Control and Cybernetics

vol. **26** (1997) No. 4

On solution of fuzzy equations

by

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Abstract: We present necessary and sufficient condition for the fuzzy equations of the type $f(X; A_1, \ldots, A_n) = C$ to admit a solution X when the parameters $A_1, \ldots, A_n, C$ are fuzzy. This condition is studied for the case of $n = 1$.

Keywords: Algebra, extension principle, fuzzy equations, fuzzy numbers

1. Introduction

Let f be a function defined on the cartesian product $Y_0 \times Y_1 \times \ldots \times Y_n$ and taking values in a set Z. The function f will be extended to fuzzy sets by means of the Zadeh's extension principle.

We will consider a single fuzzy equation with one fuzzy variable and the parameters, having the following form

$$f(X; A_1, \ldots, A_n) = C, \tag{1}$$

where X is a fuzzy variable and $A_1, \ldots, A_n, C$ are fuzzy parameters.

Fuzzy sets $X, A_1, \ldots, A_n, C$ are defined by their membership functions

$$\mu_X : Y_0 \to [0,1], \quad \mu_{A_i} : Y_i \to [0,1] \ (i = 1, 2, \ldots, n), \quad \mu_C : Z \to [0,1];$$

$C = f(X; A_1, \ldots, A_n)$ is the fuzzy set calculated according to the extension principle

$$(\forall z \in Z) \ \mu_C(z) = \sup_{y_0 \in Y_0, y \in U} \{\min(\mu_X(y_0), \mu_{A_1}(y_1), \ldots, \mu_{A_n}(y_n)) \mid z = f(y_0; y)\}, \tag{2}$$

where $y = (y_1, \ldots, y_n), U = Y_1 \times \ldots . \times Y_n$.

Note that, in (2), it is assumed, that given $z \in Z$, there exists $y_0 \in Y_0$ and $y \in U$ such that $z = f(y_0; y)$. But if that were not the case $\mu_C(z)$ would be defined as equal to 0.

We assume that the function f and the fuzzy sets $A_1, \ldots, A_n, C$ are given. We want to find fuzzy set X.

Solving fuzzy equations of the type (1) has a long story in fuzzy set theory (see Buckley, 1992). As it is known (Buckley, Qu, 1990), many fuzzy equations do not have solutions. In consequence of it, other types of solutions to fuzzy equations are investigated, not based on the extension principle. In Buckley, Qu (1991), the authors introduce two new solution procedures; one of them is based on the unified extension and the other on the possibility theory. In particular, the fuzzy quadratic equation, with real fuzzy number parameters, always has a (new) solution. Except for the need of such particular examinations one has to emphasize that the problem of solvability of fuzzy equations, when the solution concept is based on the extension principle, is far from complete. The very general sufficient conditions for the existence of solutions to such equations in the form of a simple dependence relative to the given $A_1, \ldots, A_n, C$ and f, are still unknown. In the domain of fuzzy relation equations, the problem of their solvability is simpler and much better examined (Gottwald, 1994).

The present paper is based on the results from Sanchez (1984). The properties from Sanchez (1984) will be presented, through the corresponding notation from equation (1), as properties related to equation (1) with the arbitrary number of fuzzy parameters and not only two parameters (A_1 and C). We present the necessary and sufficient condition for the fuzzy equation (1) to admit a solution. We consider the fuzzy equation $f(X; A_1) = C$, where A_1 and C are fuzzy numbers.

2. Resolution of the fuzzy equation $f(X; A_1, \ldots, A_n) = C$

Let us use the notation

$$A = (A_1, \ldots, A_n), \quad \mu_A(y) = \min(\mu_{A_1}(y_1), \ldots, \mu_{A_n}(y_n)),$$

and let $\mathcal{F}(V)$ be the class of the fuzzy sets defined on V.

Equation (1) can be written down in the form

$$A * X = C,$$

where "$*$" denotes the mapping f. The above notation of A, $\mu_A(y)$ and equation (1) with the help of operator $*$, allows to present the results from Sanchez (1984) as the results related to arbitrary equation (1). The proofs of theorems and properties presented below are similar to the corresponding proofs of theorems from Sanchez (1984) and are omitted.

For $X_1, X_2 \in \mathcal{F}(Y_0)$, we have

$$X_1 \subseteq X_2 \implies f(X_1; A_1, \ldots, A_n) \subseteq f(X_2; A_1, \ldots, A_n),$$

$$f(X_1 \cup X_2; A_1, \dots, A_n) = f(X_1; A_1, \dots, A_n) \cup f(X_2; A_1, \dots, A_n), \quad (3)$$
$$f(X_1 \cap X_2; A_1, \dots, A_n) \subseteq f(X_1; A_1, \dots, A_n) \cap f(X_2; A_1, \dots, A_n). \quad (4)$$

A direct consequense of (3) and (4) is

COROLLARY 2.1 *For given fuzzy sets* $A_i \in \mathcal{F}(Y_i)$ $(i = 1, \dots, n)$ *and* $C \in \mathcal{F}(Z)$, *we have*

a) *the set of solutions of* $f(X; A_1, \dots, A_n)X \subseteq C$ *is a lattice,*

b) *the set of solutions of* $f(X; A_1, \dots, A_n) = C$, *if non-void, is an upper semi-lattice.*

For fuzzy sets $D \in \mathcal{F}(Z)$, following Sanchez (1984) (and also Pedrycz (1987)), we define the fuzzy set $D \overline{*} A$ on Y_0 by

$$(\forall y_0 \in Y_0) \quad \mu_{D\overline{*}A}(y_0) = \inf_{z \in Z, y \in U} \{\mu_A(y) \alpha \mu_D(z) \mid f(y_0; y) = z\},$$

where

$$\mu_A(y)\alpha\mu_D(z) = \begin{cases} 1 & if \quad \mu_A(y) \leq \mu_D(z) \\ \mu_D(z) & if \quad \mu_A(y) > \mu_D(z). \end{cases}$$

For given $C \in \mathcal{F}(Z)$ we will adopt notation $\tilde{X} = C \overline{*} A$.

THEOREM 2.1 *For given fuzzy sets* $A_i \in \mathcal{F}(Y_i)$ $(i = 1, \dots, n)$ *and* $C \in \mathcal{F}(Z)$, *we have*

$$f(\tilde{X}; A_1, \dots, A_n) \subseteq C.$$

In other terms, $\tilde{X}$ is a particular solution to $f(X; A_1, \dots, A_n) \subseteq C$.

THEOREM 2.2 *For given fuzzy sets* $A_i \in \mathcal{F}(Y_i)$ $(i = 1, \dots, n)$ *and for every set* $X \in \mathcal{F}(Y_0)$, *we have*

$$X \subseteq f(X; A_1, \dots, A_n) \overline{*} (A_1, \dots, A_n).$$

Note that when $f(X; A_1, \dots, A_n) = C$, we have $X \subseteq \tilde{X}$.

COROLLARY 2.2 *For given fuzzy sets* $A_i \in \mathcal{F}(Y_i)$ $(i = 1, \dots, n)$ *and* $C \in \mathcal{F}(Z)$,

$$f(X; A_1, \dots, A_n) \subseteq C \quad \textit{iff} \quad X \subseteq \tilde{X}.$$

The equation $f(X; A_1, \dots, A_n) \subseteq C$ *has always a greatest solution given by* $\tilde{X}$.

THEOREM 2.3 *For given fuzzy sets* $A_i \in \mathcal{F}(Y_i)$ $(i = 1, \dots, n)$ *and* $C \in \mathcal{F}(Z)$, *the equation (1)*

$$f(X; A_1, \dots, A_n) = C$$

has a solution if, and only if

$$f(\tilde{X}; A_1, \dots, A_n) = C. \quad (5)$$

Moreover, if the fuzzy set $\tilde{X}$ *is a solution, then it is the greatest one.*

Remark. When there is a unique solution to (1), this solution equals $C\overline{*}A$. In the case of multiple fuzzy sets X_i, satisfying (1), the set $C\overline{*}A$ is a solution to (1) and

$$C\overline{*}A = \bigcup_i X_i.$$

3. The case of fuzzy numbers

We assume that all given fuzzy sets are (real) fuzzy numbers. Let D be a fuzzy number with membership function μ_D. Then there are three real numbers $d_1 < d_2 < d_3$, denoted $(d_1/d_2/d_3)$, such that: (1) $\mu_D(x) = 0$ outside (d_1, d_3) and it equals one at d_2; (2) $y = \mu_D(x)$ is continuous and monotonically increasing from zero to one on $[d_1, d_2]$; (3) $y = \mu_D(x)$ is continuous and monotonically decreasing from one to zero on $[d_2, d_3]$. The number D is a triangular fuzzy number, if the graph $y = \mu_D(x)$ is a straight line on $[d_1, d_2]$ and on $[d_2, d_3]$.

For given sets we denote

$$(a_{i1}/a_{i2}/a_{i3}) \text{ for } A_i \ (i = 1, \ldots, n), \qquad (c_1/c_2/c_3) \text{ for } C.$$

When $A_1, \ldots, A_n, C$ are fuzzy numbers, a solution X to equation (1) must not be a fuzzy number, but it must be a normal fuzzy set, i.e. whose membership function μ_X satisfies the condition $\max_{x \in R} \mu_X(x) = 1$. In fact

THEOREM 3.1 *If the fuzzy equation $f(X; A_1, \ldots, A_n) = C$ has a solution, then*
a) *the set $\overline{Y}_0 = \{y_0 \in R : \ f(y_0; a_{12}, \ldots, a_{n2}) = c_2\} \neq \emptyset$; and*
b) *every solution X to this equation is a normal fuzzy set such that*

$$\mu_X(y_0) = 1 \implies y_0 \in \overline{Y}_0.$$

Proof. Let X be a solution to the fuzzy equation. Substitute $z = c_2$ in (2), to obtain

$$(\exists y_0 \in R) \ \mu_X(y_0) = 1 \quad \wedge \quad (\exists y_i \in R) \ \mu_{A_i}(y_i) = 1 \ (i = 1, \ldots, n).$$

But A_i $(i = 1, \ldots, n)$ are fuzzy numbers. Hence, $y_i = a_{i2}$ $(i = 1, \ldots, n)$. Finally we get $\overline{Y}_0 \neq \emptyset$ and $y_0 \in \overline{Y}_0$.

Example. We consider quadratic equation $A_1 X^2 + A_2 = C$, where A_1, A_2 and C are triangular fuzzy numbers.

Additionally, we assume that $a_{11} > 0$ and $c_i^* = c_i - a_{i2} \geq 0$ for $i = 1, 2, 3$. Then, the quadratic equation considered (Theorem 9 in Buckley, Qu (1990)) has solutions $X_1 \geq 0$ and $X_2 = -X_1$ iff

$$a_{11} c_2^* > c_1^* a_{12} \text{ and } a_{13} c_2^* < c_3^* a_{12}.$$

The solutions X_1 and X_2 are fuzzy numbers such that

$$\mu_{X_1}(x_2) = 1 \text{ and } \mu_{X_2}(-x_2) = 1, \text{ where } x_2 = \sqrt{c_2^*/a_{12}}.$$

The solution $\tilde{X} = C\overline{*}(A_1, A_2)$ is not a fuzzy number. We have

$$\mu_{\tilde{X}}(x) = 1 \quad \text{iff} \quad x = \pm x_2.$$

4. The equation with two parameters

We consider equation with two fuzzy parameters $f(X; A_1) = C$, where A_1 and C are a fuzzy numbers. We assume:

a) function $f : R^2 \to R, f(R; R) = R$, is continuous and monotonically increasing, i.e. $(\forall x_1, x_2, y_1, y_2 \in R)$
$x_1 < x_2 \to f(x_1; y_1) < f(x_2; y_1) \quad \wedge \quad y_1 < y_2 \to f(x_1; y_1) < f(x_1; y_2)$,

b) there exist $x_i^* \in R$ $(i = 1, 2, 3)$ solving the equations $f(x_i^*; a_{1i}) = c_i$; these numbers are uniquely determined.

1) First we consider the fuzzy set $\tilde{X} = C\overline{*}A_1$. We have

$$(\forall x \in R) \quad \mu_{\tilde{X}}(x) = \inf_{y \in R} \mu_{A_1}(y) \alpha \mu_C(f(x; y)).$$

Hence, $\mu_{\tilde{X}}(x) = 1$ iff $(\forall y \in R)\ \mu_{A_1}(y) \leq \mu_C(f(x; y))$. Clearly, we must have $x = x_2^*$. The set $\tilde{X}$ is a normal fuzzy set iff

$$(\forall y \in R) \quad \mu_{A_1}(y) \leq \mu_C(f(x_2^*; y)). \tag{6}$$

Hence, if $\tilde{X}$ is a fuzzy number, we obtain

$$x_1^* < x_2^* < x_3^*. \tag{7}$$

Now we describe the membership function $\mu_{\tilde{X}}$. We assume that conditions (6),(7) hold. Let $x \in R$.

$(\forall x \in (-\infty, x_1^*) \cup (x_3^*, +\infty)) \quad \mu_{\tilde{X}}(x) = 0,$

$\mu_{\tilde{X}}(x_2^*) = 1,$

$(\forall x \in [x_1^*, x_2^*)) \quad (\exists u_1 \in [a_{11}, a_{12}))$
$\quad \mu_{\tilde{X}}(x) = \mu_C(f(x; u_1)) = \mu_{A_1}(u_1); \quad f(x; u_1) \in [c_1, c_2),$

$(\forall x \in (x_2^*, x_3^*]) \quad (\exists u_2 \in (a_{12}, a_{13}])$
$\quad \mu_{\tilde{X}}(x) = \mu_C(f(x; u_2)) = \mu_{A_1}(u_2); \quad f(x; u_2) \in (c_2, c_3].$

In general, $\tilde{X}$ does not have be a fuzzy number.

2) Now we show that if $\tilde{X}$ is a fuzzy number, then $f(\tilde{X}; A_1) = C$, i.e. condition (5) is satisfied.

Let $\tilde{C} = f(\tilde{X}; A_1)$. From the extension principle

$$(\forall z \in R) \quad \mu_{\tilde{C}}(z) = \sup_{x, y \in R} \{\min(\mu_{\tilde{X}}(x), \mu_{A_1}(y)) \mid z = f(x; y)\}.$$

We are interested in the relationship between C and $\tilde{C}$. Let $z \in R$.

$(\forall z \in (-\infty, c_1] \cup [c_3, +\infty)) \quad \mu_{\tilde{C}}(z) = \mu_C(z) = 0,$

$\mu_{\tilde{C}}(c_2) = \mu_C(c_2) = 1,$

$(\forall z \in (c_1, c_2)) \quad (\exists x_1 \in (x_1^*, x_2^*)) \quad (\exists y_1 \in (a_{11}, a_{12}))$

$\mu_{\tilde{C}}(z) = \mu_{\tilde{X}}(x_1) = \mu_{A_1}(y_1) \quad \wedge \quad z = f(x_1; y_1)$.

But we know that $\mu_{\tilde{X}}(x_1) = \mu_{A_1}(u_1) = \mu_C(f(x_1; u_1))$. Function $y = \mu_{A_1}(x)$ is monotically increasing on $[a_{11}, a_{12}]$. Hence $y_1 = u_1$ and $\mu_{\tilde{C}}(z) = \mu_C(z)$.

$(\forall z \in (c_2, c_3)) \ (\exists x_2 \in (x_2^*, x_3^*)) \ (\exists y_2 \in (a_{12}, a_{13}))$
$\mu_{\tilde{C}}(z) = \mu_{\tilde{X}}(x_2) = \mu_{A_1}(y_2) \wedge z = f(x_2; y_2)$

But we know that $\mu_{\tilde{X}}(x_2) = \mu_{A_1}(u_2) = \mu_C(f(x_2; u_2))$. Function $y = \mu_{A_1}(x)$ is monotically decreasing on $[a_{12}, a_{13}]$. Hence $y_2 = u_2$ and $\mu_{\tilde{C}}(z) = \mu_C(z)$.

This implies that $C = \tilde{C}$.

5. Concluding remarks

Contrary to pessimistic comment in Sanchez (1984), we have shown that condition (5) is interesting in the considerations of solvability problems for equation (1). We are planning in the future the complete investigation of the condition (5). We would like to obtain a very general condition, in the form of a simple dependence on $A_1, \ldots, A_n, C$ and f, for the existence of solution to (1). Such a form characterizes condition (6). We are also planning to consider fuzzy equations, when the extension principle is used in its general form with a given t-norm t.

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