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Strong stability of a model of an overhead crane

by

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Abstract: In this paper, a strong stability result is given for a model of an overhead crane which consists of a motorized platform moving along an horizontal beam with a flexible cable, holding a load mass M. A non uniform stability result is also shown.
Keywords: boundary control, feedback stabilization, flexible structure

1. Introduction

We consider the following system :

$$\left\{ \begin{array}{lll} y_{tt}(x,t) - (ay_x)_x(x,t) = 0 & 0 < x < 1, & t > 0, \\ -a(0)y_x(0,t) + my_{tt}(0,t) = F(t) & & t > 0, \\ a(1)y_x(1,t) + My_{tt}(1,t) = 0 & & t > 0, \\ y(x,0) = y_0(x) \; y_t(x,0) = y_1(x) & 0 < x < 1, & \end{array} \right. \tag{1}$$

where

- y is a scalar function of the variables x and t (space and time variables).
- m and M are given physical constants (masses).
- a is a given function.
- F is a scalar control force depending on time.

This system models an overhead crane which consists of a motorized platform moving along an horizontal beam with a flexible cable, holding a load mass M, under the following assumptions:

- The cable is completely flexible and non-stretching.
- The length of the cable is constant.
- Displacements are small.
- Friction is neglected.
- The masses m and M are point masses.
- The angle of the cable with respect to the vertical axis is small everywhere

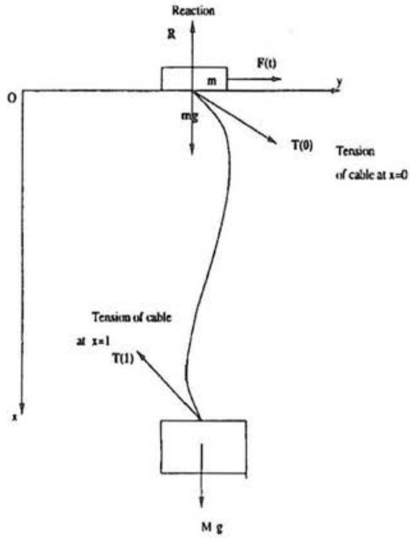

Figure 1. Overhead crane in 2-d.

Our goal is to find an appropriate feedback law for which the system (1) is well-posed and can be stabilized. In d'Andréa-Novel, Boustany, Conrad and Rao (1994), the authors studied this system with $F(t) = -\alpha y(0,t) - f(y_t(0,t))$. They neglect the acceleration of the load $y_{tt}(1,t)$ with respect to the gravity acceleration g, and obtain the strong stability of the hybrid system by using LaSalle's invariance principle. In Rao (1993) or Rao (1994), LaSalle's invariance principle and Lyapunov approach were used to stabilize the hybrid system under the condition $m << M$. A decay estimate of the energy associated to the system is also given.

In this paper, we consider the case where the controls $F(t)$ depend, as above, only on position and velocity of the platform i.e. $y(0,t)$ and $y_t(0,t)$, but we take into account the two masses $m > 0$ and $M > 0$.

The paper is divided into five sections. Section 2 provides feedback laws and energy associated to system (1). Section 3 gives a result on the well-posedness of the system (1). In Section 4 a strong stability theorem is obtained by using the method of energy. It is also shown that the system (1) cannot be uniformly stable. In Section 5, we conclude.

2. Determination of the feedback laws

In this section, we give the feedback laws such that the system (1) will be follows:

$$E(t) = \frac{1}{2}\int_0^1 (y_t^2(x,t) + a y_x^2(x,t))dx + \frac{\alpha}{2}y^2(0,t) + \frac{m}{2}y_t^2(0,t) + \frac{M}{2}y_t^2(1,t), \qquad (2)$$

where α is a positive number.

REMARK 2.1 *The term $\alpha y^2(0,t)$ is necessary to get stabilization to zero, and can be replaced by $\alpha\,(y(0,t)-c)^2$ if the objective is to drive the system to a nonzero constant position.*

We compute formally the derivative of the energy and integrate by parts. We obtain

$$\frac{dE}{dt}(t) = y_t(0,t)(F(t) + \alpha y(0,t)).$$

The choice $F(t) = -\alpha y(0,t) - f(y_t(0,t))$, under the condition that f is a non decreasing function with $f(0) = 0$, leads to

$$\frac{dE}{dt}(t) \le 0.$$

The system (1) is then dissipative.

In the next section, we use semi-group theory (see Pazy, 1983, Curtain and Zwart, 1995, Cazenave, Haraux, 1990) to prove the well-posedness of the system (1). We transform this system into a system of the type of $U_t + AU = 0$, where A is a maximal monotone operator on H, then we apply Hille-Yosida theorem.

3. Wellposedness

In the sequel of this paper, we assume that

- $a \in H^1(0,1)$, $a(x) \ge a_0 > 0$.
- $f(y_t(0,t)) = \beta y_t(0,t)$, where $\beta > 0$.

Let $H = H^1(0,1) \times L^2(0,1) \times I\!R^2$ be the energy space. H is a Hilbert space for the inner product:

$$\begin{aligned}((y,z,u,v),(y',z',u',v'))_H &= \int_0^1 a y_x y'_x + z z' dx + \alpha y(0) y'(0) \\ &\quad + m u u' + M v v'. \qquad (3)\end{aligned}$$

REMARK 3.1 *The norm associated to $(.,.)_H$ is equivalent to the usual product norm in H.*

Let y be a regular solution of (1). We introduce the following auxiliary terms

$$z(x,t) = y_t(x,t) \qquad u(t) = y_t(0,t),$$

$$v(t) = y_t(1,t) \qquad U = (y, z, u, v)$$

We write the system (1) into the form:

$$U_t(t) + (A+B)U(t) = 0, \quad \mathrm{U}(0) = U_0 \in H, \tag{4}$$

where the operators A and B are defined as follows:

$$D(A) =$$
$$\{U = (y,z,u,v) \in H^2(0,1) \times H^1(0,1) \times I\!R^2 | z(0) = u \, and \, z(1) = v\}$$
$$AU =$$
$$(-z, -(ay_x)_x, \frac{1}{m}\{\alpha y(0) - a(0)y_x(0)\}, \frac{1}{M}a(1)y_x(1)), \quad \forall U \in D(A).$$
$$BU = (0, 0, \frac{\beta}{m}u, 0), \quad \forall U \in H.$$

We use semi-group theory to show the well-posedness of the system (4). First, we have the following lemma:

LEMMA 3.1 *The operator $A+B$ is maximal monotone on H and its domain $D(A+B)$ is dense in H.*

Proof: First, we prove that A is a maximal monotone operator on H. A is monotone on H since for all $U \in D(A)$ we have $(AU, U) = 0$.
A is maximal, indeed, let $U_0 = (y_0, z_0, u_0, v_0) \in H$, we will find $U = (y,z,u,v) \in D(A)$ such that

$$U + AU = U_0.$$

This equation can be written as follows

$$\begin{cases} y - (ay_x)_x = y_0 + z_0 \\ (1 + \frac{\alpha}{m})y(0) - \frac{1}{m}a(0)y_x(0) = u_0 + y_0(0) \\ y(1) + \frac{1}{M}a(1)y_x(1) = v_0 + y_0(1). \end{cases} \tag{5}$$

We multiply the first equation of (5) by a test function $\phi \in C^\infty(0,1)$ and integrate by parts on $[0,1]$. We get:

$$L(y,\phi) = l(\phi), \tag{6}$$

where

$$\begin{aligned} L(y,\phi) &= \int_0^1 y\phi(x) + ay_x\phi_x(x)dx + y(0)\phi(0)(m+\alpha) + My(1)\phi(1) \\ l(\phi) &= \int_0^1 \phi(y_0 + z_0)(x)dx. \end{aligned} \tag{7}$$

It is easy to see that L is coercive and continuous on $H^1(0,1)$, and l is continuous on $H^1(0,1)$. So by the Lax-Milgram theorem, there exists one and only one $y \in H^1(0,1)$ such that (6) holds for all $\phi \in H^1(0,1)$. We will prove that y is the unique solution of (5). Considering a test function $\phi \in C_0^\infty(]0,1[)$, (6) can be written as follows:

$$\int_0^1 a y_x \phi_x(x) + y\phi(x)dx = \int_0^1 \phi\,(y_0 + z_0)\,(x)dx.$$

This equality leads to the equation

$$y - (ay_x)_x = y_0 + z_0, \tag{8}$$

in the sense of distributions. Since $y_0 + z_0 \in L^2(0,1)$ and $y \in L^2(0,1)$, the equality (8) is true in $L^2(0,1)$. Then $y \in H^2(0,1)$, since $a \in H^1(0,1)$, (see Mifdal, 1997). The boundary conditions at $x = 0$ and $x = 1$ of (5) can be obtained by integrating by parts the equality (6), with test function $\phi \in H^1(0,1)$, and using (8).

Hence, A is a maximal operator on H.

Moreover, it is obvious that B is continuous on H and monotone. This completes the proof of Lemma 3.1 (see Brézis, 1973). ■

Now, we apply Hille-Yosida theorem to the system:

$$\begin{cases} U_t + (A+B)U = 0 \\ U(0) = U_0 \end{cases}$$

THEOREM 3.1 *1. For all initial data $U_0 = (y_0, z_0, u_0, v_0) \in D(A)$, there exists a unique solution of the system (4) $U(x,t) = (y(x,t), z(x,t)u(t), v(t)) \in D(A)$. Moreover, we have the regularity:*

$$\begin{aligned} y &\in C^0(0,\infty; H^2(0,1)) \cap C^1(0,\infty\,;\, H^1(0,1)) \\ &y(0,.),\ y(1,.) \in C^2(0,\infty; R). \end{aligned} \tag{9}$$

2. For all initial data $U_0 = (y_0, z_0, u_0, v_0) \in H$, there exists a unique weak solution of the system (4) $U(x,t) = (y(x,t), z(x,t), u(t), v(t)) \in H$. Moreover, we have the regularity:

$$y \in C^0(0,\infty\,;\, H^1(0,1)) \cap C^1(0,\infty\,;\, L^2(0,1)). \tag{10}$$

REMARK 3.2 *The problems (1) and (4) are equivalent for all $U_0 = (y_0, z_0, u_0, v_0) \in D(A)$ (see Mifdal, 1997).*

4. Stability

4.1. Strong stability

In this subsection we prove strong stability of (4) by using the invariance principle of LaSalle. We first give the lemmas which will be used in the proof of the main result of this subsection.

LEMMA 4.1 *The canonical injection of $D(A)$ into H is compact.*

The proof of Lemma 4.1 is obvious.

LEMMA 4.2 *The function ϕ, being the solution of the system*

$$\begin{cases} \phi_x(x) + \phi|\frac{a_x}{a}|(x) = 1 & 0 < x < 1 \\ \phi(1) = -M \end{cases} \tag{11}$$

satisfies the following properties

- $\phi(x) \leq -M$, *and* $\phi(1) = -M$.
- $\phi_x(x) \geq 1 \qquad 0 < x < 1$.
- $a(\frac{\phi}{a})_x(x) \geq 1, \qquad 0 < x < 1$.

Proof: The function ϕ, solution of the system (11), can be written as follows

$$\phi(x) = \left\{ \int_1^x exp\left(\int_1^s |\frac{a_\sigma}{a}|(\sigma)d\sigma \right) ds - M \right\} exp\left(-\int_1^x |\frac{a_s}{a}|(s)ds \right) \leq 0. \tag{12}$$

This leads to the properties

1. $\phi_x = 1 - \phi|\frac{a_x}{a}| \geq 1$,

2. $\forall x \in [0,1]\ \phi(x) \leq \phi(1) = -M$.

3. $a(\frac{\phi}{a})_x = \phi_x - \phi\frac{a_x}{a}$
   $\geq \phi_x + \phi|\frac{a_x}{a}| = 1.$

■

LEMMA 4.3 *Let y be a solution of*

$$\begin{cases} y_{tt}(x,t) - (ay_x)_x(x,t) = 0 & 0 < x < 1 \quad t > 0 \\ a(0)y_x(0,t) - \alpha y(0,t) = 0 & t > 0 \\ ay_x(1,t) + My_{tt}(1,t) = 0 & t > 0 \\ y_t(0,t) = 0 & t > 0 \\ y(x,0) = y_0(x)\ y_t(x,0) = y_1(x) & \end{cases} \tag{13}$$

with y verifing (9). Then, y vanishes identically.

Proof: We integrate the first equation of (13) in x and t to get

$$\begin{aligned} \int_0^1 (y_t(x,T) - y_t(x,0))dx &= \int_0^T (ay_x(1,t) - ay_x(0,t))dt \\ &= \int_0^T (-My_{tt}(1,t) - \alpha y(0,t))dt \\ &= M[y_t(1,0) - y_t(1,T)] - \alpha T y(0,t). \end{aligned}$$

By using triangular inequality and Young's inequality, we obtain

$$|\int_0^1 (y_t(x,T) + My_t(1,T)| \leq \int_0^1 |y_t(x,T)|dx + M|y_t(1,T)|$$

$$
\begin{aligned}
&+\frac{1}{2\epsilon}M|y_t(1,T)|^2\\
&\leq \frac{\epsilon}{2}(1+M)+\frac{1}{\epsilon}E(T).
\end{aligned}
$$

$$
|\int_0^1 (y_t(x,0)+My_t(1,0)| \leq \frac{\epsilon}{2}(1+M)+\frac{1}{\epsilon}E(0).
$$

If we take $\epsilon_1 = \epsilon(1+M)$ and $C_\epsilon = \frac{2}{\epsilon}$, then

$$
\begin{aligned}
\alpha T|y(0,t)| &\leq \epsilon_1 + \frac{C_\epsilon}{2}(E(0)+E(T))\\
&\leq \epsilon_1 + C_\epsilon E(0).
\end{aligned}
$$

This inequality is available for all T positive, so we deduce that

$$
y(0,t) = 0 \ \forall t > 0. \tag{14}
$$

Now, consider the solution ϕ of (11), multiply the first equation of (13) by ϕy_x and integrate in x and t. We obtain:

$$
\begin{aligned}
&\int_0^T \int_0^1 y_{tt}\phi y_x(x,t)dxdt =\\
&[\int_0^1 y_t\phi y_x(x,t)dx]_0^T - \int_0^T\int_0^1 y_t\phi y_{xt}(x,t)dxdt\\
&= [\int_0^1 \phi(x)y_ty_x(x,t)dx]_0^T - \frac{1}{2}\int_0^T\int_0^1 \phi(x)(y_t^2)_x(x,t)dxdt\\
&= [\int_0^1 \phi(x)y_ty_x(x,t)dx]_0^T - \frac{1}{2}[\int_0^T \phi(x)y_t^2(x,t)dt]_0^1\\
&+\frac{1}{2}\int_0^T\int_0^1 y_t^2(x,t)\phi(x)_x dxdt\\
&\int_0^T\int_0^1 \phi(x)y_x(ay_x)_x(x,t)dxdt =\\
&\frac{1}{2}\int_0^1\int_0^T \frac{\phi(x)}{a}((ay_x)^2)_x(x,t)dxdt\\
&= \frac{1}{2}\int_0^T [\frac{\phi(x)}{a}(ay_x)^2(x,t)dt]_0^1\\
&-\frac{1}{2}\int_0^T\int_0^1 a(\frac{\phi(x)}{a})_x ay_x^2(x,t)dxdt.
\end{aligned}
$$

We combine these terms and use the equality $y_t(0,t) = 0$ to get:

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_0^1 a\left(\frac{\phi}{a}\right)_x(x)ay_x^2(x,t)+\phi_x(x)y_t^2(x,t)dxdt + \frac{M}{2}\int_0^T y_t^2(1,t)dt\\
&= -[\int_0^1 \phi(x)y_ty_x(x,t)dx]_0^T + \frac{1}{2}[\int_0^T \frac{\phi}{a}(x)(ay_x)^2(x,t)]_0^1.
\end{aligned}
$$

Using the properties of ϕ in the second member, $ay_x(0) = \alpha y(0) = 0$, and neglecting the second term in the second member which is negative, we get:

$$\int_0^T E(t)dt \leq -\int_0^1 \phi[y_t y_x]_0^T dx. \tag{15}$$

Hence

$$\begin{aligned} \int_0^T E(t)dt &\leq C(E(T) + E(0)) \\ &\leq 2CE(0). \end{aligned}$$

From $y_t(0,t) = 0$, we deduce that the energy is constant. Hence

$$E(T) \leq \frac{2}{T} CE(0),$$

and then

$$E(t) = 0 \ \forall\, t \geq 0,$$

i.e

$$y \equiv 0. \qquad \blacksquare$$

THEOREM 4.1 *Let $U_0 \in H$. We have*

$$\lim_{t\to\infty} E(t) = 0 \tag{16}$$

Proof: Since $D(A)$ is dense in H, we can take $U_0 \in D(A)$. So, let $U_0 = (y_0, z_0, u_0, v_0) \in D(A)$. We have (see Cazenave, Haraux, 1990)

- $\omega(U_0) \neq \emptyset$,
- $S(t)\omega(U_0) \subset D(A)$,
- $E(t)$ is constant on $\omega(U_0)$,

where $\omega(U_0)$ is the ω-limit set of U_0, and $S(t)$ is the C^0-semi-group generated by the operator $A + B$.

If we prove that $\omega(U_0) = \{0\}$, then the distance between $S(t)U_0$ and $\omega(U_0)$ converges to zero, which then implies

$$\lim_{t\to\infty} E(t) = 0. \tag{17}$$

So let us show that $\omega(U_0) = \{0\}$.
Let $U_1 = (y_1, z_1, u_1, v_1) \in \omega(U_0) \subset D(A)$ and let $\tilde{U}(x,t) = (\tilde{y}, \tilde{z}, \tilde{u}, \tilde{v})(x,t)$ be the solution of (4). We have:

$$\tilde{U}(x,t) = S(t)U_1 \in \omega(U_0)\ \forall t \geq 0,$$

so, the energy function $E(t)$ is constant, and then $\tilde{y}_t(0,t) = 0$. Theorem 4.1 is

REMARK 4.1 *System (1) with $F(t) = -\alpha y(0,t) - \beta y_t(0,t)$ can be written in the following form*

$$u_t = \mathcal{A}u + \mathcal{B}w,$$

with $w = -\mathcal{B}^* u$, where

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{d}{dx}\left(a\frac{d}{dx}\right) & 0 & 0 & 0 \\ \frac{a(0)}{m}\frac{d}{dx}|_{x=0} - \frac{\alpha}{m}()\,|_{x=0} & 0 & 0 & 0 \\ -\frac{a(1)}{M}\frac{d}{dx}|_{x=1} & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{\beta}{m}} \\ 0 \end{pmatrix}$$

$\mathcal{A}$ has a compact resolvant and generates a semi-group on H, and $\mathcal{B}$ is bounded on H. So, strong stability of system $u_t = \mathcal{A}u - \mathcal{B}\mathcal{B}^* u$ amounts to prove that the only solution of $\mathcal{B}^* e^{\mathcal{A}t} u = 0$ for any $t \geq 0$ is $u = 0$, Slemrod (1989). This is exactly what is contained in Lemma 4.3. See also Benchimol (1978) for a rather general treatment of necessary and sufficient conditions to obtain strong or weak stabilizability of contraction semi-groups, and You (1988) for an application to a vibrating plate.

4.2. Nonuniform stability

In the system (4), we have a compact perturbation of the linear operator A on Hilbert space H. Using a compactness perturbation argument due to Russell (1975), we will prove the nonuniform stability of (4).

DEFINITION 4.1 *Let A be an operator generating a C^0 semi-group $(S(t))_{t\geq 0}$, A is uniformly stable if there exists $M > 0$ and $\omega > 0$ such that*

$$\|S(t)\| \leq Me^{-\omega t} \ \forall t \geq 0.$$

We recall that if A is a linear operator, with dense domain on Hilbert space H and if we denote by $(.,.)$ the inner product on H, we define the adjoint A^* as follows

$$D(A^*) = \{\phi \in H | \exists c \geq 0 \ |(\phi, Au)| \leq c\|u\| \ \forall u \in D(A)\},$$

$$(A^* v, u) = (v, Au), \ \forall u \in D(A), \ \forall v \in D(A^*).$$

LEMMA 4.4 *The operator A is skew-adjoint on H.*

Proof:

A is skew-symmetrical, in fact from the proof of Lemma 3.1, we deduce that for all U and V in $D(A)$

$$(A(U+V), U+V)_H = 0,$$

which leads to:

$$(AU, V)_H = -(U, AV)_H \ \forall\ V \in D(A)\ \forall\ U \in D(A).$$

Morever $D(A) \subset D(A^*)$. In fact let $\Phi \in D(A)$. We show that $\Phi \in D(A^*)$. Let $U \in D(A)$, we have

$$\begin{aligned} \|(\Phi, AU)_H\| &= \|-(A\Phi, U)\| \\ &\leq \|A\Phi\|_H \|U\|_H, \end{aligned}$$

which implies the inclusion of $D(A)$ into $D(A^*)$.

Conversely, let $x \in D(-A^*)$, and put $g = x - A^*x \in H$.

Since A is maximal monotone on H, there exists $y \in D(A)$ such that $g = y + Ay$.

We use the fact that $A^* = -A$ on $D(A)$, we put $z = x - y$ to obtain:

$$(z, z) = (A^*z, z) \leq 0.$$

Hence $z = 0$ i.e. $x = y \in D(A)$.

REMARK 4.2 *The operator A, which is maximal monotone and skew adjoint, generates a group of isometries that we denote* $(S_A(t))_{t\in I\!R}$.

Now, we can give the result of nonuniform stability

THEOREM 4.2 *The problem (4) is not uniformly stable.*

Proof: Let $U = (y(x,t), z(x,t), u(t), v(t)) \in D(A)$ be the solution of the problem (4) with initial condition $U(0) = U_0 \in D(A)$. We introduce the auxiliary terms:

$$\begin{aligned} \tilde{y}(x,t) &= y(x,-t),\ \tilde{z}(x,t) = -z(x,-t), \\ \tilde{u}(t) &= -u(-t),\ \tilde{v}(t) = -v(-t), \\ \tilde{U}(x,t) &= (\tilde{y}(x,t), \tilde{z}(x,t), \tilde{u}(t), \tilde{v}(t)). \end{aligned}$$

We verify easily that

$$\left\{ \begin{array}{c} \frac{d\tilde{U}}{dt} + (A-B)\tilde{U} = 0 \ \ t > 0 \\ \tilde{U}(x,0) = (y(x,0), -z(x,0), -u(0), -v(0)),\ 0 < x < 1 \end{array} \right. \tag{18}$$

This solution can be written as:

$$\tilde{U}(x,t) = \tilde{S}(t)\tilde{U}(x,0) \quad t > 0 \quad 0 < x < 1,$$

where $\tilde{S}(t)$ is the C^0 semi-group which is associated to $A - B$. The operator A generates a group of isometries, the operator B is compact. So, from Russell's theorem, Russel (1975), there exist no reals $\gamma < 1$ and $T > 0$ such that we have:

$$\|S(T)\| \leq \gamma,$$

$\|\tilde{S}(T)\| \leq$ (19)

On the other hand, we have $\forall t \geq 0$

$$\|U(x,0)\| = \|\tilde{U}(x,0)\|, \tag{20}$$

$$\|U(x,t)\| = \|\tilde{U}(x,-t)\|. \tag{21}$$

We deduce

$$\|S(t)\| = \|\tilde{S}(-t)\|.$$

Finally, from (19) we deduce that there exist no reals $\gamma < 1$ and $T > 0$ such that we have:

$$\|S(T)| \leq \gamma. \tag{22}$$

■

5. Conclusion

The hybrid system modeling an overhead crane can be stabilized strongly but not uniformly when the action on the platform depends only on its position $y(0,t)$ and velocity $y_t(0,t)$. To obtain the uniform stability, we may take into account the rotation velocity of the cable at $x = 0$ (see Mifdal, 1997).

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