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A silent versus partially noisy duel under arbitrary moving and under general assumptions on the payoff function

by

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Abstract: In the paper a duel is considered in which Player I has one silent bullet, Player II has one partially noisy bullet, the accuracy functions are the same and players can move as they like. It is assumed that the maximum speed of Player I is greater than that of Player II.

Keywords: mixed duel, game of timing, zero-sum game

1. Introduction

Consider a game which will be called $(1,1)$ game. Two Players: I and II fight a duel. They can move as they want. The maximum speed of Player I is v_1, the maximum speed of Player II is v_2, and it is assumed that $v_1 > v_2 \geq 0$. The players have one bullet each and this fact is known to both of them. It also known that Player II does not hear the shot of Player I and that Player I hears the shot of Player II with probability p, $0 \leq p < 1$.

At the beginning of the duel the players are at distance 1 from each other. Let $P(s)$ be the probability of succeeding (destroying the opponent) by Player I (II) when the distance between them is $1-s$, $s \leq 1$. The function $P(s)$ will be called accuracy function. It is assumed that

(i) P is increasing and has continuous second derivative in $[0,1]$.

(ii) $P(s) = 0$ for $s \leq 0$, $P(1) \leq 1$.

Player I gains $k > 0$ if only he succeeds, gains $-l < 0$ if only Player II succeeds, gains w if both succeed, and gains 0 if none of them succeeds, $-l \leq w \leq k$. The duel is a zero sum game.

Game is over if at least one of the players succeeds or both bullets are shot. Otherwise, the duel lasts to infinity and the payoff is zero.

Suppose that Player II has fired his shot and misses and that Player I has heard this shot. In this case the best what Player I can do is to reach Player II in pursuit and to succeed with maximal probability $P(1)$. Since we are looking for optimal strategies we assume this behaviour of Player I in the paper.

As it will be seen from the sequel, we can suppose without loss of generality that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

The special case of this duel with $k = l = 1$, $w = 0$, $P(1) = 1$ was solved in Trybuła (1993).

For definitions and results in the theory of games of timing see Fox and Kimeldorf (1969), Karlin (1959), Kimeldorf (1983), Orłowski and Radzik (1985), Restrepo (1957), Styszyński (1974), Vorob'ev (1984).

2. Auxiliary game of timing

To solve the game $(1,1)$ presented in previous section the determining optimal strategies in the following auxiliary game $(1,1)^*$ will be necessary. Consider the one bullet silent versus the partially noisy duel with accuracy function $P(t)$ the same for both players. It is assumed that Player I approaches Player II with constant velocity $v = 1$ all the time, even after firing of his bullet. Player I gains $k > 0$ if only he succeeds etc., similarly as in the duel $(1,1)$ defined in previous section.

Denote by $K_0(s,t)$ the expected gain of Player I if he fires at time $s \in [0,1]$ and Player II fired at time $t \in [0,1]$. It is assumed that

$$K_0(s,t) = \begin{cases} kP(s) & \text{if } s < t, \\ (k-l)P(s) - (k-l-w)P^2(s) & \text{if } s = t, \\ -lP(t) + kp(1-P(t))c + k(1-p)(1-P(t))P(s) & \text{if } s > t, \end{cases}$$

$c = P(1)$.

As it is easy to see, $K_0(s,t)$ is the expected payoff in the duel in which Player II is not allowed to fire after the shot of Player I. Player I is allowed to fire after the Player II but if he has heard the shot of his opponent he has to act in the same way as in the duel $(1,1)$.

Denote by ξ_0^a the strategy of Player I in the game $(1,1)^*$ in which he fires at a random moment s distributed according to a density $f_1(s)$ in the interval $[a,1]$, $0 < a < 1$, and according to probability α, $0 < \alpha < 1$ at the point 1. This distribution is chosen in such a way that if $t \in [a,1)$ then

$$K_0(\xi_0^a;t) = \int_a^t kP(s)f_1(s)ds \tag{1}$$
$$+\int_t^1 (-lP(t) + kp(1-P(t))c + k(1-p)(1-P(t))P(s))f_1(s)ds$$
$$+(kc - (kc+l)P(t))\alpha = \text{const.}$$

Here $K_0(\xi_0^a;t)$ is the expected gain of Player I if he applies the strategy ξ_0^a and Player II fires at time t.

We have

$$\begin{aligned}\frac{\partial K_0(\xi_0^a;t)}{\partial t} &= [(k+1)P(t) - kp(1-P(t))c - k(1-p)(1-P(t))P(t)]f_1(t) \quad (2)\\ &\quad -P'(t)\int_t^1 (l+kpc+k(1-p)P(s))f_1(s)ds - (kc+l)P'(t)\alpha = 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial K_0(\xi_0^a:t)}{\partial t^2} &= [(k+l)P(t) - kp(1-P(t))c - k(1-p)(1-P(t))P(t)]f_1'(t) \quad (3)\\ &+ P'(t)[k+2l+2kpc-k(1-p)(1-3P(t)]f_1(t)\\ &- P''(t)\int_t^1 (l+kpc+k(1-p)P(s))f_1(s)ds - (kc+l)P''(t)\alpha = 0.\end{aligned}$$

By eliminating from the equations (2) and (3) the integral we obtain

$$\frac{f_1'(t)}{f_1(t)} = \frac{P''(t)}{P'(t)} + \frac{k+2l+2kpc-k(1-p)(1-3P(t))}{k(1-p)P^2(t)+[l+kp(1+c)]P(t)-kpc}P'(t) = 0$$

with the solution

$$f_1(t) = \frac{CP'(t)}{(P(t)-P_1)^E(P(t)-P_2)^F} \tag{4}$$

where C is constant and

$$\begin{aligned}P_1 &= \frac{1}{2k(1-p)}\left[-l-kp(1+c)+\sqrt{(l+kp(1+c))^2+4k^2p(1-p)c}\right] \quad (5)\\ &= \frac{2kpc}{l+kp(1+c)+\sqrt{(l+kp(1+c))^2+4k^2p(1-p)c}},\end{aligned}$$

$$P_2 = \frac{1}{2k(1-p)}\left[-l-kp(1+c)+\sqrt{(l+kp(1+c))^2+4k^2p(1-p)c}\right], \tag{6}$$

$$E = \frac{3}{2} + \frac{\frac{l}{2} - \frac{kp(1-c)}{2}}{\sqrt{(l+kp(1+c))^2+4k^2p(1-p)c}}, \tag{7}$$

$$F = \frac{3}{2} - \frac{\frac{l}{2} - \frac{kp(1-c)}{2}}{\sqrt{(l+kp(1+c))^2+4k^2p(1-p)c}}, \tag{8}$$

$c = P(1)$. From (5) it follows that $0 < P_1 < 1$.

Obviously, the constant C in (4) satisfies the equation

$$C\int_a^1 \frac{P'(t)dt}{(P(t)-P_1)^E(P(t)-P_2)^F} + \alpha = 1. \tag{9}$$

Let η_0^a be the strategy of Player II in the game $(1,1)^*$ in which he chooses at random a moment t for his shot according to the density $f_2(t)$ in $[a,1]$ to obtain

$$\begin{aligned} K_0(s;\eta_0^a) &= \int_a^s (-lP(t)+kp(1-P(t))c+k(1-p)(1-P(t))P(s))f_2(t)dt \qquad (10) \\ &+ \int_s^1 kP(s)f_2(t)dt = \text{const.} \end{aligned}$$

if $s \in [a,1]$, where $K_0(s;\eta_0^a)$ is the expected gain of Player I if Player II applies the strategy η_0^a and Player I fires at time s.

In the same way as before we obtain

$$f_2(t) = \frac{DP'(t)}{(P(t)-P_1)^F(P(t)-P_2)^E}, \tag{11}$$

where

$$D\int_a^1 \frac{P'(t)dt}{(P(t)-P_1)^F(P(t)-P_2)^E} = 1. \tag{12}$$

From (2) and (4) it follows that

$$\frac{\partial K_0(\xi_0^a;t)}{\partial t}\frac{1}{P'(t)}\,|_{t=1-} = \frac{Ck(1-p)}{(c-P_1)^{E-1}(c-P_2)^{F-1}} - (kc+l)\alpha = 0, \tag{13}$$

$$\frac{\partial K_0(s;\eta_0^a)}{\partial s}\frac{1}{P'(s)}\,|_{s=a} = \frac{-Dk(1-p)}{(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-1}} + k = 0. \tag{14}$$

From (9), (12), (13) and (14) we determine unknown parameters C, D, α, a, $0<\alpha<1$, $P_1 < P(a) < e$. We shall prove that such solution exists and is unique for any $P(t)$ satisfying our conditions.

By introducing the constant D from (14) to equation (12) we obtain

$$(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-1}\int_a^1 \frac{P'(s)ds}{(P(s)-P_1)^F(P(s)-P_2)^E} = 1-p. \tag{15}$$

Denote by $\varphi(P(a))$ the left hand side of above equation. We have

$$\begin{aligned} \frac{d\varphi(P(a))}{da} &= \{[(F-1)(P(a)-P_1)^{F-2}(P(a)-P_2)^{E-1} \\ &+(E-1)(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-2}\int_a^1 \frac{P'(s)ds}{(P(s)-P_1)^F(P(s)-P_2)^E} \\ &- \frac{1}{(P(a)-P_1)(P(a)-P_2)}\Big\} P'(a) = \frac{(1-p)P(a)+p-1}{(P(a)-P_1)(P(a)-P_2)}P'(a) > 0 \end{aligned}$$

for any a being a solution of (15), such that $P_1 < P(a) < c$, and any p, $0 \le p < 1$. Moreover

$$\varphi(P_1+) = \frac{1}{P_1+\frac{p}{1-p}} > 1-p, \ \ \varphi(1) = 0$$

if $0 \le p < 1$. Then there always exists the unique solution a, $P_1 < P(a) < c$, of the equation (15) if $0 \le p < 1$.

Moreover from (9) and (13) we obtain

$$\left(\frac{(kc+l)(c-P_1)^{E-1}(c-P_2)^{F-1}}{k(1-p)}\int_a^1 \frac{P'(t)dt}{(P(t)-P_1)^E(P(t)-P_2)^F}+1\right)\alpha = 1. \quad (16)$$

From the above and (13) it follows that if $0 \le p < 1$ there always exists the unique solution (C, α) of the equation (9) and (13) such that $0 < \alpha < 1$.

We shall prove now that $K_0(\xi_0^a; t) = K_0(s; \eta_0^a)$ for $a \le s \le 1$, $a \le t < 1$. We get

$$\begin{aligned} K_0(\xi_0^a; a) &= [-lP(a) + k(1-P(a))c](1-\alpha) \quad (17)\\ &+ [kc - (kc+l)P(a)]\alpha + k(1-p)(1-P(a))\int_a^1 P(s)f_1(s)ds \\ &= \int_a^1 kP(s)f_1(s)ds + [(k-l)c - kc^2]\alpha = K_0(\xi_0^a; 1-), \end{aligned}$$

$$K_0(1-; \eta_0^a) = \int_a^1 (kc - (l+kc)P(t))f_2(t)dt = kP(a) = K_0(a; \eta_0^a). \quad (18)$$

From (17) it follows that

$$\begin{aligned} &[kp + k(1-p)P(a)]\int_a^1 P(s)f_1(s)ds \quad (19)\\ &= kpc - (l+kpc)P(a) + [kc(p-1)P(a) + kc(c-p) + lc]\alpha \end{aligned}$$

and

$$\begin{aligned} K_0(\xi_0^a; 1-) &= \int_a^1 kP(s)f_1(s)ds + [(k-l)c - kc^2]\alpha \\ &= kP(a) = K_0(a; \eta_0^a) \end{aligned}$$

if and only if

$$\alpha = \frac{k(P(a)-P_1)(P(a)-P_2)}{(kc^2+lc)(1-P(a))}. \quad (20)$$

On the other hand

$$\begin{aligned} &\frac{\partial K_0(\xi_0^a; t)}{\partial t}\frac{1}{P'(t)}\Big|_{t=a} = \frac{Ck(1-p)}{(P(a)-P_1)^{E-1}(P(a)-P_2)^{F-1}} - \\ &- \int_a^1 (l + kpc + k(1-p)P(s))f_1(s)ds - (kc+l)\alpha = 0 \end{aligned}$$

and by (19) we get

$$\begin{aligned} &\frac{C(1-p)k}{(P(a)-P_1)^{E-1}(P(a)-P_2)^{F-1}} - (l+kpc)(1-\alpha) - (kc+l)\alpha \\ &-(1-p)\frac{kpc - (l+kpc)P(a) + [kc(p-1)P(a) + kc(c-p) + lc]\alpha}{p + (1-p)P(a)} = 0. \end{aligned}$$

Using (13) we obtain

$$\frac{(kc+l)(c-P_1)^{E-1}(c-P_2)^{F-1}}{(P(a)-P_1)^{E-1}(P(a)-P_2)^{F-1}} = (l+kc)(1-\alpha) \tag{21}$$
$$+(1-p)\frac{kpc-(l+kpc)P(a)+[kc(p-1)P(a)+kc(c-p)+lc]\alpha}{p+(1-p)P(a)} + (kc+l)\alpha$$

Moreover, from (10) and (11) we have

$$\begin{aligned} \frac{\partial K_0(s;\eta_0^a)}{\partial s}\frac{1}{P'(s)}\Big|_{s=1} &= -\frac{D(1-p)k}{(c-P_1)^{F-1}(c-P_2)^{E-1}} \\ &+ k(1-p)\int_a^1 (1-P(t))f_2(t)dt = 0 \end{aligned} \tag{22}$$

and by (18)

$$\int_a^1 (1-P(t))f_2(t)dt = \frac{kP(a)+l}{kc+l}. \tag{23}$$

Then from (14), (21) and (22)

$$\frac{(P(a)-P_1)^{F-1}(P(a)-P_2)^{E-1}}{(c-P_1)^{F-1}(c-P_2)^{E-1}} = (1-p)\frac{kP(a)+l}{kc+l}. \tag{24}$$

But

$$(P-P_1)(P-P_2) = P^2 + \frac{l+kp(1+c)}{k(1-p)}P - \frac{pc}{1-p}$$

and

$$(c-P_1)(c-P_2) = \frac{kc^2+lc}{k(1-p)}.$$

Moreover, $E+F=3$. Then, dividing (21) by (24) and taking into account the above equations we obtain

$$\alpha = \frac{k(1-p)P^2(a)+[l+kp(1+c)]P(a)-kpc}{(kc^2+lc)(1-p)(1-P(a))} \tag{25}$$

which is the same as (20). Thus $K_0(\xi_0^a;t) = K_0(s;\eta_0^a)$ for $\alpha \le s \le 1$, $\alpha \le t < 1$.

LEMMA 2.1 *For a being the solution of equation (15) the strategy ξ_0^a is maximin and the strategy η_0^a is minimax in the game $(1,1)^*$. The value of the game is $v_{11}^0 = kP(a)$.*

Proof. We have proven that

$$K_0(\xi_0^a;t) = kP(a)$$

for $a \le t < 1$. Moreover

$$\begin{aligned} K_0(\xi_0^a;1) &= \int_a^1 kP(s)f_1(s)ds + [(k-l)c-(k-l-w)c^2]\alpha \\ &\ge \int_a^1 kP(s)f_1(s)ds + [kc(1-c)-lc]\alpha = \lim_{t\to 1-} K_0(\xi_0^a;t) = kP(a). \end{aligned}$$

If $t < a$ we have

$$\begin{aligned} K_0(\xi_0^a:t) &= \int_a^1 (-lP(t)+k(1-P(t))c+k(1-p)(1-P(t))P(s))f_1(s)ds \\ &+ [kc-(kc+l)P(t)]\alpha \\ &> \int_a^1 (-lP(a)+k(1-P(a))c+k(1-p)(1-P(\alpha))P(s))f_1(s)ds \\ &+ [kc-(kc+l)P(a)]\alpha = K_0(\xi_0^a;a) = kP(a). \end{aligned}$$

Thus

$$K_0(\xi_0^a;\eta) \le kP(a)$$

for any strategy η of Player II.

On the other hand

$$K_0(s;\eta_0^a) = kP(a)$$

for $a \le s \le 1$, and if $s < a$ then

$$K_0(s;\eta_0^a) = kP(s) < kP(a).$$

Therefore

$$K(\xi;\eta_0^a) \le kP(a)$$

for any strategy ξ of Player I what ends the proof of the lemma. ∎

3. Solution of the duel $(1,1)$

We shall now consider the duel $(1,1)$ defined at the beginning of the paper. For any given natural n such that $1/n \le 1-\alpha$ let the constants a_k be defined as follows

$$a_0 = a, \; \int_{a_{k-1}}^{a_k} f_1(s)ds = \frac{1}{n}, \; k = 1,\ldots,n_0, \; a_{n_0+1} = 1,$$

where n_0 is defined from the inequalities

$$1-\alpha-\frac{1}{n} \le \frac{n_0}{n} < 1-\alpha.$$

Define the strategy ξ_ϵ of Player I in the game $(1,1)$ as follows:

If Player I did not hear the shot of Player II he moves back and forth with the maximal speed in the following manner: At first between 0 and a_1, then between 0 and $a_2, \ldots$, finally between 0 and a_{n_0+1}. At the k-th step, $k = 1, \ldots, n_0+1$, he can fire his shot at random only if he is between the points a_{k-1} and a_k ($a_0 = a$) and goes forward, and he fires it with probability density $f_1(s)$. If he fired at the k-th step, he reaches the point a_k, evades to 0 and never approaches Player II. If Player I did not fire between the points 0 and 1 and survives, he fires when he is at 1, as soon as possible.

If Player I heard the shot of Player II and has a bullet he reaches Player II in pursuit and succeeds with the maximum probability $P(1)$.

The strategy η_0 of Player II is defined as follows: If Player I reaches the point t the first time and his velocity is $v_1(\tau)$, τ being time, fire at random with density $v_1(\tau)f_2(t(\tau))$. Otherwise do not fire.

It is assumed that function $v_1(\tau)$ is piecewise continuous.

THEOREM 3.1 *The strategy ξ_ϵ is ϵ-maximin and the strategy η_0 is minimax in the game $(1,1)$. The value of the game is $v_{11} = kP(a)$ where a is the solution of the equation (15).*

Proof. Assume that Player I applies the strategy ξ_ϵ. We say that Player II fires at (k, a') if he fires when Player I is at the point a' and if this happens during the first player's approach to a_k or his first evasion from a_{k-1}.

Denote by $K(\xi_\epsilon; k, a')$ the expected gain of Player I if he applies the strategy ξ_ϵ and Player II fires at (k, a'). We obtain

$$\begin{aligned}
K(\xi_\epsilon; k, a') &\geq \int_a^{a_{k-1}} kP(s)f_1(s)ds \\
&+ \int_{a_k}^1 (-lP(a') + kp(1-P(a'))c + k(1-p)(1-P(a'))P(s))f_1(s)ds \\
&+ [-lP(a') + k(1-P(a'))c]\alpha - \frac{l}{n} \\
&\geq \int_a^{a_{k-1}} kP(s)f_1(s)ds + \int_{a_k}^1 (-lP(a_k) + kp(1-P(a_k))c + \\
&+ k(1-p)(1-P(a_k))P(s))f_1(s)ds + [-lP(a_k) + k(1-P(a_k))c]\alpha - \frac{l}{n} \\
&\geq \int_a^{a_k} kP(s)f_1(s)ds + \int_{a_k}^1 (-lP(a_k) + kp(1-P(a_k))c + \\
&+ k(1-p)(1-P(a_k))P(s))f_1(s)ds + [-lP(a_k) + k(1-P(a_k))c]\alpha - \epsilon \\
&= kP(a) - \epsilon
\end{aligned}$$

where $\epsilon = \frac{k+l}{n}$, $k = 1, \ldots, n_0+1$.

If Player II fires only when Player I reaches 1 the best for him is to fire as soon as possible. For such a strategy (call it η) we obtain

$$\begin{aligned} K(\xi_\epsilon;\eta) &= \int_a^1 kP(s)f_1(s)ds + [(k-l)c - (k-l-w)c^2]\alpha \\ &\geq \int_a^1 kP(s)f_1(s)ds + [kc(1-c) - lc]\alpha \\ &= \lim_{t\to 1-} K(\xi_0^a;t) = kP(a). \end{aligned}$$

From the above it follows that $K(\xi_\epsilon;\eta) \geq P(a) - \epsilon$ for any strategy of Player II.

On the other hand, denote by
$\dot{a}$ the farthest point reached by Player I before he fires,
a' the point from which he fires,
$\dot{a}'$ the farthest point reached by Player II after he fires.

We have

$$a' \leq \dot{a},\ a' \leq \dot{a}'.$$

For such a strategy, say ξ, of Player I we have, if $a < \dot{a}$

$$\begin{aligned} K(\xi;\eta_0) &= \int_a^{\dot{a}} (-lP(t) + kp(1-P(t))c + k(1-p)(1-P(t))P(a'))f_2(t)dt \\ &+ \int_{\dot{a}}^{\max(\dot{a},\dot{a}')} (kP(a') - l(1-P(a'))P(t))f_2(t)dt \\ &\leq \int_a^{\dot{a}} (-lP(t) + kp(1-P(t))c + k(1-p)(1-P(t))P(\dot{a}))f_2(t)dt \\ &+ \int_{\dot{a}}^1 kP(\dot{a})f_2(t)dt = kP(a). \end{aligned}$$

If $\dot{a} < a < \dot{a}'$ then

$$\begin{aligned} K(\xi;\eta_0) &= kP(a') - l(1-P(a'))\int_a^{\dot{a}'} P(t)f_2(t)dt \\ &\leq kP(a) - l(1-P(a))\int_a^{\dot{a}'} P(t)f_2(t)dt \leq kP(a). \end{aligned}$$

At the end, when $\max(\dot{a},\dot{a}') \leq a$

$$K(\xi,\eta_0) = kP(a') \leq kP(a).$$

Then

$$K(\xi,\eta_0) \leq kP(a)$$

for any strategy of Player I. The theorem is proved. ■

Duels under arbitrary moving, as far as the author knows, were never considered before, except in the papers of the author (see Trybuła 1990; 1991; 1992; 1993).

For other results in the theory of duels see Cegielski (1986; 1986), Orłowski and Radzik (1985), Teraoka (1976; 1979).

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