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Sufficient optimality conditions and the dual dynamic programming for the calculus of variations

by

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Abstract: The paper concerns an application of the idea of the dual dynamic programming to the classical sufficient optimality conditions for the problem of the calculus of variations. By making use of the dual Hamilton-Jacobi inequality, we prove sufficient optimality conditions for a relative minimum.

Keywords: calculus of variations, dual Hamilton-Jacobi inequality, sufficient optimality conditions, relative minimum

1. The dual dynamic programming

Let an interval $[a, b]$, two points r, s in R^n and a function $L: [a,b] \times R^n \times R^n \to R$ be given. We consider *the basic problem in the calculus of variations*, called *the problem V* in the sequel, which has the form:

$$\text{minimize } J(x) := \int_a^b L(t, x(t), \dot{x}(t))dt$$

$$\text{subject to } x(a) = r,\ x(b) = s$$

where $x: [a, b] \to R^n$ is an absolutely continuous function. Besides, we assume that L is measurable with respect to $L \times B$, where $L \times B$ denotes the σ-algebra of subsets of $[a, b] \times R^n \times R^n$ generated by product sets $M \times N$, where M is a Lebesgue measurable subset of $[a, b]$ and N is a Borel subset of $R^n \times R^n$. In this section, an absolutely continuous function $x: [a, b] \to R^n$ is called a feasible arc if $x(b) = s$ and $J(x)$ is finite. We say that a feasible arc $\bar{x}$ is an optimal arc for the problem V if $\bar{x}(a) = r$ and, for all feasible arcs x such that $x(a) = r$, the inequality $J(x) \geq J(\bar{x})$ is satisfied.

We can find in the literature two methods of dynamic programming which lead to necessary and sufficient optimality conditions for nonlinear control problems. These conditions are expressed in terms of a value function or some function associated with the dual value function. Now, we shall briefly discuss both methods for the problem V.

In the first of these methods (classical one – see Fleming, Rishel, 1975) we define the value function $S(t,x)$ on some set $T \subset R^{n+1}$ as

$$S(t,x) := \inf \int_t^b L(\tau, x(\tau), \dot{x}(\tau)) d\tau \tag{1}$$

where the infimum is taken over those feasible arcs $x(\tau), \tau \in [t,b]$, whose graphs start at the point $(t,x) \in T$ and are contained in the set T (we shall say that x lies in T). If the function $S(t,x)$ is differentiable, then it satisfies the partial differential equation of the form

$$S_t(t,x) + H\left(t, x, S_x(t,x)\right) = 0 \tag{2}$$

where the Hamiltonian $H(t,x,y) := \sup\{\langle y, \dot{x}\rangle - L(t,x,\dot{x}) \, : \, \dot{x} \in R^n\}$. Moreover, we have the partial differential equation of dynamic programming

$$\sup\{S_t(t,x) + S_x(t,x)\dot{x} - L(t,x,\dot{x}) \, : \, \dot{x} \in R^n\} = 0 \tag{3}$$

which implies the Hamilton-Jacobi inequality being the basis of proofs in Clarke, Zeidan (1986).

In the other (non-classical – see Nowakowski, 1992) method of dynamic programming, instead of the (t,x)-space, we consider the (t,p)-space. We then define the dual value function $S_D(t,p)$ on a set $P \subset R^{n+1}$ of the space of variables (t,p) by the formula

$$S_D(t,p) := \inf \int_t^b L(\tau, x(\tau), \dot{x}(\tau)) d\tau \tag{4}$$

where the infimum is taken over those feasible arcs $x(\tau), \tau \in [t,b]$, whose graphs start at the point $(t, x(t,p))$ and lie in the set T defined above; $x(t,p)$ is a measurable, locally bounded function defined on the set P, such that $(t, x(t,p)) \in T$ for $(t,p) \in P$. Next, we define on the set P a new function $V(t,p)$ by the formula $V(t,p) := -S_D(t,p) - x(t,p)p$. If the function $V(t,p)$ is differentiable, then it satisfies the partial differential equation of the form

$$V_t(t,p) + H(t, -V_p(t,p), p) = 0 \tag{5}$$

where the Hamiltonian $H(t,v,p) := \sup\{\langle p, \dot{x}\rangle - L(t,v,\dot{x}) \, : \, \dot{x} \in R^n\}$. Moreover, we have the dual partial differential equation of dynamic programming

$$\max\{V_t(t,p) + p\dot{x} - L(t, -V_p(t,p), \dot{x}) \, : \, \dot{x} \in R^n\} = 0 \tag{6}$$

which implies the dual Hamilton-Jacobi inequality given in Section 3.

Note that in the non-classical method, we may resign from some assumptions which are made in the case of the first of the methods mentioned above. Namely, examining (4)-(6) instead of (1)-(3), we need not require the set T to have nonempty interior or the function $S(t,x)$ to be differentiable in T. Moreover, if we can solve the problem V by using (2), (3), then we can also do it by means of the dual dynamic programming (Theorem 1.1). Indeed, the set T then has nonempty interior and the function $S(t,x)$ is of class C^1 on an open subset Q of T. Put $P := \{(t,p) : a \le t \le b, p = S_x(t,x),\ (t,x) \in Q\}$ and let $x(t,p)$, $(t,p) = (t, S_x(t,x))$, be the value of an optimal arc whose graph starts at the point (t,x). Further, we adopt the following definitions: $V_t(t,p) := S_t(t,x(t,p))$, $-V_p(t,p) := x(t,p)$, $-W(t,-V_p(t,p)) := S_D(t,p) = S(t,x(t,p))$. Substituting these quantities in (3), we obtain that a function $V(t,p) = W(t,-V_p(t,p)) + V_p(t,p)p$ satisfies (6). Conversely, if there exists a function $V(t,p) = W(t,-V_p(t,p))+V_p(t,p)p$ satisfying (6) and such that the closure of $T := \{(t,x) : x = -V_p(t,p),\ t \in [a,b], (t,p) \in P\}$ has nonempty interior in which $S(t,x)$ is of class C^1, then from (6) we analogously get (3).

We shall now formulate and prove sufficient optimality conditions for the problem V by applying the dual dynamic programming. They are the analogues of the Proposition 2.1(ii) in Nowakowski (1992).

Given a set $T \subset R^{n+1}$ and a set $P \subset R^{n+1}$ of variables (t,p), $t \in [a,b]$, which has nonempty interior, let us take a measurable, locally bounded function $x(t,p)$ defined on the set P, such that $(t, x(t,p)) \in T$ for $(t,p) \in P$. Besides, assume that, for each feasible arc x lying in T, there exists a function $p(t)$ of bounded variation, lying in P and such that $x(t) = x(t,p(t))$. For the set T and the function $x(t,p)$ thus defined, we consider the function $S_D(t,p)$ given by formula (4). Then $S_D(t,p) = S(t,x(t,p))$ for $(t,p) \in P$ and the following proposition is satisfied:

PROPOSITION 1.1 *Let $W(t,p) = Z(t,x(t,p))$ be a real-valued function defined on the set P, such that $W(b,p) = 0$, whereas $(t_0,x_0) \in T$ - a given point. Suppose that, for all functions $p(t)$, $t \in [t_0,b]$, of bounded variation, lying in P and such that the mapping $t \to x(t) := x(t,p(t))$, $t \in [t_0,b]$, $x(t_0) = x_0$, is a feasible arc lying in T, the function*

$$\overline{W}(t,p(t)) := W(t,p(t)) - \int_t^b L(s,x(s),\dot{x}(s))ds$$

is nondecreasing on $[t_0,b]$.

If $\bar{p}(t)$, $t \in [t_0,b]$, is absolutely continuous and $\bar{x}(t) := x(t,\bar{p}(t))$, $t \in [t_0,b]$, $\bar{x}(t_0) = x_0$, is a feasible arc lying in T, such that the function

$$\overline{W}(t,\bar{p}(t)) := W(t,\bar{p}(t)) - \int_t^b L(s,\bar{x}(s),\dot{\bar{x}}(s))ds$$

is constant on $[t_0,b]$, then $\bar{x}$ is an optimal arc for the problem V on the interval $[t_0,b]$, and $W(t_0,\bar{p}(t_0)) = S_D(t_0,\bar{p}(t_0))$.

Proof. Let the functions $p(t)$ and $\bar{p}(t)$, $t \in [t_0, b]$, be as in the proposition. Then, making use of the monotonicity of the function $\overline{W}$ along $p(t)$ and $\bar{p}(t)$, on $[t_0, b]$ we get

$$Z(t_0, x_0) \leq \int_{t_0}^{b} L(s, x(s), \dot{x}(s))ds$$

and

$$Z(t_0, x_0) = \int_{t_0}^{b} L(s, \bar{x}(s), \dot{\bar{x}}(s))ds,$$

where $x(t) = x(t, p(t))$, $\bar{x}(t) = x(t, \bar{p}(t))$ are feasible arcs satisfying the respective assumptions. In view of the above, we have that $W(t_0, \bar{p}(t_0)) = S_D(t_0, \bar{p}(t_0))$ and $\bar{x}$ is an optimal arc for the problem

$$\min \left\{ \int_{t_0}^{b} L(s, x(s), \dot{x}(s))ds \; : \; x(t), \; t \in [t_0, b], \right.$$
$$\left. \text{feasible arcs lying in } T \text{ such that } x(t_0) = x_0. \right\}$$

∎

We have demonstrated that a sufficient condition for the feasible arc $\bar{x}$ to be optimal is the existence of a function $W\ (t, p)$ possesing certain properties. It turns out that the requirements that the function $\overline{W}\ (t, p(t))$ associated with it be nondecreasing along any $p(t)$ and constant along $\bar{p}(t)$ may be reduced to the verification of a smaller set of conditions. This is dealt with by the theorem below which is, at the same time, the answer to the question "when does the solution $V\ (t, p)$ of equation (6) satisfy the sufficient conditions for optimality in Proposition 1.1?"

THEOREM 1.1 *Let $V\ (t, p)$, $t \in [a, b]$, be a C^1 - function defined on a set P, satisfying equation (6). Denote by T the set*

$$T := \{(t, x) \; : \; x = -V_p(t, p), \quad t \in [a, b], \quad (t, p) \in P\}.$$

Assume further that there is a function $W\ (t, x)$ defined on T, such that $W(b, x) = 0$ and, for all functions $p(t)$ of bounded variation lying in P and such that $x(t) = -V_p(t, p(t))$ for almost all $t \in [a, b]$ is a feasible arc lying in T, the mapping $t \to W(t, -V_p(t, p(t)))$ is absolutely continuous. Besides, assume that

$$V(t, p) = W(t, -V_p(t, p)) + V_p(t, p)p \tag{7}$$

for $(t, p) \in P$. Let $x(t)$, $t \in [a, b]$, be a feasible arc lying in T, such that there is a function $p(t)$ of bounded variation lying in P and satisfying the condition $x(t) = -V_p(t, p(t))$ for almost all $t \in [a, b]$. Then the function

$$\overline{W}(t, p(t)) := -W\left(t, -V_p(t, p(t))\right) - \int_{t}^{b} L(s, x(s), \dot{x}(s))ds$$

is nondecreasing with respect to t. Let $\bar{x}(t)$, $t \in [a,b]$, $\bar{x}(a) = r$, *be a feasible arc lying in* T *and let* $\bar{p}(t)$, $t \in [a,b]$, *be an absolutely continuous function lying in* P, *satisfying the condition* $\bar{x}(t) = -V_p(t, \bar{p}(t))$ *for* $t \in [a,b]$ *and such that the mapping* $t \to W(t, -V_p(t, \bar{p}(t)))$ *is absolutely continuous. Besides, suppose that, for almost all* $t \in [a,b]$,

$$V_t(t, \bar{p}(t)) + \bar{p}(t)\dot{\bar{x}}(t) - L\left(t, -V_p(t, \bar{p}(t)), \dot{\bar{x}}(t)\right) = 0. \tag{8}$$

Then $\bar{x}$ *is an optimal arc for the problem* V *relative to all feasible arcs* $x(t)$, $t \in [a,b]$, $x(a) = r$, *lying in* T *and such that the corresponding function* $p(t)$ $(x(t) = -V_p(t, p(t))$ *for a.a.* $t \in [a,b])$ *is of bounded variation. Moreover, the function* $S\left(t, x(t, \bar{p}(t))\right) = -W\left(t, -V_p(t, \bar{p}(t))\right)$, *where* $x(t,p) = -V_p(t,p)$, *is the dual value function along* $\bar{p}(t)$.

Proof. Let $x(t)$ and $p(t)$ be the functions satisfying the assumptions of the theorem. Then, making use of (7), we obtain

$$V_t(t, p(t)) = \frac{d}{dt} W\left(t, -V_p(t, p(t))\right) + p(t) \frac{d}{dt} V_p(t, p(t))$$

for a.a. $t \in [a,b]$. Since $\frac{d}{dt} V_p(t, p(t)) = -\dot{x}(t)$ and, by the definition of the function $\overline{W}(t, p(t))$, we have $\frac{d}{dt} W\left(t, -V_p(t, p(t))\right) = -\frac{d}{dt}\overline{W}\left(t, p(t)\right) + L(t, x(t), \dot{x}(t))$ for a.a. $t \in [a,b]$, therefore, on the ground of (6), $\frac{d}{dt}\overline{W}(t, p(t)) \geq 0$ for a.a. $t \in [a,b]$. This means that the function $\overline{W}(t, p(t))$ is a nondecreasing function of variable t.

Let now the functions $\bar{x}(t)$ and $\bar{p}(t)$ be as in the theorem. Then, using (8), (7) and the definition of the function $\overline{W}(t, p(t))$, we get

$$-W\left(t, -V_p(t, \bar{p}(t))\right) = -W\left(b, -V_p(b, \bar{p}(b))\right) + \int_t^b L(s, \bar{x}(s), \dot{\bar{x}}(s))ds$$

for a.a. $t \in [a,b]$. Hence, and from the assumption $W\ (b, x) = 0$ we have that $\overline{W}(t, \bar{p}(t)) = 0$, i.e. that it is constant on $[a,b]$.

Consequently, satisfied are the assumptions of Proposition 1.1 which implies that the arc $\bar{x}$ is optimal for the problem V relative to the corresponding x's, and $-W\left(t, -V_p(t, \bar{p}(t))\right) = S\left(t, x(t, \bar{p}(t))\right)$. ■

Theorem 1.1, called the verification theorem, indicates what should be done in order to find out if the feasible arc $\bar{x}$ is a relative optimal arc for the problem V. Well, now it should be checked whether: there exists a solution of equation (6), the conditions $W\ (b, x) = 0$ and (7) are satisfied, equality (8) holds.

2. Sufficiency and the Jacobi condition

Let us consider the problem V formulated in the preceding section, under the assumption that $x : [a,b] \to R^n$ is a continuously differentiable function. Such a function x is called a smooth arc. A smooth arc x is a feasible arc if it satisfies the boundary conditions, i.e. $x(a) = r$ and $x(b) = s$. We shall first present the classical necessary and sufficient conditions for optimality in the problem V. With that end in view, let us explain the notions we are going to use.

DEFINITION 2.1 *A feasible arc $\bar{x}$ is called a strong local minimum for the problem V if there exists a positive number ϵ such that, for all feasible arcs x satisfying $|x(t) - \bar{x}(t)| < \epsilon$ for all $t \in [a, b]$, the inequality $J(x) \geq J(\bar{x})$ holds. We say that $\bar{x}$ is a weak local minimum for V if the above condition holds for each feasible arc x satisfying $|x(t) - \bar{x}(t)| < \epsilon$ and $|\dot{x}(t) - \dot{\bar{x}}(t)| < \epsilon$ for all $t \in [a, b]$.*

In the optimality conditions given below we also use some matrix inequalities in the following sense. For an $n \times n$ symmetric matrix M, the inequality $M > 0$ ($M < 0$) means that $yMy > 0$ ($yMy < 0$) for all nonzero y in R^n. Then M is said to be positive (negative) definite. Similarly, $M \geq 0$ ($M \leq 0$) means that $yMy \geq 0$ ($yMy \leq 0$) for all y in R^n. In these cases, M is said to be positive (negative) semidefinite. Obviously, each positive (negative) definite matrix is also positive (negative) semidefinite. Moreover, each positive or negative definite matrix is nonsingular.

Suppose that the function L is of class C^2 and $\bar{x}$ is a weak local minimum for the problem V. The best-known necessary condition satisfied by $\bar{x}$ was given by Euler [1744]:

$$\frac{d}{dt} L_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) = L_x(t, \bar{x}(t), \dot{\bar{x}}(t)) \text{ for } t \in [a, b]. \tag{9}$$

Legendre [1786] demanded, in addition, that

$$L_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) \geq 0 \text{ for } t \in [a, b] \text{ (positive semidefiniteness).} \tag{10}$$

The Jacobi necessary condition (11) [1837] states that (if in (10) the strict inequality is satisfied) in the interval (a, b) there is no point c for which there would exist a nontrivial solution h on $[a, c]$ of the equation

$$\frac{d}{dt}\left[L_{\dot{x}\dot{x}}(t)\dot{h}(t) + L_{\dot{x}x}(t)h(t)\right] - L_{x\dot{x}}(t)\dot{h}(t) - L_{xx}(t)h(t) = 0$$

satisfying the conditions $h(a) = h(c) = 0$, where, for example, $L_{\dot{x}\dot{x}}(t)$ denotes in short $L_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))$. The above differential equation is called the Jacobi equation, and we say that a point $c \in (a, b)$ for which a nontrivial solution h exists and satisfies the conditions $h(a) = h(c) = 0$ is conjugate to a. Consequently, the Jacobi condition goes as follows:

$$\text{there is no point in } (a, b) \text{ conjugate to } a. \tag{11}$$

It turns out that a slight strengthening of necessary conditions (9), (10), (11) makes them sufficient. Denote by (12) and (13) the following conditions

$$L_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) > 0 \text{ for } t \in [a, b] \text{ (positive definiteness),} \tag{12}$$

$$\text{there is no point in } (a,b] \text{ conjugate to } a. \tag{13}$$

Then we have

THEOREM 2.1 *If, for a feasible arc $\bar{x}$, conditions (9), (12) and (13) are satisfied, then $\bar{x}$ is a weak local minimum for the problem V.*

Conditions (9), (10) and (11) are also necessary conditions for $\bar{x}$ to be a strong local minimum for the problem V. Another necessary condition was formulated by Weierstrass [1879]:

$$E(t,\bar{x}(t),\dot{\bar{x}}(t),w) \geq 0 \text{ for } t \in [a,b] \text{ and for all } w \in R^n, \tag{14}$$

where E is a real-valued function defined by the formula

$$E(t,x,\dot{x},w) := L(t,x,w) - L(t,x,\dot{x}) - (w-\dot{x})L_{\dot{x}}(t,x,\dot{x}).$$

A strong form of this condition leads to sufficient conditions of the form:

$$\begin{aligned} &E(t,x,\dot{x},w) \geq 0 \text{ for } (t,x,\dot{x}) \text{ such that } t \in [a,b],\ |x-\bar{x}(t)| < \epsilon \text{ and} \\ &\qquad |\dot{x}-\dot{\bar{x}}(t)| < \epsilon \text{ for all } t \in [a,b], \text{ and for all } w \in R^n. \end{aligned} \tag{15}$$

The sufficient conditions for a strong local minimum are as follows:

THEOREM 2.2 *If, for a feasible arc $\bar{x}$, conditions (9), (12), (13) and (15) are satisfied, then $\bar{x}$ is a strong local minimum for the problem V.*

The unified proof of the above theorems can be found in Clarke, Zeidan (1986). It consists in the indication of a function of class C^1 satisfying the Hamilton-Jacobi inequality (such is the value function $S(t,x)$). In this way, the authors weaken the assumptions concerning the smoothness of the function L (i.e. from C^3 to C^2) and make the proof simpler than those known so far. Namely, they do not refer to field theorems of the theory of differential equations, which is the basic component of the usual proofs. Besides, they obtain that the sufficient conditions are also satisfied in the case of the optimality of $\bar{x}$ relative to all absolutely continuous feasible arcs (this fact does not follow from simple approximation) since the Hamilton-Jacobi inequality remains true then. Finally, this proof is easily carried over to the situation in which the arc $\bar{x}$ is piecewise smooth, which is not a feature of the classical approach.

The essential role in proving Theorems 2.1 and 2.2 is played by the Riccati inequality. The point is that, under certain assumptions, the Jacobi sufficient condition (13) is equivalent to the existence of a solution to this inequality. So, let us recall this fact proved in Clarke, Zeidan (1986).

Assume that the conditions of Theorem 2.1 are satisfied and define the three continuous $n \times n$ – matrix-valued functions defined on the interval $[a,b]$:

$$\begin{aligned} A(t) &:= -L_{\dot{x}\dot{x}}^{-1}(t)L_{\dot{x}x}(t), \\ B(t) &:= L_{\dot{x}\dot{x}}^{-1}(t), \\ C(t) &:= L_{xx}(t) - L_{x\dot{x}}(t)L_{\dot{x}\dot{x}}^{-1}(t)L_{\dot{x}x}(t), \end{aligned}$$

where, for instance, $L_{\dot{x}\dot{x}}(t)$ denotes in short $L_{\dot{x}\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))$. We then have

PROPOSITION 2.1 *There exists a symmetric solution $Q(t)$ on $[a,b]$ of the matrix Riccati inequality*

$$\dot{Q}(t) + Q(t)A(t) + A^T(t)Q(t) + Q(t)B(t)Q(t) - C(t) < 0. \tag{16}$$

Finally, let us note that both the Jacobi condition (13) and the existence of a solution to Riccati inequality (16) guarantee the positive definiteness of the second variation of the functional J.

3. The main results

Consider the problem V where the notion of a feasible arc is understood as in Section 2 and the function L is of class C^2. Moreover, for a given function $\bar{y} : [a,b] \to R^n$ and a positive number ϵ, we adopt the notation

$$N(\bar{y}; \epsilon) := [a,b] \times N_\epsilon(\bar{y}) \tag{17}$$

where $N_\epsilon(\bar{y}) := \bigcup_{t\in[a,b]} \{y \in R^n : \ |y - \bar{y}(t)| < \epsilon\}$.

Now, we shall apply the dual dynamic programming to the sufficient conditions for optimality given in the previous section. We shall use the concept of a strong relative minimum which is defined below.

DEFINITION 3.1 *Let a subset $T \subset R^{n+1}$ and a feasible arc $\bar{x}$ lying in T be given. We say that $\bar{x}$ is a strong relative minimum for the problem V, i.e. $\bar{x}$ is a strong minimum with respect to the set T (in other words, with respect to all feasible arcs x lying in T) if, for all feasible arcs x lying in T, the inequality $J(x) \geq J(\bar{x})$ is satisfied.*

We shall now formulate and prove a proposition which plays the role analogous to Proposition 3 in Clarke, Zeidan (1986). Namely, it is the basis of proofs of sufficient optimality conditions.

PROPOSITION 3.1 *Let $\bar{p}(t)$, $t \in [a,b]$, be an absolutely continuous function. Assume that there exist a positive number ϵ as well as functions $V(t,p)$ and $W(t,x)$ defined on $N(\bar{p};\epsilon)$ and T, respectively, where*

$$T := \{(t,x) \in [a,b] \times R^n : \ x = -V_p(t,p), \ (t,p) \in N(\bar{p};\epsilon)\},$$

such that $V(t,p)$ is of class C^1 on $N(\bar{p};\epsilon)$; for all absolutely continuous functions $p(t)$ lying in $N(\bar{p};\epsilon)$ and such that $x(t) = -V_p(t,p(t))$ for $t \in [a,b]$ is a feasible arc lying in T, the mapping $t \to W(t, -V_p(t,p(t)))$ is absolutely continuous. Moreover, assume that, for all $(t,p) \in N(\bar{p},\epsilon)$, (7) holds. Let $\bar{x}$ be a feasible arc lying in T and satisfying the condition $\bar{x}(t) = -V_p(t,\bar{p}(t))$ for $t \in [a,b]$. Assume further that, for a.a. $t \in [a,b]$, for all $p \in N_\epsilon(\bar{p})$ and for all $\dot{x} \in R^n$, we have the inequality

$$V_t(t,p) + p\dot{x} - L(t, -V_p(t,p), \dot{x}) \leq V_t(t,\bar{p}(t)) + \bar{p}(t)\dot{\bar{x}}(t) - L(t,\bar{x}(t),\dot{\bar{x}}(t)). \tag{18}$$

Then $\bar{x}$ is a strong minimum for the problem V with respect to all feasible arcs x lying in T, such that there exists an absolutely continuous function $p(t)$ lying in $N(\bar{p};\epsilon)$ and satisfying the condition $x(t) = -V_p(t,p(t))$ for $t \in [a,b]$.

Proof. Let x be a feasible arc lying in T, such that there exists an absolutely continuous function $p(t)$ lying in $N(\bar{p};\epsilon)$ and satisfying the condition $x(t) = -V_p(t,p(t))$ for $t \in [a,b]$. Then, making use of (7), we obtain, that for a.a. $t \in [a,b]$,

$$V_t(t,p(t)) = \frac{d}{dt}W(t,-V_p(t,p(t))) + p(t)\frac{d}{dt}V_p(t,p(t))$$

and

$$V_t(t,\bar{p}(t)) = \frac{d}{dt}W(t,-V_p(t,\bar{p}(t))) + \bar{p}(t)\frac{d}{dt}V_p(t,\bar{p}(t))$$

Hence and from inequality (18) we have that, for a.a. $t \in [a,b]$,

$$\begin{aligned} &L(t,\bar{x}(t),\dot{\bar{x}}(t)) - \frac{d}{dt}W(t,-V_p(t,\bar{p}(t))) \\ &\le L(t,x(t),\dot{x}(t)) - \frac{d}{dt}W(t,-V_p(t,p(t))). \end{aligned}$$

Making use of the above inequality and of the boundary conditions, i.e. $x(a) = \bar{x}(a) = r$ and $x(b) = \bar{x}(b) = s$, we obtain $J(x) \ge J(\bar{x})$. Consequently, $\bar{x}$ is a strong minimum for the problem V with respect to the corresponding x's. ■

The above proposition is the dual version of Proposition 3 in Clarke, Zeidan (1986) in which we do not require any longer that the value function $W(t,x)$ be smooth. The dual Hamilton-Jacobi inequality (18) occurring in it follows directly from differential equations (5), (6). Using Proposition 3.1, we shall prove the following

THEOREM 3.1 *Let $\bar{x}$ be a feasible arc, and let $\bar{p}(t) := L_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))$. Assume that conditions (9), (12), (15) are satisfied and there exists a C^1-symmetric solution $Q(t)$ of the inequality*

$$-\dot{Q}(t) + A(t)Q(t) + Q(t)A^T(t) + B(t) - Q(t)C(t)Q(t) < 0. \tag{19}$$

Then there exists a positive number ϵ such that $\bar{x}$ is a strong minimum for the problem V with respect to all feasible arcs x lying in $T := \{(t,x) \in [a,b] \times R^n : x = \bar{x}(t) + Q(t)(p - \bar{p}(t)),\ (t,p) \in N(\bar{p};\epsilon)\}$, such that there exists an absolutely continuous function $p(t)$ lying in $N(\bar{p};\epsilon)$ and satisfying the condition $x(t) = \bar{x}(t) + Q(t)(p(t) - \bar{p}(t))$ for $t \in [a,b]$.

Proof. Let $\bar{x}$ and Q be the functions satisfying the assumptions of the theorem. Then we define

$$V(t,p) := -\langle \bar{x}(t), p - \bar{p}(t)\rangle - \frac{1}{2}\langle p - \bar{p}(t), Q(t)(p - \bar{p}(t))\rangle$$

where $(t,p) \in N(\bar{p};\epsilon_1)$ for some positive ϵ_1. We shall demonstrate that the function $V(t,p)$ so defined satisfies the conditions of Proposition 3.1.

By the Euler equation, $\dot{\bar{p}}(t) = L_x(t,\bar{x}(t),\dot{\bar{x}}(t))$ on $[a,b]$, therefore the functions $\bar{p}$ and V are of class C^1 and, for all $(t,p) \in N(\bar{p};\epsilon_1)$, we have

$$\begin{aligned} V_t(t,p) &= -\langle \dot{\bar{x}}(t), p-\bar{p}(t)\rangle + \langle \bar{x}(t), \dot{\bar{p}}(t)\rangle + \langle \dot{\bar{p}}(t), Q(t)(p-\bar{p}(t))\rangle \\ &\quad -\frac{1}{2}\left\langle p-\bar{p}\,(t), \dot{Q}(t)(p-\bar{p}(t))\right\rangle, \\ V_p(t,p) &= -\bar{x}(t) - Q(t)(p-\bar{p}(t)). \end{aligned}$$

Denote by T_1 the set

$$T_1 := \{(t,x) \in [a,b]\times R^n : x = \bar{x}(t) + Q(t)(p-\bar{p}(t)),\ \ (t,p)\in N(\bar{p};\epsilon_1)\}$$

and define a function $x(t,p)$ on $N(\bar{p};\epsilon_1)$ by the formula

$$x(t,p) := -V_p(t,p).$$

Then $(t,x(t,p)) \in T_1$ for $(t,p) \in N(\bar{p};\epsilon_1)$ and $x(t,\bar{p}(t)) = \bar{x}(t)$ on $[a,b]$.

Consider the equation

$$p = L_{\dot{x}}(t,-V_p(t,p),z). \tag{20}$$

A solution of (20) for an arbitrarily fixed $t=\gamma$ from the interval $[a,b]$ and for $p=\bar{p}(\gamma)$ is the point $z(\gamma,\bar{x}(\gamma)) = \dot{\bar{x}}(\gamma)$. Moreover, condition (12) is satisfied, that is, $L_{\dot{x}\dot{x}}(\gamma,\bar{x}(\gamma),\dot{\bar{x}}(\gamma)) > 0$. On the ground of the implicit function theorem, there exists exactly one continuous function $z(t,x(t,p))$ defined on the neighbourhood of the point $(\gamma,\bar{x}(\gamma))$, satisfying equation (20) and $z(\gamma,\bar{x}(\gamma)) = \dot{\bar{x}}(\gamma)$. Applying the Heine-Borel theorem, we may assume that z is defined on the neighbourhood of the graph of $\bar{x}$, that is, for $(t,x) \in T_2 := \{(t,x)\in[a,b]\times R^n :\ x = \bar{x}(t)+Q(t)(p-\bar{p}(t)),\ \ (t,p)\in N(\bar{p};\epsilon_2)\}$ where $\epsilon_2 < \epsilon_1$ is some positive number. By diminishing ϵ_2, if necessary, we may further assume that, for all $(t,x)\in T_2$ and for all $\dot{x}\in \operatorname{Im} T_2$, the condition

$$L_{\dot{x}\dot{x}}(t,x,\dot{x}) > 0 \tag{21}$$

is satisfied, where $\operatorname{Im} T_2$ is the image of the set T_2, determined by the function $z(t,x(t,p))$. Then, making use of (21) and (20), we obtain that, for a fixed $(t,p)\in N(\bar{p};\epsilon_2)$, the function

$$\dot{x} \to \langle p,\dot{x}\rangle - L(t,-V_p(t,p),\dot{x})$$

is concave on the set $\operatorname{Im} T_2$ with respect to $\dot{x}$ and has the gradient equalling zero at the point $\dot{x} = z(t,x(t,p))$. Consequently, we have the equality

$$\begin{aligned} F(t,p) &:= \max\{V_t(t,p) + \langle p,\dot{x}\rangle - L(t,-V_p(t,p),\dot{x})\ :\ \dot{x}\in \operatorname{Im} T_2\} \quad (22) \\ &= V_t(t,p) + \langle p, z(t,x(t,p))\rangle - L(t,-V_p(t,p),z(t,x(t,p))). \end{aligned}$$

So, substituting in (22) the derivatives $V_t(t,p)$ and $V_p(t,p)$, computed earlier, we get

$$\begin{aligned} F(t,p) \ &:= \ -\langle \dot{\bar{x}}(t), p-\bar{p}(t)\rangle + \langle \bar{x}(t), \dot{\bar{p}}(t)\rangle + \langle \dot{\bar{p}}(t), Q(t)(p-\bar{p}(t))\rangle \\ &- \ \frac{1}{2}\left\langle p-\bar{p}(t), \dot{Q}(t)(p-\bar{p}(t))\right\rangle + \langle p, z(t,x(t,p))\rangle \\ &- \ L(t, \bar{x}(t)+Q(t)(p-\bar{p}(t)), z(t,x(t,p))). \end{aligned}$$

Now, calculating F_p and F_{pp}, we have that F_p and F_{pp} are continuous with respect to (t,p), and

$$\begin{aligned} F_p(t,\bar{p}(t)) &= 0, \\ F_{pp}(t,\bar{p}(t)) &= -\dot{Q}(t) + A(t)Q(t) + Q(t)A^T(t) + B(t) - Q(t)C(t)Q(t). \end{aligned}$$

Since Q is a solution of (19), therefore the Taylor formula implies that, for each (s,p) near $(t,\bar{p}(t))$, the inequality

$$F(s,p) \le F(s,\bar{p}(s))$$

is satisfied. Applying Heine-Borel theorem, we have that, for some positive number ϵ_3 and for every $(t,p) \in N(\bar{p};\epsilon_3)$,

$$F(t,p) \le F(t,\bar{p}(t)) \tag{23}$$

holds. The above inequality and definition (22) of the function $F(t,p)$ imply (18) for a.a. $t \in [a,b]$, for all $p \in N_{\epsilon_3}(\bar{p})$ and for all $\dot{x} \in \operatorname{Im} T_2$.

Let now $(t,p) \in N(\bar{p};\epsilon)$ where $\epsilon := \min\{\epsilon_1,\epsilon_2,\epsilon_3\}$ and let $\dot{x}$ be any point in R^n. Then, using condition (15) and inequality (23), we obtain

$$\begin{aligned} &V_t(t,p) + \langle p,\dot{x}\rangle - L(t,-V_p(t,p),\dot{x}) \le V_t(t,p) + \langle p,z\rangle - L(t,-V_p(t,p),z) \\ &= F(t,p) \le F(t,\bar{p}(t)) = V_t(t,\bar{p}(t)) + \langle \bar{p}(t), \dot{\bar{x}}(t)\rangle - L(t,\bar{x}(t),\dot{\bar{x}}(t)), \end{aligned}$$

that is, the dual Hamilton-Jacobi inequality (18) is satisfied for a.a. $t \in [a,b]$, for all $p \in N_\epsilon(\bar{p})$ and for all $\dot{x} \in R^n$.

Denote by T the set

$$T := \{(t,x) \in [a,b]\times R^n : \ x = \bar{x}(t) + Q(t)(p-\bar{p}(t)), \ \ (t,p) \in N(\bar{p};\epsilon)\}.$$

Then, $\bar{x}$ lies in T and satisfies the condition $\bar{x}(t) = -V_p(t,\bar{p}(t))$ for $t \in [a,b]$. Moreover, (7) holds for a function $W(t,x)$ defined on T, such that

$$W(t,-V_p(t,p)) := \langle \bar{p}(t), \bar{x}(t)\rangle + \frac{1}{2}\langle p+\bar{p}(t), Q(t)(p-\bar{p}(t))\rangle$$

where $(t,p) \in N(\bar{p};\epsilon)$. Note that, for any absolutely continuous function $p(t)$ lying in $N(\bar{p};\epsilon)$ and such that $x(t) = \bar{x}(t) + Q(t)(p(t)-\bar{p}(t))$ for $t \in [a,b]$ is a feasible arc lying in T, the mapping $t \to W(t,-V_p(t,p(t)))$ is absolutely continuous. Consequently, in virtue of Proposition 3.1, $\bar{x}$ is a strong minimum for the problem V with respect to all feasible arcs x lying in T and such that there exists an absolutely continuous function $p(t)$ lying in $N(\bar{p};\epsilon)$ and satisfying the condition $x(t) = \bar{x}(t) + Q(t)(p(t)-\bar{p}(t))$ for $t \in [a,b]$. ∎

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