's defection is critical. Suppose n_1 is the number of players whose defection from S_1 is critical, n_2 is the number of players whose defection from S_2 is critical and so on up to n_p being the number of players whose defection from S_p is critical. Then

$$TJP(i) = \sum_{k=1}^{p} \frac{1}{n_k}$$

(or zero if i is a null player). Then the *Johnston index of* i, denoted here by $JI(i)$, is obtained by normalizing $TJP(i)$ (taking into account all players).

The total *Deegan Packel* of i, denoted here by $TDPP(i)$, is the number obtained as follows:

Suppose $S_1, \cdots, S_j$ are the minimal winning coalitions to which i belongs. Suppose n_1 is the number of players in S_1, n_2 is the number in S_2, and so on up to n_j being the number of players in S_j. Then

$$TDPP(i) = \sum_{k=1}^{j} \frac{1}{n_k}$$

(or zero if i is a null player). Then the *Deegan-Packel index* of i, denoted here by $DPI(i)$, is obtained by normalizing $TDPP(i)$ (taking into account all players).

The different indices can produce wildly different measurements. For instance, using the four indices cited above, in the United States federal system there are 537 voters: 435 members of the House of Representatives, 100 members of the Senate, the vice president, and the president. The vice president plays the role of tiebreaker in the Senate, and the president has veto power that can be overridden by a two-thirds vote of both the House and the Senate. Thus, for a bill to pass it must be supported by either:

218 or more representatives and 51 or more senators (with or without the vice president) and the president.

218 or more representatives and 50 senators and the vice president and the president.

290 or more representatives and 67 or more senators (with or without either the vice president or the president).

The power held by the president is: 0.16 (SSI), 0.038 (BI), 0.138 (SI), 0.77 (JI) and 0.0037 (DPI). The variability among them is huge, but, is there any limitation, or can the power of the president take any value in [0,1]? In the next section we shall give minimal requirements for power indices suitable for describing political models, arbitration problems, bargaining situations, etc. In this context, we plan to add to (1) new properties for an arbitrary power index.

3. Internal properties

We begin by considering monotonic's property. We will prove, for the first four indices cited above, that each power of them is a vector in the imputation set that preserves the order of desirability relation. The desirability relation $\succeq$ relative to game $\mathcal{W}$, is defined as

$$i \succeq j \quad \Leftrightarrow \quad S \cup \{j\} \in \mathcal{W} \quad \text{implies} \quad S \cup \{i\} \in \mathcal{W}.$$

An immediate consequence of this definition is that if $(N, \mathcal{W})$ is a weighted game then $i \succeq j$ implies $w(i) \geq w(j)$, i.e., the desirability relation preserves the order of the weights. The desirability relation on individuals goes back at least to Isbell (1958) and the collection of games for which the desirability relation is complete to be a natural collection, and has been investigated in Hu (1965), Carreras (1984), Ostmann (1987), Krohn and Sudhölter (1995), Carreras and Freixas (1995), and Taylor and Zwicker (1996).

Formally, a power index ψ is *monotonic* if

$$i \succeq j \quad \text{if, and only if} \quad \psi(i) \geq \psi(j). \tag{2}$$

As Deegan and Packel (1982, pp. 247-248) point out, their index violates monotonicity. In the weighted majority game

$$[6; 4, 2, 1, 1, 1, 1]$$

the Deegan-Packel index has a value of 3/20 for each of the four last players, and 7/80 for the second player. The Deegan-Packel index assumes that only minimal winning coalitions be formed. This limitation yields this behaviour.

The second requirement deals with equal treatment among players. It seems reasonable to require that a permutation θ of N such that $\theta v = v$ (an automorphism) preserves power. We will say that a power index ψ is *anonymous* if

$$\psi(i) = \psi(\theta(i)) \quad \text{for all } i \in N \tag{3}$$

for every automorphism θ. It should be noted that this third axiom only has incidence if the game is not complete, since otherwise the orbits (two players i, j belong to the same orbit if an automorphism exists such that $\theta(i) = j$), denoted here by $\mathcal{O}$, and indifferent classes (two players i, j belong to the same indifferent class if $i \succeq j$ and $j \succeq i$) coincide.

The set of automorphisms of a given game v is a subgroup of the symmetric group $\mathcal{S}(\mathcal{N})$. The set of automorphisms of a given game v is *transitive* if the game has a unique orbit.

A player i is *null* if $i \notin S$ for all $S \in \mathcal{W}$. We shall denote $\mathcal{D}$ to the class of null players (if any). We will declare the power index ψ as satisfying null property if

$$i \in \mathcal{D} \quad \text{imply} \quad \psi(i) = 0. \tag{4}$$

THEOREM 3.1 *The Shapley-Shubik, Banzhaf, Johnston indices and the nucleolus are imputations which satisfy monotonicity, anonymity and null properties.*

Proof: Monotonicity. For Shapley-Shubik and Banzhaf indices this property will be proved in Proposition 4.2.

Johnston index: Suppose $i \succeq j$ and $S_1, \ldots, S_p$ are the winning coalitions for which j's defection is critical. Then, $S_1, \ldots, S_p$ are winning coalitions for which i's defection is critical, consequently, $TJP(i) \geq TJP(j)$ and $JI(i) \geq JI(j)$.

Nucleolus: Set $\nu_j = SI(j)$ and $\nu_i = SI(i)$. Suppose $\nu_j > \nu_i$ and $i \succeq j$. For every coalition $\{j\} \subseteq S \subseteq N - \{i\}$ we can associate a coalition S' obtained replacing j by i in S. This gives:

$$\begin{aligned} e(\nu, S') &= e(\nu, S) + (\nu_i - \nu_j) + (v(S) - v(S')) \\ &\leq e(\nu, S) + (\nu_i - \nu_j) < e(\nu, S). \end{aligned}$$

Since the imputation ν we define a new imputation ν' whose components are:

$$\begin{aligned} \nu'_k &= \nu_k \qquad k \neq i, j \\ \nu'_i &= \nu'_j = \frac{\nu_i + \nu_j}{2}. \end{aligned}$$

Comparing the excesses for both imputations, we have three possibilities:

a) if $\{i, j\} \subseteq S$ then $e(\nu, S) = e(\nu', S)$,

b) if $S \subseteq N - \{i, j\}$ then $e(\nu, S) = e(\nu', S)$,

c) for every coalition $\{j\} \subseteq S \subseteq N - \{i\}$ and every coalition S' defined as above, we have

$$\begin{aligned} e(\nu', S) &= e(\nu, S) + \frac{\nu_i - \nu_j}{2} \\ e(\nu', S') &= e(\nu, S') + \frac{\nu_j - \nu_i}{2}. \end{aligned}$$

Thus, $e(\nu, S') < e(\nu', S') \leq e(\nu', S) < e(\nu, S)$, and $e(\nu') \leq_L e(\nu)$. Therefore, ν would not be the nucleolus.

Anonymity. A permutation θ is an automorphism of a game v if $\theta v = v$. Thus, the four indices are anonymous.

Null property. A null player does not belong to minimal winning coalitions, so his or her defection from a winning coalition is not critical. Then, Shapley-Shubik, Banzhaf and Johnston indices have this property. The Schmeidler index is reasonable in the sense of Milnor (see Maschler, Peleg and Shapley, 1979), then

$$\nu_i \leq \max_{S: i \in S} [v(S) - v(S - \{i\})], \qquad \text{for all } i \in N.$$

In particular, a null player i receives 0. q.e.d.

For every game $(N, \mathcal{W})$ we define a set $\mathcal{K}$, formed by assignments x with real components x_i for each player, satisfying:

$$\begin{aligned} x_1 + \ldots + x_n &= 1 \\ x_1 \geq 0, \ldots, x_n &\geq 0 \\ x_i \geq x_j & \qquad \text{if } i \succeq j \end{aligned} \tag{5}$$

$$x_i = x_j \qquad \text{if } i, j \in \mathcal{O}$$
$$x_i = 0 \qquad \text{if } i \in \mathcal{D}.$$

Theorem 3.1 says that with the exception of the Deegan-Packel index, the above mentioned indices are in the set $\mathcal{K}$. Either owing to problems of arbitration bargaining or political situations it seems adequate to consider power indices belonging to the set $\mathcal{K}$.

There are several properties that introduce relationships between how power is distributed in different games, one of them deal with monotonicity and you should notice that neither the Shapley-Shubik index nor the Banzhaf index is monotonic in changes of voting weights. Fisher and Schotter (1979) present and discusse cases of weighted majority games in which a player can increase his voting weight and still "receive" less voting power than before. Besides they demonstrated that this "paradox of redistribution" cannot be excluded if the set of players is large enough (about $n = 5$). For instance, Felsenthal and Machover (1994) noticed the following paradoxical result for the Banzhaf index. Consider the weighted majority game

$$[8; 5, 3, 1, 1, 1].$$

The Banzhaf index of the first player turn out to be 9/19. Now suppose that the player with weight 5 gives one of his "votes" to the player with weight 4. This results in the weighted system

$$[8; 4, 4, 1, 1, 1].$$

It now turns out that the first player has Banzhaf index 1/2. But 1/2 is greater than 9/19.

4. The interval notion of power

Of the various attempts to make the notion of power indices precise, none has yet emerged as being clearly correct. Instead Taylor and Zwicker (1995) considered two "interval notions of power" that assign intervals of real numbers to the players. The set $\mathcal{K}$ allows us to define for each player i an interval formed by all possible values of x_i as x ranges over all $\mathcal{K}$. We will call it the i's *Wide interval*, $\mathcal{I}(i)$. For the wide interval we can ensure that the four indices cited above belong to it.

Examples 4.1: (a) In the example considered in first section, the three first players are indifferent to each other, and therefore, belong to the same indifferent class. The fourth player is dominated by other players by desirability relation . Set $\mathcal{K}$ consists of the single point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, and each interval is reduced to their components in $\mathcal{K}$.

(b) The present Catalonian Parliament can be described by means of the weighted majority game

$$[68; 60, 34, 17, 13, 11].$$

Briefly, party 1 is C.i.U., party 2 is P.S.C., party 3 is P.P., party 4 is E.R.C., and party 5 is I.C. Party 1 is the maximum by the $\succeq$ relation. The other parties form an indifferent class. Set $\mathcal{K}$ is:

$$\begin{aligned} x_1 &+ \ldots + x_5 = 1 \\ x_1 &\geq 0, \ldots, x_5 \geq 0 \\ x_1 &\geq x_j \quad \text{if } j = 2,3,4,5 \\ x_i &= x_j \quad \text{if } i,j \geq 2. \end{aligned}$$

All possible values of x_1 as x ranges over all $\mathcal{K}$ gives rise to the wide interval $[1/5, 1]$ for the first player, while for the remaining players the interval is $[0, 1/5]$.

(c) In the United States federal system the President dominates every senator and every representative; this fact, together with efficiency, anonymity and desirability relation (see conditons (5)) gives rise to $\mathcal{I} = [1/536, 1]$. Then, 1/536 is the lower bound for a reasonable power index and also seems adequate the existence of power indices which concentrate the most of the power on the President, e.g. 0.77 by Johnston index.

(d) The 4-player game defined by the minimal winning coalitions $\{1,2\}$ and $\{3,4\}$ is an improper and a non-complete game, and therefore, a non-weighted game (for weighted majority games the desirability relation is total). The automorphism group is the diedrical group of 8 elements which are: Id, t_{12}, t_{34}, $t_{12} \circ t_{34}$, $t_{13} \circ t_{24}$, $t_{14} \circ t_{23}$, $t_{24} \circ t_{34} \circ t_{14}$, $t_{14} \circ t_{34} \circ t_{24}$, (where t_{ij} denote the transposition between players i and j) and "$\circ$" stands for the composition). Consequently, the group has a unique orbit and therefore the game is transitive. Thus, set $\mathcal{K}$ is reduced to $\{(1/4, 1/4, 1/4, 1/4)\}$, and the wide interval is reduced to $\{1/4\}$ for each player.

The situation in the last example can be generalized. If the game is transitive, then by anonymity $\mathcal{K}$ is reduced to one point and automorphism's group is a semidirect product of groups and proof can be found in Freixas (1994).

5. Generating power indices

The semivalues defined for cooperative games and introduced by Dubey, Neyman and Weber (1981) satisfy linear, symmetric, positive and projection axioms. For a simple game a semivalue can be expressed by weighting coefficients:

$$\Psi(i) = \sum_{\substack{i \notin S \\ S \notin \mathcal{W} \\ S \cup \{i\} \in \mathcal{W}}} p_s \qquad \text{with } p_s \geq 0, \ \sum_{s=0}^{n-1} p_s \binom{n-1}{s} = 1. \tag{6}$$

The coefficients p_s can be interpreted as probability weights. In particular, the Shapley-Shubik index SSI, is the only efficient semivalue and is defined by the coefficients $p_s = \frac{1}{n\binom{n-1}{s}}$. Setting $p_s = \frac{1}{2^{n-1}}$ for all k gives rise to the Banzhaf value, while the normalized Banzhaf index is obtained after normalizing the Banzhaf value.

Thus, the semivalues may be identified, for a fixed n, with the points of the intersection of a hyperplane in the n-space with the positive orthant, and this gives $n-1$ degrees of freedom. It might be considered that any semivalue also introduces, in fact, some kind of external information in the game, which concentrates on the evaluation of the marginal contributions to coalitions in terms of their size – the identical treatment given to coalitions of equal size guarantees anonymity property and does not seem a too serious limitation.

The Banzhaf value index does not discriminate between sizes, but the Shapley value strongly favours marginal contributions to extreme coalitions.

We shall say that a semivalue Ψ is *regular* if $p_s > 0$ for $s = 0, 1, \ldots, n-1$, and a semivalue Ψ is *decisive* if $\sum_{i=1}^{n} \Psi(i) > 0$ for all simple games. These definitions are equivalents and we need them to define well-defined power indices.

LEMMA 5.1 *A semivalue is decisive if, and only if, it is regular.*

Proof: Let Ψ be a semivalue. ($\Leftarrow$) In a simple game, by definition, there exists at least one winning coalition and one losing coalition; from regularity it follows that Ψ is decisive. ($\Rightarrow$) Suppose Ψ is not regular. Then, it exist some s with $p_s = 0$. Consider the weighted majority game $v = [s+1; 1, 1, \ldots, 1]$, we have in this game, $v(T) = 0$ if $|T| \leq s$, $v(S) = 1$ if $|S| \geq s+1$. Thus, $\Psi(i) = 0$ for all player and Ψ would not be decisive. q.e.d.

Since a weighted majority game was used in the preceding proof, it follows that only the regular semivalues are decisive for simple games. Regular semivalues can be *normalized* taking the ratio between the individual semivalue and the sum of such members extended to all players (not null for regular semivalues). For instance, normalizing the Banzhaf semivalue, obtained setting $p_s = \frac{1}{2^{n-1}}$ in (6) for all s, we get the normalized index of Banzhaf.

It is straightforward to check that the normalized index of a regular semivalue satisfies efficiency, anonymity and null properties. The following property states the monotonicity of regular semivalues.

PROPOSITION 5.1 *Every normalized regular semivalue is monotonic.*

Proof: We shall prove this proposition for all semivalues. Let $i, j \in N$ and let Ψ be a semivalue. For any simple game v, write

$$\begin{aligned}
\Psi v(i) &= \sum_{S \subseteq N-\{i,j\}} p_s[v(S \cup i) - v(S)] + p_{s+1}[v(S \cup j \cup i) - v(S \cup j)], \\
\Psi v(j) &= \sum_{S \subseteq N-\{i,j\}} p_s[v(S \cup j) - v(S)] + p_{s+1}[v(S \cup i \cup j) - v(S \cup i)],
\end{aligned}$$

Substracting these equalities yields

$$\Psi v(i) - \Psi v(j) = \sum_{S \subseteq N - \{i,j\}} (p_s + p_{s+1})[v(S \cup i) - v(S \cup j)] \tag{7}$$

and then, if $i \succeq j$ in v, it follows that $\Psi(i) - \Psi(j) \geq 0$. q.e.d.

As a consequence every normalized regular semivalue belongs to $\mathcal{K}$. By means of normalized regular semivalues, we can choose for every player power indices that vary in a certain subinterval in the wide interval.

This idea led us to consider a more restrictive notion of interval based on marginal contributions. If $(N, \mathcal{W})$ is a simple game, then for every player i, i's *Marginal interval*, $\mathcal{M}(i)$ is the set of all possible values of $\psi(i)$ as ψ ranges over all normalized regular semivalues for $(N, \mathcal{W})$. Of course, this interval is contained in the wide interval and contains Shapley-Shubik and Banzhaf indices for every player and for every game. For instance, it is easy to see that for the Parliament of Catalonia, the marginal interval for the first player is $(1/2, 1)$ while for the remaining parties we get $(0, 1/8)$ each.

6. Strict monotonicity

If the set of coalitions for which i is crucial includes the set of coalitions for which j is crucial, then the monotonic property (2) assures the ith player a payment not inferior to the jth player payment. The question we can now ask is, if the inclusion is strict, must the payment received by both players also be strict?

Formally, the *strict* desirability relation is defined as follows: $i \succ j$ if, and only if, $i \succeq j$ and there exists $S \subseteq N - \{i,j\}$ such that $S \cup \{j\} \notin \mathcal{W}$ but $S \cup \{i\} \in \mathcal{W}$.

We shall say a power index ψ is *strictly monotonic* when for every pair of players and every game, we have

$$i \succ j \quad \text{if and only if} \quad \psi(i) > \psi(j) \tag{8}$$

It is easy to see that Johnston index is strictly monotonic. Nevertheless, there are indices, apart from Deegan-Packel index for which this property fails. For instance, taking the weighted game $w = [6; 4, 2, 1, 1]$ we get for players 2 and 3, $2 \succ 3$, but $SI(2) = SI(3) = 0$.

PROPOSITION 6.1 *Every normalized regular semivalue is strictly monotonic*

Proof: From Proposition 4.2 the normalized regular semivalues are monotonic, and if we have $i \succ j$, there is some $T \subseteq N - \{i,j\}$ such that $v(T \cup i) > v(T \cup j)$, while $v(S \cup i) \geq v(S \cup j)$ for the remaining coalitions $S \subseteq N - \{i,j\}$. By using (7) and the regularity condition, it follows that $\psi v(i) > \psi v(j)$. q.e.d.

In particular, Shapley-Shubik and Banzhaf indices are strictly monotonic. The property of strict monotonicity seems suitable for problems of arbitration or in bargaining situations. Admitting this fact, accepted for most power indices, the defined set $\mathcal{K}$ could be replaced by

$$\begin{aligned} x_1 &+ \ldots + x_n = 1 \\ x_1 &\geq 0, \ldots, x_n \geq 0 \\ x_i &> x_j \quad \text{if } i \succ j \\ x_i &= x_j \quad \text{if } i, j \in \mathcal{O} \\ x_i &= 0 \quad \text{if } i \in \mathcal{D}. \end{aligned} \tag{9}$$

7. Conclusions

Several authors have proposed various power indices on the basis of different axiomatic grounds and bargaining models. It is interesting to note how different indices vary for each simple game. Using the most widely used of them we have observed that the variability among them is huge. For example, to measure the power of the president in the context of the United States federal system, it emerges that according to the normalized Banzhaf index, the president has less than 1 percent of the power. The Johnston index, however, suggests that the president has 77 percent of the power. The Shapley-Shubik and Schmeidler indices are 16 and 14 percent respectively.

On the basis of different common internal properties among the widest power indices, we have constructed, for every game, a subset $\mathcal{K}$ of $\mathbf{R}^n$ which contains all reasonable power indices. This definition has made it possible to define for each player an interval of possible values to which power indices belong. By means of normalized regular semivalues, we can choose power indices that vary in a certain subinterval.

The power indices approach works by assigning to each player a real number that would reflect their expectation of bargaining in the game. The interval-based approach lies in the exclusion of certain values that should not be taken as power indices of any solution.

Two different notions of interval have been proposed by Taylor and Zwicker (1995). A basic difference beteween their intervals and ours, is that in our case we can assure that Shapley-Shubik and Banzhaf indices belong to the intervals being considered.

In conclusion, we wish to draw attention to some features of normalized regular semivalue versatility, apart from Shapley-Shubik and the normalized Banzhaf indices. We can, however, design "centralizing" semivalues, which reward the medium-sized coalitions. Or "individualistic" semivalues for which p_k is closest to zero for high values of k. Or "sympathetic" ones, if p_k is closest to zero for low values of k.

When considering "political games" (e.g., among parties), we note that oversized coalitions are not usually formed – in Germany only in three years of almost 50 the two major parties CDU/CSU and SPD worked together in a government coalition on the federal level –, so that an individualistic semivalue would reflect the distribution of power more accurately. If, on the other hand, we look at "economic" games (e.g., among consumers), small groups of players do not seem to be powerful enough to influence the market rules: in this case, perhaps a sympathetic semivalue would describe the situation better. We think that considerations of this kind open a wide range of possibilities for the application of semivalues to real problems.

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The paper by Josep Freixas and Gianfranco Gambarelli triggered off a discussion concerning power indices and their properties. Thus, the notes by Manfred J. Holler, Hannu Nurmi, František Turnovec, Jacek W. Mercik and Honorata Sosnowska present the opinions on the matter and should perhaps also be viewed as expressions of views within a broader context of application of formal methods to some practical societal situations.