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# The conjugate points sufficient conditions for an optimal control

by

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**Abstract:** The paper concerns an application of the idea of field theory and the concept of "concourse of flights" to the sufficient optimality conditions for the optimal control problems stated in terms of focal and conjugate points. The concept of concourse of flights was begun by Young (1969), and later extended by Nowakowski (1988). In the paper the definition of a focal and conjugate point of a field of extremals is given. Using these concepts, we prove that the existence of a field of extremals without conjugate points implies the existence of concourse of flights and consequently we obtain the second order sufficient conditions for the generalized problem of Bolza. Another approach to the concept of focal and conjugate points is given by Zeidan (1983, 1984).

**Keywords:** sufficient conditions, generalized problem of Bolza, field of extremals

## 1. Preliminaries and assumptions

Let us consider the generalized problem of Bolza:

$$\text{minimize } J(x,u) = \int_0^T L(t, x(t), u(t))\, dt + l(x(T)) \tag{1}$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. in } [0,T], \quad x(0) = 0, \tag{2}$$

$$u(t) \in Q \subset \Re^m \text{ a.e. in } [0,T], \text{ where } Q \text{ is a compact set} \tag{3}$$

Here $x : [0,T] \to \Re^n$ is an absolutely continuous function, $u : [0,T] \to \Re^m$ is a Lebesgue measurable function, $L : [0,T] \times \Re^n \times \Re^m \to \Re$, $f : [0,T] \times \Re^n \times \Re^m \to \Re^n$, $l : \Re^n \to \Re \cup \{+\infty\}$, $f : [0,T] \times \Re^n \times \Re^m \to \Re^n$, $l : \Re^n \to \Re \cup \{+\infty\}$ and $Q$ is compact.

We assume that $f$, $l$, $L$ and $u$ satisfy the following hypotheses:

**H1.** the functions $L$, $f$ are continuous with respect to all variables, the function $l$ is lower semicontinuous and not identifically $+\infty$, there exist the following derivatives: $f_x$, $L_x$, $f_{xx}$, $L_{xx}$ and they are continuous

**H2.** there exists a neighbourhood of $(0,0,0)$ such that the function

$H(t,x,y) = \sup\{\, y f(t,x,u) - L(t,x,u) \mid u \in Q \,\}$ has continuous partial derivatives $H_y(t,x,y)$, which is a Lipschitzian function with respect to $x$, and $H_x(t,x,y)$, which is a Lipschitzian function with respect to $y$.

The consequence of these assumptions is the existence of local solutions of the following Hamilton equations:

$$\frac{dx}{dt} = H_y(t,x,y), \frac{dy}{dt} = -H_x(t,x,y), x(0,\varsigma) = \varsigma \tag{4}$$

where $\varsigma$ belongs to some open set in $\Re^n$, which will be defined later.

## 2. The local sufficient condition

A family of solutions of (4) will be named canonical extremals of our problem. We shall distinguish one of them, namely that for which $x(0,0) = 0$, denoting it by $\bar{x}(t)$ and the canonical trajectory $\bar{y}(t)$, and the control function $\bar{u}(t)$, corresponding to it (i.e. $\bar{x}(t), \bar{u}(t)$ satisfy (2)).

We assume that:

**H3.** the control function $\bar{u}(t)$ is piecewise continuous and for $t \in [0,T]$ the generalized Jacobian $\partial_y H_y(t,\bar{x}(t),\bar{y}(t))$ in the sense of Clark of the function $H_y(t,\bar{x}(t),\bar{y}(t))$ has the maximal rank $n$.

The last hypothesis allow us to state a local one to one and smooth embedding theorem.

THEOREM 2.1 *There exist $\delta > 0$ and a neighbourhood $N$ of the point $(0,0)$, such that the extremals $x(t)$ of (4), restricted to $(0,\delta)$, cover $N$ simply.*

**Proof.** This is basically a "three map" proof. We construct three separate maps.

Map $n°1$. Let us consider the extremal $\bar{x}(t)$ in a neighbourhood of $(0,0)$. There exists a canonical trajectory $\bar{y}(t)$ such that $\frac{d\bar{y}(t)}{dt} = -H_x(t,\bar{x}(t),\bar{y}(t))$. Denoting $y_0 = \bar{y}(0)$, we have $\dot{\bar{x}}(0) = H_y(0,0,y_0)$.

By H3, the generalized Jacobian $\partial_y H_y(0,0,y_0)$ has the maximal rank. By the generalized implicit theorem (Clarke, 1983), there exist a neighbourhood $K$ of $(0,0,y_0)$ and a neighbourhood $N$ of $(0,0,\dot{\bar{x}}(0))$ and a one-to-one map $K$ onto $N$. Our map $n°1$ is thus a map $(t,x,y) \to (t,x,\dot{x})$.

Map $n°2$. Consider the canonical extremals $x(t,w,v)$, $y(t,w,v)$, $t \in (0,\delta)$, ($\delta$ is determined by the neighbourhood $K$) with the initial values $x(0,w,v) = w$, canonical extremals exist and are unique, by H2. For these extremals, consider the following system of equations

$$t = t, x(t, w, v) = x, y(t, w, v) = y, \quad t \in [0, \delta].$$

At $t = 0$ the Jacobian in $w, v$ is the identity matrix, and so nonsingular. By the implicit function theory, the above equations have unique solutions locally. After diminishing $K$ suitably, we can determine our second map defined by these solutions, in the form

$$(t, w, v) \to (t, x, y).$$

Map $n°3$. The same arguments allows us to define our third map

$$(t, x, v) \to (t, w, v)$$

in suitable domains. All the three maps are one-to-one. Moreover, if we diminish the initial domain sufficiently, the image of each map can be mapped by the previous map, and the final image will be in $N$.

The composite map $T$. We can now arrange the three maps and combine them by writing

$$\begin{aligned} (t, x, v) \to (t, w, v) &\to (t, x, y) \to (t, x, \dot{x}), \\ (t, x, v) &\to (t, x, \dot{x}) \end{aligned}$$

If we inverse the above map, we shall obtain a map $T$: $(t, x, \dot{x}) \to (t, x, v)$. For a given $(t, x, \dot{x}) \in N$, there is just one $v$ for which the equations

$$t = t, x = \dot{x}, \dot{x} = p(t, x, v)$$

have solutions. The map $T$ realizes the required covering. ■

In order to study the existence of an extremal joining two given points, we assume the local restriction of the maps in (4). By a change of scale of the form $(t, x, y) \to (at, bx, cy)$, where $a, b, c$ are positive constants, we now arrange that there exists a neighbourhood of $(0, 0, y_0)$ such that, for any $(t, x, y)$ in this neighbourhood, we have the following inequalities: $|H_y(t, x, y)| \leq 1$, $|w| \leq 1$ for all $w \in \partial_y H_y(t, x, y)$, $|s| \leq 1$ for all $s \in \partial_x H_y(t, x, y)$, $|z| \leq 1$ for all $z \in \partial_t H_y(t, x, y)$.

By a *$\delta$-trajectory*, we shall mean the solution $x(t)$ of (4) that corresponds to the interval $[0, \delta]$. Further, we shall term *local $\delta$-pencil* of $\delta$-trajectories, the family of $\delta$-trajectories beginning at $t = 0$, whose derivatives $\dot{x}$ at $t = 0$ satisfy $|\dot{x}(0) - \dot{\bar{x}}(0)| \leq \delta$. The set of points $(t, x)$ for which $t \in [0, \delta]$ and $\left|\frac{x - \bar{x}(0)}{t} - \dot{\bar{x}}(0)\right| < \delta$ will be termed a local angle about $(0, \bar{x}(0), \dot{\bar{x}}(0))$.

LEMMA 2.1 *If $x(t)$ is a $\delta$-trajectory on the interval $[0, \delta]$, then for any $t, t_1, t_2$ from the interval $[0, \delta]$, we have*

$$\left|\frac{x(t_2) - x(t_1)}{t_2 - t_1} - \dot{x}(t)\right| \leq \max\{3|t_2 - t|, 3|t_1 - t|\} \tag{5}$$

**Proof.** We may set $t = 0$. Then we have
$\frac{x(t_2)-x(t_1)}{t_2-t_1} - \dot{x}(0) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} H_y(\tau, x(\tau), y(\tau))\, d\tau - H_y(0,0,y_0)$
$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} [H_y(\tau, x(\tau), y(\tau)) - H_y(0, x(\tau), y(\tau))]\, d\tau$
$+ \frac{1}{t_2-t_1}\int_{t_1}^{t_2} [H_y(0, x(\tau), y(\tau)) - H_y(0,0,y(\tau))]\, d\tau$
$+ \frac{1}{t_2-t_1}\int_{t_1}^{t_2} [H_y(0,0,y(\tau)) - H_y(0,0,y_0)]\, d\tau.$

By the mean-value theorem of Leburg (Clarke, 1983), for $\tau \in (t_1, t_2)$, we obtain for $i = 1, 2, \ldots, n$

$$H_{y_i}(\tau, x(\tau), y(\tau)) - H_{y_i}(0,0,y_0) \in \langle \partial_t H_{y_i}(\tilde{t}_i, x(\tau), y(\tau)), \tau\rangle + \langle \partial_x H_{y_i}(0, \tilde{x}_i, y(\tau)), x(\tau)\rangle + \langle \partial_y H_{y_i}(0,0,\tilde{y}_i), y(\tau) - y_0\rangle,$$

where $\tilde{t}_i \in (0,\tau), \tilde{x}_i \in (0, x(\tau)), \tilde{y}_i \in (y_0, y(\tau))$, and $\partial_t H_{y_i}, \partial_x H_{y_i}, \partial_y H_{y_i}$ denote the generalized gradients in the sense of Clark of the function $H_{y_i}(t,x,y)$.

In view of above, we have that for $\tau \in (t_1, t_2)$ and for $i = 1, 2, \ldots, n$,

$$|H_{y_i}(\tau, x(\tau), y(\tau)) - H_{y_i}(0,0,y_0)| \le \tau + |x(\tau)| + |y(\tau) - y_0|.$$

Using the relations

$$x(t) = \int_0^t H_y(t, x(t), y(t))\, dt, \quad y(t) - y_0 = -\int_0^t H_x(t, x(t), y(t))\, dt,$$

we would have,

$$|x(t)| \le t, |y(t) - y_0| \le t$$

and in consequence

$$\left|\frac{x(t_2) - x(t_1)}{t_2 - t_1} - \dot{x}(0)\right| \le \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} |H_y(\tau, x(\tau), y(\tau)) - H_y(0,0,y_0)|\, d\tau \le$$
$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} 3\tau d\tau = \frac{3\left(t_2^2 - t_1^2\right)}{2(t_2 - t_1)} = \frac{3(t_2 + t_1)}{2} \le \max\{3t_1, 3t_2\}.$$

■

Two $\delta$-trajectories $\Gamma_1, \Gamma_2$ will be termed *markedly deflected* if they possess, respectively, line elements $(t_1, x_1, p_1), (t_2, x_2, p_2)$ such that either $|p_2 - p_1| \ge 12\delta$ or $(t_1, x_1) = (t_2, x_2)$ and $|p_2 - p_1| > 6\delta$.

LEMMA 2.2 *Two markedly deflected trajectories cannot intersect at more than one point.*

**Proof.** Suppose that there exist $t_1 \neq t_2$ of the interval $[0, \delta]$ such that $x_1(t_1) = x_2(t_1)$ and $x_1(t_2) = x_2(t_2)$. Denoting $\bar{p}_1 = \dot{x}_1(t_1), \bar{p}_2 = \dot{x}_2(t_1), \hat{p}_1 = \dot{x}_1(t_2), \hat{p}_2 = \dot{x}_2(t_2)$, we have $|\bar{p}_2 - \bar{p}_1| > 6\delta$ and $|\hat{p}_2 - \hat{p}_1| > 6\delta$. By Lemma 2.1, for $t = t_1$ we obtain $\left|\frac{x_1(t_2)-x_1(t_1)}{} - \dot{x}_1(t_1)\right| < 3\delta$ and $\left|\frac{x_2(t_2)-x_2(t_1)}{} - \dot{x}_2(t_1)\right| <$

$3\delta$. Arguing as before, we find that $|\bar{p}_2 - \bar{p}_1| \leq 6\delta$, which contradicts the definition of markedly deflected trajectories. ■

We are now in a position to establish, for differential equations (4), the existence of a solution of the boundary-value problem in a local angle.

THEOREM 2.2 *(Local Angle Theorem) Let $(t_1, x_1)$ lie in the local $\left(\frac{\delta}{12}\right)$-angle about $(0, 0, \dot{\bar{x}}(0))$. Then the points $(0,0)$ and $(t_1, x_1)$ can be joined by a trajectory of the local 3 $\delta$-pencil, and by no trajectory not belonging to that pencil.*

**Proof.** It will be sufficient that the points in question can be joined by a trajectory of a local $\left(\frac{\delta}{2}\right)$-pencil. Then two trajectories, one from the $6\delta$-pencil and one from the $\left(\frac{\delta}{2}\right)$-pencil will be markedly deflected, and they will have only one common point $(0,0)$. The existence of the trajectory from a local $\frac{\delta}{2}$-pencil, joining the points $(0,0)$ and $(t_1, x_1)$ follows from the distortion theorem (Young, 1969).

Let us denote

$$S = \left\{ p \,\middle|\, |p - \dot{\bar{x}}(0)| < \frac{\delta}{2} \right\}, H = \left\{ p \,\middle|\, |p - \dot{\bar{x}}(0)| < \frac{\delta}{12} \right\}.$$

We have that the boundary of the set $H$ is distant at least $\frac{5\delta}{12}$ from the boundary of the set $S$. Let us define the following continuous map $T : S \to S\prime$

$$T(p) = \frac{x(t_1, p)}{t_1}$$

where $t_1 \in \left(0, \frac{\delta}{12}\right)$ and $x(t,p)$ is the $\frac{\delta}{12}$-trajectory satysfying $x(0) = 0$ and $\dot{x}(0) = p$. By Lemma 2.1,

$$|T(p) - p| = \left| \frac{x(t_1, p)}{t_1} - p \right| \leq 3t_1 \leq 3 \cdot \frac{\delta}{12} < \frac{5}{12}\delta.$$

By the distortion theorem, $H \subset S\prime$. This, in particular, implies $\frac{x_1}{t_1} \in S\prime$, which means that, for $p \in S$, we have $T(p) = \frac{x_1}{t_1}$, so $x(t_1, p_1) = x_1$. This completes the proof. ■

THEOREM 2.3 *(Minimum Property of Well Directed Local Extremals).*

*There exist a neighbourhood $N$ of $(0,0)$ and a local angle about $(0, 0, \dot{\bar{x}}(0))$ such that, for any extremal $C_0$ in $N$ with one end at $(0,0)$ and with the derivative at some relevant t in the local angle, and for any other admissible trajectory $C$ lying in $N$ with the same ends as $C_0$, we have*

$$J(C_0) \leq J(C)$$

*where $J(C), (J(C_0))$ denote the values of functional (1) restricted to the trajec-*

**Proof.** The existence of the neighbourhood $N$ and of the local angle, such that $N$ is covered one-to-one by graphs of the extremals satisfying (4), follows from Theorems 2.1 and 2.2. Then we have the existence of a local spray of flights described in Nowakowski (1988). Thus the assertion of the theorem follows from (Nowakowski, 1988, Theorem 4). ■

## 3. Focal and conjugate points

We assume the following hypothesis:

**H4.** there exists a division of the interval $[0, T]$ into subintervals $[t_i, t_{i+1}]$, $i = 0, \ldots, q$, such that, for each $i = 0, \ldots, q$, there exist an open set $Q_i \subset \Re^{m_i}$ of parameters $\sigma^i$, containing zero, functions $t_i\left(\sigma^i\right), t_{i+1}\left(\sigma^i\right), (t_i(0) = t_i, t_{i+1}(0) = t_{i+1})$ of the class $C^1$, canonical extremals

$$x\left(t, \sigma^i\right), y\left(t, \sigma^i\right), t \in \left[t_i\left(\sigma^i\right), t_{i+1}\left(\sigma^i\right)\right], \sigma^i \in Q_i. \tag{6}$$

which are smooth functions of both variables and $x(t, 0) = \bar{x}(t)$. Moreover, we assume that at $\sigma^i = 0$ the $m_i \times 2n$ Jacobian matrix $(x_{\sigma^i}, y_{\sigma^i})$ has rank $m_i$ for some $t_0 \in (t_i, t_{i+1})$, $i = 0, \ldots, q$.

For $i = 0, \ldots, q$, let us denote:

$T_i$ a set covered by graphs of trajectories $x\left(t, \sigma^i\right), t \in [t_i, t_{i+1}], \sigma^i \in Q_i$,

$S_i^- = \left\{\left(t, \sigma^i\right) \left| t = t_i\left(\sigma^i\right) \geq t_i, \sigma^i \in Q_i\right.\right\}$,

$S_i = \left\{\left(t, \sigma^i\right) \left| t_i\left(\sigma^i\right) < t < t_{i+1}\left(\sigma^i\right), \sigma^i \in Q_i\right.\right\}$,

$S_i^+ = \left\{\left(t, \sigma^i\right) \left| t = t_{i+1}\left(\sigma^i\right) \leq t_{i+1}, \sigma^i \in Q_i\right.\right\}$,

$[S_i] = S_i^- \cup S_i \cup S_i^+$,

$E_i^- = \left\{(t, x) \left| x = x\left(t, \sigma^i\right), \left(t, \sigma^i\right) \in S_i^-\right.\right\}$,

$E_i = \left\{(t, x) \left| x = x\left(t, \sigma^i\right), \left(t, \sigma^i\right) \in S_i\right.\right\}$,

$E_i^+ = \left\{(t, x) \left| x = x\left(t, \sigma^i\right), \left(t, \sigma^i\right) \in S_i^+\right.\right\}$,

$[E_i] = E_i^- \cup E_i \cup E_i^+$.

By $\Sigma_i$ we denote a canonical family

$$x\left(t, \sigma^i\right), u\left(t, \sigma^i\right), y\left(t, \sigma^i\right), \left(t, \sigma^i\right) \in S_i. \tag{7}$$

For $(t, x) \in [E_i]$, we denote sets

$$\begin{aligned} Y_{\Sigma_i}(t, x) &= \left\{y(t, x) \left| (t, x) \in [E_i], x = x\left(t, \sigma^i\right)\right.\right\}, \\ U_{\Sigma_i}(t, x) &= \left\{u(t, x) \left| (t, x) \in [E_i], x = x\left(t, \sigma^i\right)\right.\right\}. \end{aligned} \tag{8}$$

DEFINITION 3.1 *A set $T_i$ will be called a relative exact set for the family $\Sigma_i$ if, for each bounded rectifiable curve $C \subset T_i$ with end points $(t_1, x_1), (t_2, x_2)$,* *of the description of* $C$*), at almost every point of* $C$ *takes the same value for all* $y(t,x) \in Y_{\Sigma_i}(t,x), u(t,x) \in U_{\Sigma_i}(t,x)$*, we have*

$$\int_C \{L(t,x,u(t,x)) - y(t,x) \cdot f(t,x,u(t,x))\}\, dt + y(t,x)\, dx$$
$$= I(t_1,x_1) - I(t_2,x_2) \tag{9}$$

*for each admissible pair* $y(t,x) \in Y_{\Sigma_i}(t,x), u(t,x) \in U_{\Sigma_i}(t,x), (t,x) \in T_i$.

Let further $\gamma_i$ denote the extremal $\bar{x}(t)$ restricted to the interval $[t_i, t_{i+1}]$. The extremal $\gamma_i$ is then embedded in the family of extremals (7).

DEFINITION 3.2 *By a focal point of our embedding we mean a point of* $\gamma_i$ *at which the real Jacobian matrix* $x_{\sigma^i}$ *has rank less than* $m_i$.

When a family of solutions to (4) satisfies H4 and in addition the expression $y \cdot \frac{\partial x}{\partial \sigma^i} + \int_t^{t_{i+1}} \frac{\partial}{\partial \sigma^i} L(\tau, x(\tau,\sigma^i), u(\tau,\sigma^i)) d\tau$ vanishes at the point $(t_{i+1}(\sigma^i), x(t_{i+1}(\sigma^i), \sigma^i))$ for all $y \in Y_{\Sigma_i}(t,x)$, $i = 0, \ldots, q$, we call such an embedding of $\gamma_i$ canonical. Now, we can formulate

THEOREM 3.1 *Let* $\gamma_i$ *have a canonical embedding without focal points. Then there exists a neighbourhood* $W$ *of* $\gamma_i$ *such that for any admissible trajectory* $x(t)$ *whose graph lies in* $T_i \cap W$ *with the same ends as* $\gamma_i$ *we have*

$$\int_{t_i(0)}^{t_{i+1}(0)} L(t, \bar{x}(t), \bar{u}(t))\, dt \le \int_{t_i(0)}^{t_{i+1}(0)} L(t, x(t), u(t))\, dt.$$

**Proof.** The consequence of the assumption about the existence of a canonical embedding $x(t,\sigma^i), u(t,\sigma^i), u(t,\sigma^i)$ without focal points is that the matrix $x_{\sigma^i}(t,0)$ has rank $m_i$ for all $t \in [t_i, t_{i+1}]$. By the implicit function theorem (Clarke, 1983), there exist a neighbourhood $U_{\tilde{t}}$ for all $\tilde{t} \in [t_i, t_{i+1}]$, a neighbourhood $V_0$ of $\sigma^i = 0$ and a map $\sigma^i : U_{\tilde{t}} \to V_0$, such that $x(t, \sigma^i(t)) = x$. Since $\sigma^i \in Q_i \subset \Re^{m_i}$ and $x \in \Re^n, m_i < n$, the covering of the strip $t \in U_{\tilde{t}}$ of $(t,x)$ by the extremals $x(t,\sigma^i(t))$ is not one to one. This covering will be descriptive (Young, 1969). Each small arc of $x(t,\sigma^i)$ is the image of an arc of $(t,\sigma^i(t))$ in $U_{\tilde{t}} \times V_0$. Let us denote by $W_{\tilde{t}}$ the embedding of $\gamma_i$ restricted to the image under the map $x(t,\sigma^i(t))$ of the set $U_{\tilde{t}} \times V_0$. The neighborhood $W$ of $\gamma_i$ has the form $W = \bigcup_{\tilde{t} \in [t_i, t_{i+1}]} W_{\tilde{t}}$.

In view of the assumption preceding the theorem (Lemma 4 in Nowakowski, 1988), we have that the identity

$$y \cdot \frac{\partial x}{\partial \sigma^i} + \int_t^{t_{i+1}(\sigma^i)} \left[\frac{\partial \tilde{L}_i}{\partial \sigma^i}\right] d\tau \equiv 0$$

holds in $[S_i]$, where $\tilde{L}_i = L(t, x(t,\sigma^i), u(t,\sigma^i))$. Then $\Sigma_i$ is an exact spray of flights. The final inequality thus follows from Theorem 4 published in Nowakowski (1988).

We suppose one more hypothesis to be satisfied:

**H5.** the function $l^+(\sigma^q) = l(x(T,\sigma^q))$ has a continuous derivative $l^+_{\sigma^q}$ in $T_q$. The map $S_q^+ \to E_q^+, \left(E_q^+ = \{(T,x)\,|x = x(T,\sigma^q), \sigma^q \in Q_q\}\right)$ has the following property: given any bounded rectifiable curve $C$ in $E_q^+$, there exists a rectifiable curve $\Gamma$ in $S_q^+$ such that $C$ is its image under the map $(T,\sigma^q) \to (T, x(T,\sigma^q))$ and the ends of $C$ are the images of those of $\Gamma$.

Arguing analogously as in Nowakowski (1988), we get:

LEMMA 3.1 *Under the assumption H5, the set $E_q^+$ is a relative exact set to $\Sigma_q$.*

Our next step is to fit together many different sprays of flights. Analogously as in Nowakowski (1988) and in Young (1969) we have:

DEFINITION 3.3 *A finite or countable sequence of spray of flights in $T$*

$$\Sigma_1, \Sigma_2, \ldots, \Sigma_N, \ldots$$

*will be termed a chain of flights if $E_i^+ \subset E_{i+1}^-$ for $i = 1, 2, \ldots$.*

If $E_1^+$ of $\Sigma_1$ happens to be a relative exact set then all sets $E_i$ and $E_i^-$, $i = 1, 2, \ldots, N, \ldots$, are relative exact sets, and such a chain will be termed *an exact chain of flights.*

The consequence of the above definition and of Lemma 3.1 is

THEOREM 3.2 *Let $\gamma$ have a canonical embedding (6) without focal points and let H5 be satisfied. Then the finite sequence $\Sigma_q, \Sigma_{q-1}, \ldots, \Sigma_0$ is an exact chain of flights in $T$.*

**Proof.** In view of Theorem 3.1 our canonical embedding, if we diminish it if necessary – obtaining in this way a $W$, consists of a finite number of sprays of flights. What we need to do now is to join them together. But this procedure is described in Nowakowski (1988) and in Young (1969). In consequence, we have a chain of flights in $T$ consisting of $\Sigma_q, \Sigma_{q-1}, \ldots, \Sigma_0$. By the H5 and Lemma 3.1, this chain is an exact chain of flights in $T$. ■

It is clear that the nonexistence of focal points for embedding (6) means that there are no focal points in any subinterval $[t_i, t_{i+1}]$, $i = 0, \ldots, q$, in the sense of Definition 3.1. Therefore we can formulate the global version of Theorem 3.1.

THEOREM 3.3 *Let $\bar{x}(t), t \in [0,T]$, have a canonical embedding (6) without focal points. Then there exists a neighborhood $W$ of $\bar{x}(t)$ such that for any admissible trajectory $x(t)$, $t \in [0,T], x(0) = 0$, whose graph lies in $T \cap W$ we have*

$$\int^T L(t, \bar{x}(t), \bar{u}(t))\, dt \le \int^T L(t, x(t), u(t))\, dt.$$

In the sequel, the most important case of embedding (8) will be that in which the extremals all pass through the same point $(0,0)$. In that case we speak of a pencil of extremals and the point $(0,0)$ will be termed its vertex. Thus we shall further consider only such an embedding of $\bar{x}(t), t \in [0,T]$, for which a family of canonical extremals in the subinterval $[0,t_1]$ is of the form $x(t,\sigma^0), y(t,\sigma^0), \sigma^0 \in Q_0 \subset \Re^n$ subject at $t=0$ to the initial condition

$$x(0,\sigma^0) = 0, y(0,\sigma^0) = \bar{y}(0) + \sigma^0.$$

We assume that the matrix $(x_{\sigma^0}, y_{\sigma^0})$ has rank $n$ for $\sigma^0 = 0$. By Theorems 2.1 and 2.2, there exist a neighbourhood $N$ of $(0,0)$ and a local angle with vertex $(0,0,\dot{\bar{x}}(0))$ such that the neighbourhood $N$ can be covered by $x(t,\sigma^0)$ with line elements $(t, x(t,\sigma^0), \dot{x}(t,\sigma^0))$ from the local angle. The vertex $(0,0)$ is a focal point of this embedding. Other focal points, if any, on $\bar{x}(t)$, constitute *the conjugate set of the point* $(0,0)$.

With the notion of the conjugate points just defined, we shall write the following version of Theorem 3.3.

THEOREM 3.4 *(Jacobi) Assume hypotheses (H1)-(H5) to be satisfied. Suppose that $\bar{x}(t), t \in [0,T], \bar{x}(0) = 0$, contains no conjugate points of $(0,0)$. Then there exists an open set $W_0$ containing the graph of $\bar{x}(t), t \in [0,T]$, such that, for any other admissible trajectory $x(t)$, $t \in [0,T]$, $x(0) = 0$, whose graph lies in $W_0$, we have*

$$\int_0^T L(t, \bar{x}(t), \bar{u}(t))\, dt \le \int_0^T L(t, x(t), u(t))\, dt.$$

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