Control and Cybernetics

vol. 27 (1998) No. 3

Contingent epiderivative and its applications to set-valued optimization

by

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Abstract: In the present paper we give an alternative definition of contingent epiderivative for a set-valued map. We use our concept of contingent epiderivative to formulate necessary and/ or sufficient optimality conditions for a set-valued optimization problem and to study sensitivity of a family of parametrized vector optimization problems.

Keywords: Contingent epiderivative, vector optimization, sensitivity analysis, optimality conditions

1. Introduction

The notion of contingent derivative of a set-valued mapping plays an important role in vector optimization. For instance, contingent derivative is used to formulate optimality conditions for vector optimization problems with set-valued maps (eg. Corley, 1988, Luc, 1989), and to study sensitivity (eg. Tanino, 1988a, b, Sawaragi, Nakayama and Tanino, 1985, Shi, 1993).

To derive sufficient and necessary optimality conditions Jahn and Rauh (1997) introduced the notion of contingent epiderivative of a set-valued map. In the present paper we give an alternative definition of contingent epiderivative and present some of its applications. In Section 2, we introduce the concept of contingent epiderivative and investigate its properties. In Section 3, we formulate necessary optimality conditions for Benson's proper minimality and sufficient optimality conditions for minimality for a set-valued optimization problem. In Section 4, we exploit the contingent epiderivative to study sensitivity of a family of parametrized vector optimization problems.

2. Contingent epiderivative of set-valued mappings

In this section we give a definition of contingent epiderivative for set-valued mappings.

Let X and Y be real normed spaces, and let $F: X \to Y$ be a set-valued mapping. The domain and graph of F are defined by

$$\mathrm{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\},$$

$$\mathrm{gr}(F) := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

DEFINITION **1 (Aubin and Frankowska (1990))** *Let A be a nonempty set of X and $u \in \mathrm{cl}(A)$ (closure of A) a given element. The* contingent cone $T_A(u)$ *is defined by*

$$T_A(u) = \{v \in X \mid \liminf_{h \downarrow 0} h^{-1} \mathrm{d_A}(u + hv) = 0\},$$

where $d_A(u) = \inf_{v \in A} \|u - v\|$. Equivalently, $v \in T_A(u)$ if and only if there exist sequences $\{h_n\}$ of positive real numbers and $\{v_n\} \subset X$ with $h_n \to 0$, $v_n \to v$ such that

$$u + h_n v_n \in A, \text{ for all } n \geq 1.$$

Clearly, $T_A(x)$ is a closed cone, and if A is a convex set, then $T_A(x)$ is a closed convex cone (see Aubin and Frankowska, 1990).

DEFINITION **2 (Aubin and Frankowska (1990))** *Let $F: X \to Y$ be a set-valued map, and let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$. A set-valued map $DF(\bar{x}, \bar{y}): X \to Y$ whose graph equals the contingent cone to the graph of F at $(\bar{x}, \bar{y})$, i.e.*

$$\mathrm{gr}(DF(\bar{x}, \bar{y})) = T_{\mathrm{gr}(F)}(\bar{x}, \bar{y}),$$

is called the contingent derivative of F at $(\bar{x}, \bar{y})$.

It is well known that the concept of contingent derivative is a natural extension of tangency and plays an important role in set-valued analysis (see Aubin and Frankowska, 1990). This concept has been used in set-valued optimization to formulate optimality conditions (Corley, 1988, Luc, 1989) and to study sensitivity analysis (see Tanino, 1988a, b and Shi, 1993).

In order to generalize classical optimality conditions, a concept of contingent epiderivative has been introduced by Aubin (1981) (see also Aubin and Frankowska, 1990) for extended real-valued functions. This concept has been used by Penot (1997) to study sensitivity in optimization. In a very recent paper, the concept of contingent epiderivative has been extended by Jahn and

Let K be a closed convex cone of Y and let $F: X \to Y$ be a set-valued map. The set

$$\mathrm{epi}(F) := \{(x, y) \in X \times Y \mid y \in F(x) + K\}$$

is called the epigraph of F.

DEFINITION **3 (Jahn and Rauh (1997))** *Let $(x, y) \in \mathrm{gr}(F)$. A single-valued map $D'F(x, y): X \to Y$ whose epigraph equals the contingent cone to the epigraph of F at (x, y), i.e.*

$$\mathrm{epi}(D'F(x, y)) = T_{\mathrm{epi}(F)}(x, y),$$

is called contingent epiderivative of F at (x, y).

This concept has been used in Jahn and Rauh (1997) to formulate optimality conditions for set-valued optimization problems. However, even for a simple set-valued map, this epiderivative may not exist.

EXAMPLE **1** *Let $X = R^1, Y = R^2, K = R^2_+$. Define a set-valued mapping as follows*

$$F(x) = \begin{cases} \{y = (\xi_1, \xi_2) \in R^2 \mid \xi_1^2 + \xi_2^2 \le x^2\}, & \text{if } 0 \le x \le 1; \\ \emptyset, & \text{if } x < 0 \text{ or } x > 1. \end{cases}$$

Let $x_0 = 1, y_0 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

$$T_{\mathrm{epi}(F)}(x_0, y_0) = \{(x, y) \in R^1 \times R^2 \mid y = (\xi_1, \xi_2),\ \xi_1 + \xi_2 \ge -\sqrt{2}x\}.$$

It is easy to show that the contingent epiderivative $D'F(x_0, y_0)$ does not exist.

In what follows we shall give another definition of contingent epiderivative for a set-valued map. Since no confusion arises, we use the same name as in Jahn and Rauh (1997).

Let A be a subset of Y and $K \subset Y$ be a closed convex pointed cone.

A point $a_0 \in A$ is a *minimal point* (*an efficient point*) of A with respect to K (see Jahn, 1986), $a_0 \in \mathrm{Min}A$, if

$$(A - a_0) \cap (-K) = \{0\}.$$

Define $(F + K)(x) = F(x) + K$, for $x \in X$.

DEFINITION **4** *Let $(x, y) \in \mathrm{gr}(F)$. We say that the set-valued map $D_\uparrow F(x, y): X \to Y$ defined by*

$$D_\uparrow F(x, y)(u) := \mathrm{Min}D(F + K)(x, y)(u),$$

is the contingent epiderivative of F at (x, y).

The set-valued map F is said to be contingently epidifferentiable at (x, y) if $\mathrm{dom}(D_\uparrow F(x, y)) = \mathrm{dom}(D(F + K)(x, y))$

The idea of Definition 4 is based on Definition 6.1.2 in Aubin and Frankowska (1990). Independently, this definition was also introduced in Chen and Jahn (1998).

Clearly, if the contingent epiderivative $D^{'}F(x,y)$ exists, by noting that $\mathrm{epi}(F) = \mathrm{gr}(F+K)$, we obtain

$$D^{'}F(x,y)(u) + K = D(F+K)(x,y)(u), \text{ for } u \in \mathrm{dom}(D(F+K)(x,y)).$$

Hence

$$D_{\uparrow}F(x,y)(u) = D^{'}F(x,y)(u).$$

EXAMPLE 2 *Let us consider Example 1 again. It is clear that*

$$D_{\uparrow}F(x_0,y_0)(u) = \{y = (\xi_1,\xi_2) \in R^2 \mid \xi_1 + \xi_2 = -\sqrt{2}u\}.$$

Let $(x,y) \in \mathrm{gr}(F)$. By Proposition 2.1 of Tanino (1988a), we have

$$DF(x,y)(u) + K \subset D(F+K)(x,y)(u), \text{ for all } u \in X. \tag{1}$$

By similar arguments as those used in the proof of Proposition 2.1 of Tanino (1988a), we can also prove that

$$D(F+K)(x,y)(u) + K = D(F+K)(x,y)(u), \text{ for all } u \in X.$$

PROPOSITION 1 *Let $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$. If K has a compact base, then*

$$D_{\uparrow}F(\bar{x},\bar{y})(u) \subset \mathrm{Min}DF(\bar{x},\bar{y})(u), \textit{ for all } u \in X. \tag{2}$$

Proof. Since K has a compact base, by Theorem 2.1 of Tanino (1988a),

$$D_{\uparrow}F(x,y)(u) \subset DF(x,y)(u), \text{ for all } u \in X. \tag{3}$$

¿From (1) and (3) it follows that

$$D_{\uparrow}F(x,y)(u) \subset \mathrm{Min}(DF(x,y)(u) + K) = \mathrm{Min}DF(x,y)(u), \text{ for all } u \in X.$$

■

DEFINITION 5 *A set-valued map $F: X \to Y$ is K-convex if the epigraph of F is convex, i.e. for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$,*

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + K.$$

If F is K-convex, $D(F+K)(x,y)(\cdot)$ is clearly a closed convex process, $D(F+K)(x,y)(u)$ is closed and convex for each $u \in \mathrm{dom}(D(F+K)(x,y))$, and $D(F+K)(x,y)(0)$ is a closed convex cone.

PROPOSITION 2 *Let $F: X \to Y$ be a K-convex set-valued map and let $(\bar{x},\bar{y}) \in$*

(i) *For every* $\lambda > 0$ *and* $u \in X$,
$$\lambda D_{\uparrow}F(\bar{x},\bar{y})(u) = D_{\uparrow}F(\bar{x},\bar{y})(\lambda u);$$
(ii) *if* $D_{\uparrow}F(\bar{x},\bar{y})(0) \neq \emptyset$, *then*
$$0 \in D_{\uparrow}F(\bar{x},\bar{y})(0);$$
(iii) *if* $\mathrm{epi}(D_{\uparrow}F(\bar{x},\bar{y})) = T_{\mathrm{epi}(F)}(\bar{x},\bar{y})$, *then for every* $u_1, u_2 \in X$,
$$D_{\uparrow}F(\bar{x},\bar{y})(u_1) + D_{\uparrow}F(\bar{x},\bar{y})(u_2) \subset D_{\uparrow}F(\bar{x},\bar{y})(u_1+u_2) + K;$$
(iv) *if* $\mathrm{epi}(D_{\uparrow}F(\bar{x},\bar{y})) = T_{\mathrm{epi}(F)}(\bar{x},\bar{y})$, *then*
$$\mathrm{epi}(DF(\bar{x},\bar{y})) \subset \mathrm{epi}(D_{\uparrow}F(\bar{x},\bar{y})),$$
and the converse inclusion holds if K *has a compact base. Consequently,*
$$D_{\uparrow}F(\bar{x},\bar{y})(u) = \mathrm{Min}DF(\bar{x},\bar{y})(u), \text{ for all } u \in \mathrm{dom}(D(F+K)(\bar{x},\bar{y})).$$

Proof. (i) and (iii) are proved in Chen and Jahn (1998).

(ii) Assume on the contrary that $0 \notin D_{\uparrow}F(\bar{x},\bar{y})(0)$. Then there exists $k \in K \setminus \{0\}$ with

$$-k \in D(F+K)(\bar{x},\bar{y})(0).$$

For every $y \in D(F+K)(\bar{x},\bar{y})(0)$, since $D(F+K)(\bar{x},\bar{y})(0)$ is a closed convex cone,

$$y - k \in D(F+K)(\bar{x},\bar{y})(0).$$

Hence

$$y \notin \mathrm{Min}D(F+K)(\bar{x},\bar{y})(0).$$

Thus

$$D_{\uparrow}F(\bar{x},\bar{y})(0) = \emptyset,$$

a contradiction.

(iv) If $\mathrm{epi}(D_{\uparrow}F(\bar{x},\bar{y})) = T_{\mathrm{epi}(F)}(\bar{x},\bar{y})$, from (1), we get

$$\mathrm{epi}(DF(\bar{x},\bar{y})) \subset \mathrm{epi}(D_{\uparrow}F(\bar{x},\bar{y})).$$

The converse inclusion follows from Proposition 1. ■

Now we give the conditions ensuring the equality

$$\mathrm{epi}(D_{\uparrow}F(\bar{x},\bar{y})) = T_{\mathrm{epi}(F)}(\bar{x},\bar{y}) \tag{4}$$

LEMMA **1 (Ha (1994))** *Let* K *have a bounded base. Let* $F: X \to Y$ *be a* K*-convex set-valued map and let* $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$. *If there exists a set-valued map* $G: X \to Y$ *with bounded images such that*

$$T_{\mathrm{epi}(F)}(\bar{x},\bar{y}) \subset \mathrm{epi}(G),$$

then (4) holds.

LEMMA **2** *Let $K \subset R^n$ be a closed convex pointed cone, let $F: X \to R^n$ be a K-convex set-valued map, and let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$. If $D_\uparrow F(\bar{x}, \bar{y})(0) \neq \emptyset$, then (4) holds.*

Proof. From the definition it follows that

$$D(F+K)(\bar{x}, \bar{y})(u) = L_u \cap T_{\mathrm{epi}(F)}(\bar{x}, \bar{y}), \text{ for all } u \in \mathrm{dom}(D(F+K)(\bar{x}, \bar{y})),$$

where $L_u = \{(u, y) \in X \times Y \mid y \in Y\}$. Since L_u and $T_{\mathrm{epi}(F)}(\bar{x}, \bar{y})$ are closed and convex, by Proposition 2.5 of Luc (1989),

$$0^+ D(F+K)(\bar{x}, \bar{y})(u) = 0^+ L_u \cap T_{\mathrm{epi}(F)}(\bar{x}, \bar{y}) = D(F+K)(\bar{x}, \bar{y})(0).$$

Since $D_\uparrow F(\bar{x}, \bar{y})(0) \neq \emptyset$, by Proposition 2, $0 \in D_\uparrow F(\bar{x}, \bar{y})(0)$. Hence

$$D(F+K)(\bar{x}, \bar{y})(0) \cap (-K) = \{0\}.$$

By Corollary 4.6 of Luc (1989), $D(F+K)(\bar{x}, \bar{y})(u)$ has the domination property, i.e.,

$$D(F+K)(\bar{x}, \bar{y})(u) = D_\uparrow F(\bar{x}, \bar{y})(u) + K.$$

Thus, (4) holds. ■

PROPOSITION **3 (Shi (1993))** *Let K be a closed convex pointed cone in R^n. Let $F: R^m \to R^n$ be a K-convex set-valued map and let $\bar{x} \in \mathrm{int}(\mathrm{dom}(F))$. If $\bar{y}$ is a Benson proper minimal point of $F(\bar{x})$, then*

$$D(F+K)(\bar{x}, \bar{y})(u) = DF(\bar{x}, \bar{y})(u) + K, \text{ for all } u \in R^m.$$

Consequently,

$$D_\uparrow F(\bar{x}, \bar{y})(u) = \mathrm{Min} DF(\bar{x}, \bar{y})(u), \text{ for all } u \in R^m.$$

DEFINITION **6** *A set-valued map $F: X \to Y$ is said to be upper locally Lipschitz at $x_0 \in X$ if there exist a neighborhood U of x_0 and a positive constant M such that*

$$F(x) \subset F(x_0) + M\|x - x_0\| B_Y, \text{ for all } x \in U,$$

where B_Y is the unit ball of the space Y.

PROPOSITION **4 (Tanino (1988a))** *Let $K \subset Y$ have a compact base. Let $F: X \to Y$ be upper locally Lipschitz at $\bar{x} \in \mathrm{dom}(F)$. If $\bar{y}$ is a Benson proper minimal point of $F(\bar{x})$, then*

$$D(F+K)(\bar{x}, \bar{y})(u) = DF(\bar{x}, \bar{y})(u) + K, \text{ for all } u \in X.$$

Consequently,

$$D_\uparrow F(\bar{x}, \bar{y})(u) = \mathrm{Min} DF(\bar{x}, \bar{y})(u), \text{ for all } u \in X.$$

DEFINITION 7 *Let $F: X \to Y$ be a set-valued mapping defined on a neighborhood of $\bar{x}$ and let $\bar{y} \in F(\bar{x})$. F is called directionally compact at $(\bar{x}, \bar{y})$ in the direction $\bar{u}$ if for every sequence of positive numbers $h_n \to 0$ and every sequence $u_n \to \bar{u}$, any sequence y_n with*

$$\bar{y} + h_n y_n \in F(\bar{x} + h_n u_n), \text{ for each } n$$

contains a convergent subsequence.

If F is single-valued and Fréchet differentiable at $\bar{x}$, then F is directionally compact at $\bar{x}$ in any direction u.

Following Penot (1984), a set-valued mapping $F: X \to Y$ is called compact at $\bar{x}$ if for every sequence $x_n \to \bar{x}$, any sequence $y_n \in F(x_n)$ has a converging subsequence.

If $DF(\bar{x}, \bar{y})$ is compact at u and F is pseudo-convex at $(\bar{x}, \bar{y})$ (see Aubin and Frankowska, 1990), i.e.

$$\mathrm{gr}F \subset (\bar{x}, \bar{y}) + T_{\mathrm{gr}F}(\bar{x}, \bar{y}),$$

then F is directionally compact at $(\bar{x}, \bar{y})$ in the direction u. Indeed, for every sequence of positive numbers $h_n \to 0$ and every sequence $u_n \to u$, any sequence y_n with

$$\bar{y} + h_n y_n \in F(\bar{x} + h_n u_n), \text{ for each } n,$$

Since F is pseudo-convex at $(\bar{x}, \bar{y})$, we have

$$y_n \in DF(\bar{x}, \bar{y})(u_n).$$

Because $DF(\bar{x}, \bar{y})$ is compact at u, y_n has a converging subsequence.

PROPOSITION 5 *Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$. If F is directionally compact at $(\bar{x}, \bar{y})$ in the direction $u \in X$, then*

$$D(F + K)(\bar{x}, \bar{y})(u) = DF(\bar{x}, \bar{y})(u) + K.$$

Consequently,

$$D_{\uparrow} F(\bar{x}, \bar{y})(u) = \mathrm{Min} DF(\bar{x}, \bar{y})(u).$$

Proof. In view of (1), it suffices to prove that

$$D(F + K)(\bar{x}, \bar{y})(u) \subset DF(\bar{x}, \bar{y})(u) + K.$$

Let $y \in D(F + K)(\bar{x}, \bar{y})(u)$. From the definition, there exist a sequence of positive numbers $h_n \to 0$ and sequences $u_n \to u$, $y_n \to y$ and $d_n \in K$ such that

$$\bar{y} + h_n y_n - d_n \in F(\bar{x} + h_n u_n), \text{ for all } n.$$

By our assumption, $y_n - d_n/h_n$ contains a convergent subsequence. Without loss of generality, we may assume that $y_n - d_n/h_n$ converges to some element y_1. Hence $y_1 \in DF(\bar{x}, \bar{y})(u)$ and $d_n/h_n \to y - y_1 \in K$. Thus, $y \in DF(\bar{x}, \bar{y})(u) + K$. ■

3. Optimality conditions in set-valued optimization

Let X and Y be real normed spaces, let A be a nonempty subset of X, and let K be a convex and pointed cone of Y. Let $F: A \to Y$ be a set-valued map.

Consider a set-valued optimization problem:

$$\min_{x \in A} F(x). \tag{5}$$

DEFINITION 8 (a) *A pair* $(\bar{x}, \bar{y})$ *with* $\bar{x} \in A$ *and* $\bar{y} \in F(\bar{x})$ *is called a minimal solution of (5) if* $\bar{y}$ *is a minimal point of the set* $F(A) = \cup_{x \in A} F(x)$, *i.e*

$$(F(A) - \bar{y}) \cap (-K) = \{0\}.$$

(b) *A pair* $(\bar{x}, \bar{y})$ *with* $\bar{x} \in A$ *and* $\bar{y} \in F(\bar{x})$ *is called a Benson proper minimal solution of (5) if* $\bar{y}$ *is a Benson proper minimal point of the set* $F(A) = \cup_{x \in A} F(x)$, *i.e*

$$\text{cl}[\text{cone}(F(A) + K - \bar{y})] \cap (-K) = \{0\}.$$

Optimality conditions in set-valued optimization have been given by Corley (1988) and Luc (1989) with the aid of contingent derivatives, and by Jahn and Rauh (1997) with the aid of their concept of contingent epiderivative.

We present a necessary optimality condition for proper minimal solution of problem (5) by using the notion of contingent epiderivative introduced in previous section. We also give a sufficient condition for minimal solution of problem (5) under convexity assumption. For weak minimizers, sufficient and necessary conditions were given in Chen and Jahn (1998).

THEOREM 1 *Suppose that* K *has a weakly compact base and* $F(A)+K$ *is convex or* K *has a compact base. If* $(\bar{x}, \bar{y})$ *is a Benson proper minimal solution of (5), then*

$$D_{\uparrow} F(\bar{x}, \bar{y})(x - \bar{x}) \cap (-K \setminus \{0\}) = \emptyset, \text{ for all } x \in A.$$

Proof. Since $(\bar{x}, \bar{y})$ is a Benson proper minimal solution of (5), we have

$$\text{cl}[\text{cone}(F(A) + K - \bar{y})] \cap (-K) = \{0\}.$$

By our assumptions and Theorem 1 in Dauer and Saleh (1993), there exists a closed convex pointed cone S such that $K \setminus \{0\} \subset \text{int } S$ and

$$\text{cl}[\text{cone}(F(A) + K - \bar{y})] \cap (-\text{int } S) = \emptyset.$$

Assume that there exists $x_1 \in A$ such that

$$D_{\uparrow} F(\bar{x}, \bar{y})(x_1 - \bar{x}) \cap (-K \setminus \{0\})$$

contains an element v. Then there exist sequences $\{x_n\} \subset A$, $\{v_n\} \subset Y$ and $\{h_n\}$ of positive real numbers with $x_n \to x_1 - \bar{x}$, $v_n \to v$ and $h_n \to 0$ such that

Hence

$$\bar{y}_n = \bar{y} + h_n v_n - k_n \in F(\bar{x} + h_n x_n), \text{ for } k_n \in K.$$

Since $v \in -K \setminus \{0\} \subset -\text{int}S$, there exists N such that

$$h_n v_n \in -\text{int}S, \text{ for all } n \geq N.$$

Since $\text{int}S + K \setminus \{0\} \subset \text{int}S$,

$$\bar{y}_N - \bar{y} \in -\text{int}S.$$

Hence $(\bar{x}, \bar{y})$ is not a proper minimal solution of (5). ■

Under some additional assumptions, we obtain the following sufficient conditions.

THEOREM 2 *Let A be convex and let $F\colon A \to Y$ be K-convex. Let $\bar{x} \in A$ and $\bar{y} \in F(\bar{x})$ with $\text{epi}(D_\uparrow F(\bar{x}, \bar{y})) = T_{\text{epi}(F)}(\bar{x}, \bar{y})$. If*

$$D_\uparrow F(\bar{x}, \bar{y})(x - \bar{x}) \cap (-K \setminus \{0\}) = \emptyset, \text{ for all } x \in A,$$

then $(\bar{x}, \bar{y})$ is a minimal solution of (5).

Proof. Since F is K-convex,

$$F(x) - \bar{y} \subset D(F + K)(\bar{x}, \bar{y})(x - \bar{x}), \text{ for all } x \in A.$$

In view of the equality

$$\text{epi}(D_\uparrow F(\bar{x}, \bar{y})) = T_{\text{epi}(F)}(\bar{x}, \bar{y}),$$

we have

$$F(x) - \bar{y} \subset D_\uparrow F(\bar{x}, \bar{y})(x - \bar{x}) + K, \text{ for all } x \in A.$$

If

$$D_\uparrow F(\bar{x}, \bar{y})(x - \bar{x}) \cap (-K \setminus \{0\}) = \emptyset, \text{ for all } x \in A,$$

then

$$[D_\uparrow F(\bar{x}, \bar{y})(x - \bar{x}) + K] \cap (-K \setminus \{0\}) = \emptyset, \text{ for all } x \in A.$$

Hence

$$[F(x) - \bar{y}] \cap (-K \setminus \{0\}) = \emptyset, \text{ for all } x \in A.$$

This means that $(\bar{x}, \bar{y})$ is a minimal solution of (5). ■

4. Sensitivity analysis

In this section, we consider a family of parametrized vector optimization problems. Let Φ be a set-valued map from a normed space W to Y, where W is the parameter space. The set-valued map $P: W \to Y$ defined by

$$P(w) = \mathrm{Min}\Phi(w), \text{ for every } w \in W$$

is called the perturbation map.

Sensitivity analysis concerns differentiability properties of the perturbation map $P(\cdot)$. For vector optimization problems, contingent derivatives of P has been investigated in eg. Tanino (1988a, b), Sawaragi, Nakayama and Tanino (1985) and Shi (1993).

Below we present some sensitivity results by using the notion of contingent epiderivative introduced in Section 2.

DEFINITION 9 *We say that Φ has domination property near $\bar{w}$ if there exists a neighborhood U of $\bar{w}$ such that*

$$\Phi(w) \subset P(w) + K, \textit{ for all } w \in U.$$

THEOREM 3 *Suppose that K has a compact base and Φ has domination property near $\bar{w}$. Let $\bar{y} \in P(\bar{w})$. Then*

$$D_{\uparrow}P(\bar{w}, \bar{y})(u) \subset \mathrm{Min}D\Phi(\bar{w}, \bar{y})(u), \textit{ for all } u \in U, \tag{6}$$

and the converse inclusion holds if, in addition,

$$\mathrm{epi}(D_{\uparrow}\Phi(\bar{w}, \bar{y})) = T_{\mathrm{epi}(\Phi)}(\bar{w}, \bar{y}).$$

Proof. Since $P(w) \subset \Phi(w)$ and Φ has the domination property near $\bar{w}$, there exists a neighborhood U of $\bar{w}$ such that

$$\Phi(w) + K = P(w) + K, \text{ for all } w \in U.$$

Hence

$$D(\Phi + K)(\bar{w}, \bar{y})(u) = D(P + K)(\bar{w}, \bar{y})(u), \text{ for all } u \in U.$$

Thus

$$D_{\uparrow}P(\bar{w}, \bar{y})(u) = D_{\uparrow}\Phi(\bar{w}, \bar{y})(u). \tag{7}$$

By Proposition 1, we get (6). The converse inclusion follows from Proposition 2 (iv). ■

As a consequence of Theorem 3 and Lemma 2, we obtain the following result.

COROLLARY 1 *Let $K \subset R^n$ be a closed convex pointed cone, let $\Phi: W \to R^n$ be a K-convex set-valued map with domination property near $\bar{w}$, and let $\bar{y} \in P(\bar{w})$. If $D_{\uparrow}\Phi(\bar{w}, \bar{y})(0) \neq \emptyset$, then*

$$D_{\uparrow}P(\bar{w}, \bar{y})(u) = \mathrm{Min}\, D\Phi(\bar{w}, \bar{y})(u), \textit{ for all } u \in W$$

COROLLARY **2** *Let $K \subset R^n$ be a closed convex pointed cone, let $\Phi: R^m \to R^n$ be a K-convex set-valued map such that $\Phi(w) + K$ is closed (or $\Phi(w)$ is closed and convex) for w near $\bar{w} \in \operatorname{int}(\operatorname{dom}(\Phi))$, and let $\bar{y}$ be a Benson proper minimal point of $\Phi(\bar{w})$. Then*

$$D_{\uparrow}P(\bar{w},\bar{y})(u) = \operatorname{Min}D\Phi(\bar{w},\bar{y})(u), \ \textit{for all } u \in R^m.$$

Proof. By the assumptions and by Lemmas 4.3, 4.4 of Tanino (1988b), $\Phi(w)$ has domination property for w near $\bar{w}$. By Proposition 3 and formula (7), we get the conclusion. ■

By Proposition 4, 5 we obtain the following results.

COROLLARY **3** *Let $K \subset Y$ have a compact base and let $\Phi: W \to Y$ be upper locally Lipschitz at $\bar{w} \in \operatorname{dom}(\Phi)$ and have domination property near $\bar{w}$. If $\bar{y}$ is a Benson proper minimal point of $\Phi(\bar{w})$, then*

$$D_{\uparrow}P(\bar{w},\bar{y})(u) = \operatorname{Min}D\Phi(\bar{w},\bar{y})(u), \ \textit{for all } u \in W.$$

COROLLARY **4** *Let $\Phi: W \to Y$ have domination property near $\bar{w}$ and let $\bar{y} \in P(\bar{w})$. If Φ is directionally compact at $(\bar{w},\bar{y})$ in the direction u, then*

$$D_{\uparrow}P(\bar{w},\bar{y})(u) = \operatorname{Min}D\Phi(\bar{w},\bar{y})(u).$$

Note. After we had finished this paper, we got a copy of manuscript of Chen and Jahn (1998) where the authors give the definition of generalized contingent epiderivative which coincides with the definition of contingent epiderivative we propose. We decided to submit our paper in its present form because our presentation differs from that of Chen and Jahn (1998) in many aspects. In our paper some results are stronger (eg. the statement below Definition 4 is stronger than Theorem 4 of Chen and Jahn, 1998) and some additional properties of the contingent epiderivative are proved. As original and new applications we give necessary optimality conditions for Benson's proper minimality, sufficient conditions for minimality, and we study sensitivity of parametrized vector optimization problems.

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