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A multi-step factorization scheme for a specific class of $m - D$ polynomials

by

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Abstract: A new multi-step technique for factorizing $(m - D)$ multidimensional (multivariable) polynomials, i.e. polynomials in m complex variables, is presented. By the term multi-step, it is meant that in order to factorize an $m - D$ polynomial, other lower order polynomials should be factorized.

Keywords: multidimensional systems, multivariable polynomials, factorization

1. The factorization method

Multidimensional $(m-D)$ systems theory has received, during the recent years, much attention in the context of systems theory and modern computer science, Tzafestas (1986), Kaczorek (1986), Bose (1985). $m - D$ systems are described by appropriate multidimensional polynomials, whose factorization makes simpler the realization problem of $m - D$ systems and facilitates the application of stability criteria, Mastorakis, Theodorou and Tzafestas (1992, 1994), Mastorakis, Tzafestas and Theodorou (1991, 1994). These latter papers actually republish the material of Mastorakis (1988) and Mastorakis (1992). However, up to now, the general factorization problem (g.f.p.), for $m - D$ polynomials, i.e. the factorization of every factorizable $m - D$ polynomial, has not been solved, Tzafestas (1986), Kaczorek (1986), Bose (1985), Musser (1975), Wang and Rotchild (1975), Chakrabarti and Mitra (1977), Misra and Patel (1990), Theodorou and Tzafestas (1985), Mastorakis, Theodorou and Tzafestas (1992, 1994), Mastorakis, Tzafestas and Theodorou (1991, 1994), Mastorakis and Theodorou (1990), Mastorakis (1988, 1992).

An $m - D$ polynomial is written as follows:

$$f(z_1, \ldots, z_m) = \sum_{i_i=0}^{N_1} \cdots \sum_{i_m=0}^{N_m} a(i_1, \ldots, i_m) z_1^{i_1} \ldots z_m^{i_m} \tag{1}$$

For a specific class of $m-D$ polynomials, the following factorization scheme is obtained:

$$f(z_1,\dots,z_m) = (A(\bar{z}_a) + B(\bar{z}_a^c)) \cdot (C(\bar{z}_a) + D(\bar{z}_a^c)) \tag{2}$$

where $A(\bar{z}_a)$, $B(\bar{z}_a^c)$, $C(\bar{z}_a)$, $D(\bar{z}_a^c)$ are multidimensional polynomials. The k-tuple $\bar{z}_a$ symbolizes k of the complex variables from $z_1,\dots,z_m$, while the $(m-k)$ – tuple $\bar{z}_a^c$ symbolizes the remaining $m-k$ variables (which are not included in $\bar{z}_a$). We also consider that the constant term is included in $B(\bar{z}_a^c)$ and $D(\bar{z}_a^c)$ i.e. $A(\bar{0}) = C(\bar{0}) = 0$. From the equation (2) one has

$$f(z_1,\dots,z_m) = A(\bar{z}_a)C(\bar{z}_a) + B(\bar{z}_a^c)C(\bar{z}_a) + A(\bar{z}_a)D(\bar{z}_a^c) + B(\bar{z}_a^c)D(\bar{z}_a^c) \tag{3}$$

So, if the above type of factorization (i.e. (2)) holds, then the given polynomial is uniquely divided into three terms as follows

$$f(z_1,\dots,z_m) = f_1(\bar{z}_a) + f_2(\bar{z}_a^c) + f_R(z_1,\dots,z_m) \tag{4}$$

where the constant term is included in $f_2(\bar{z}_a^c)$, i.e. $f_1(\bar{0}) = f_R(0,\dots,0) = 0$. Therefore, if this polynomial comes from a factorization as in (2), we will obviously have:

$$f_1(\bar{z}_a) = A(\bar{z}_a)C(\bar{z}_a) \tag{5}$$

$$f_2(\bar{z}_a^c) = B(\bar{z}_a^c)D(\bar{z}_a^c) \tag{6}$$

and

$$f_R(z_1,\dots,z_m) = B(\bar{z}_a^c)C(\bar{z}_a) + A(\bar{z}_a)D(\bar{z}_a^c) \tag{7}$$

Therefore, possible polynomial factors of $f(z_1,\dots,z_m)$ are $r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$ where $h_1(\bar{z}_a)$ and $h_2(\bar{z}_a^c)$ are divisors of $f_1(\bar{z}_a)$ and $f_2(\bar{z}_a^c)$ respectively.

In order to formulate this idea in an algorithmic form, we find the set of all factors of $f_1(\bar{z}_a)$ ($f_2(\bar{z}_a^c)$) denoted $\Phi_{f_1(\bar{z}_a)}$, ($\Phi_{f_2(\bar{z}_a^c)}$, respectively). So our problem is reduced to the factorization problem of the polynomials $f_1(\bar{z}_a)$, $f_2(\bar{z}_a^c)$ and for this reason the method appears to be a multi-step one.

Arranging the above ideas in a logical order, the following algorithm is obtained:

STEP 1: Select an integer number k with $1 < k < m$.

STEP 2: Select k variables of the set of m complex variables $\{z_1,\dots,z_m\}$. These variables constitute the k-tuple $\bar{z}_a$. The remaining $m-k$ variables constitute the $(m-k)$-tuple $\bar{z}_a^c$.

STEP 3: Split the given polynomial as follows

$f(z_1,\dots,z_m) = f_1(\bar{z}_a) + f_2(\bar{z}_a^c) + f_R(z_1,\dots,z_m)$, eq. (4)

where the constant term is included in $f_2(\bar{z}_a^c)$, i.e. $f_1(0) = f_R(0,\dots,0) = 0$

STEP 4: Factorize (by this or by other method) the polynomials $f_1(\bar{z}_a)$ and $f_2(\bar{z}_a^c)$. Write the sets of their factors $\Phi_{f_1(\bar{z}_a)}$ and $\Phi_{f_2(\bar{z}_a^c)}$ respectively.

STEP 5: Form all the linear combinations $r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$ for the probable polynomial factors of $f(z_1, \ldots, z_m)$ where
$h_1(\bar{z}_a) \in \Phi_{f_1(\bar{z}_a)}$, $h_2(\bar{z}_a^c) \in \Phi_{f_2(\bar{z}_a^c)}$
(r is always an unknown real constant).

STEP 6: For each linear combination $r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$, check if it is a factor of $f(z_1, \ldots, z_m)$. If not, go to STEP 2 and change the k complex variables. If for all combinations of the k variables, which constitute the k-tuple $\bar{z}_a$, no factor of the form $r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$ is obtained, then change k $(1 < k < m)$. If for all k $(1 < k < m)$ and each time for all the combinations of k variables which constitute the k-tuple $\bar{z}_a$, no factor of the form $r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$ is obtained, then the algorithm is terminated and the polynomial is not factorizable by this method.→END.

STEP 7: Evaluate the real parameter r simultaneously by checking if the polynomial $r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$ is a factor of $f(z_1, \ldots, z_m)$.

STEP 8: Carry out the algorithmic division

$$f(z_1, \ldots, z_m) : r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c) = q(z_1, \ldots, z_m) \tag{8}$$

STEP 9: The polynomial $f(z_1, \ldots, z_m)$ has been factorized as follows

$$f(z_1, \ldots, z_m) = (r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)) \cdot q(z_1, \ldots, z_m) \tag{9}$$

→END.

Furthermore, the polynomial $q(z_1, \ldots, z_m)$ can probably be factorized with this or with other methods, Theodorou and Tzafestas (1985), Mastorakis, Theodorou and Tzafestas (1992, 1994), Mastorakis, Tzafestas and Theodorou (1991, 1994), Mastorakis and Theodorou (1990), Mastorakis (1988, 1992).

Remarks: It should be noted that if the original polynomial $f(z_1, \ldots, z_m)$ cannot be factorized according to the above method, it may be factorized via another method, Theodorou and Tzafestas (1985), Mastorakis, Theodorou and Tzafestas (1992, 1994), Mastorakis, Tzafestas and Theodorou (1991, 1994), Mastorakis and Theodorou (1990), Mastorakis (1988, 1992). Hence, the above method for checking possible factors of a given polynomial provides only sufficient conditions for factorization.

It is clear that due to application of the above method to an $m-D$ polynomial $f(z_1, \ldots, z_m)$, two other, i.e. $f_1(\bar{z}_a)$, $f_2(\bar{z}_a^c)$, which are lower degree polynomials, should be factorized. This leads to the multi-step character of the method. These simpler polynomials may be factorized by another method or algorithm, Theodorou and Tzafestas (1985), Mastorakis, Theodorou and Tzafestas (1992, 1994), Mastorakis, Tzafestas and Theodorou (1991, 1994), Mastorakis and Theodorou (1990), Mastorakis (1988, 1992).

Example: Consider the 3-D polynomial

$$f(z_1, \ldots, z_m) = z_1^2 + z_1 z_2^2 - z_1 z_3 - z_2^2 z_3 + z_1 + z_2^2$$

If we choose $\bar{z}_a = (z_1)$ (therefore $\bar{z}_a^c = (z_2, z_3)$), we find

$$f_1(\bar{z}_a) = z_1^2 + z_1, \; f_2(\bar{z}_a^c) = -z_2^2 z_3 + z_2^2, f_R(z_1, z_2, z_3) = z_1 z_2^2 - z_1 z_3$$

It is easy to factorize $f_1(\bar{z}_a)$, $f_2(\bar{z}_a^c)$.

$$f_1(\bar{z}_a) = z_1(z_1+1),\ f_2(\bar{z}_a^c) = z_2 \cdot z_2 \cdot (-z_3+1).$$

Therefore,

$$\Phi_{f_1(\bar{z}_a)} = \{1, z_1, z_1+1, z_1(z+1)\}$$

$$\Phi_{f_2(\bar{z}_a^c)} = \{1, z_2, z_3-1, z_2^2, z_2(z_3-1), z_2^2(z_3-1)\}$$

So, the factors

$$r \cdot h_1(\bar{z}_a) + h_2(\bar{z}_a^c)$$

are tested as factors of $f(z_1, z_2, z_3)$, where

$$h_1(\bar{z}_a) \in \Phi_{f_1(\bar{z}_a)},\ h_2(\bar{z}_a^c) \in \Phi_{f_2(\bar{z}_a^c)}$$

By choosing $h_1(\bar{z}_a) = z_1$, and $h_2(\bar{z}_a^c) = z_3 - 1$ we obtain the polynomial $rz_1 + (z_3-1)$ which is a factor of $f(z_1, z_2, z_3)$, since $f(z_1, z_2, 1-rz_1) = \ldots = (1+r)z_1^2 + (1+r)z_1z_2^2 = 0$, when $r = -1$.

So, one polynomial factor for the given polynomial is: $z_3 - z_1 - 1$. The division $f(z_1, z_2, z_3) \underline{\big|\ z_3 - z_1 - 1\ }$ is carried out with respect to z_3.

$$\begin{array}{l|l} z_3(-z_1 - z_2^2) + (z_1^2 + z_1z_2^2 - +z_1 + z_2^2) & z_3 - z_1 - 1 \\ \hline \underline{z_3(+z_1 - z_2^2) - (z_1^2 + z_1z_2^2 - +z_1 + z_2^2)} & -z_1 - z_2^2 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad 0 & \end{array}$$

Thus,

$$f(z_1, z_2, z_3) = (-z_1 + z_3 - 1)(-z_1 - z_2^2)$$

or

$$f(z_1, z_2, z_3) = (z_1 - z_3 + 1)(z_1 + z_2^2)$$

2. Conclusion

The presented multi-step method factorizes a special class of multidimensional $(m-D)$ polynomials. As the general factorization problem (g.f.p.) has not been solved yet, this method is useful in the realization of an $m-D$ system, if the numerator and denominator of its transfer function are factorized. The characteristic polynomial of an $m-D$ system may be factorized following the above procedure. In this case, the application of stability criteria is facilitated, Mastorakis, Theodorou and Tzafestas (1992, 1994), Mastorakis, Tzafestas and Theodorou (1991, 1994), Mastorakis and Theodorou (1990), Mastorakis (1988, 1992).

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