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Time–parametric control: Uniform convergence of the optimal value functions of discretized problems

by

Martin Gugat

University of Trier, Department of Mathematics,
D–54286 Trier, Germany,
e-mail: gugat@uni-trier.de

Abstract: The problem of time–optimal control of linear hyperbolic systems is equivalent to the computation of the root of the optimal value function of a time–parametric program, whose feasible set is described by a countable system of moment equations.

To compute this root, discretized problems with a finite number of equality constraints can be used. In this paper, we show that on a certain time–interval, the optimal value functions of the discretized problems converge uniformly to the optimal value function of the original problem.

We also give sufficient conditions for Lipschitz and Hölder continuity of the optimal value function of the original problem.

Keywords: Time–minimal control, moment problems, parametric optimization, optimal value function, discretization, continuity, uniform convergence, Hölder condition, Lipschitz condition

1. Introduction

Consider the problem of damping of vibrations of a one–dimensional medium, where the elastic behaviour of the medium is modelled by a hyperbolic partial differential equation. In Krabs (1982), it is shown that the set of controls steering the medium from a given initial position to a desired terminal state can be described as the solution set of a certain moment problem. Originally, this approach is due to Russel (1967).

In the problem of time–minimal control, an inequality constraint for the control function is added. Often an upper bound for the L^2–norm of the control is introduced. In this paper, we consider the more general case of an upper bound for the L^2–norm of the image of the control under an affine linear operator. This type of constraint is motivated by a control problem for a rotating Euler–Bernoulli beam considered in Krabs (1993), (1996), where an upper bound for the L^2–norm of the torque corresponding to the angular acceleration is introduced as an inequality constraint. The torque is given by a Volterra–operator applied to the angular acceleration which is the control function.

In the minimal controlling time, this inequality constraint is active (Krabs, 1981). In Rolewicz (1987), it is shown how problems of time–minimal control can be reduced to problems of norm–minimal control. To compute the minimal controlling time, a time–parametric control problem can be used. For a fixed time–parameter, the function defining the inequality constraint is taken as the objective function, that is minimized subject to the countable system of moment equations.

In this way, a convex time–parametric auxiliary problem is defined. The minimal time, where the optimal value function of this time–parametric problem attains the value zero is the minimal controlling time. For a fixed parameter, the problem has a countable number of equality constraints that are given by the moment problem.

For numerical computations, a discretized auxiliary problem has to be used where the countable system of equality constraints is replaced by the first N equality constraints i.e. the first N moment equations. For each fixed time–parameter, the problems are uniquely solvable. In this way a sequence of optimal value functions is defined.

In the present paper, we show the following properties of the optimal value functions:

For each fixed N, the corresponding optimal value function is continuous. The optimal value function corresponding to the original problem is continuous. Our main result is that on a given time–interval, the sequence of optimal value functions corresponding to the discretized problems converges *uniformly* to the optimal value function of the original problem.

This result is important for the stability of the numerical approach via the moment equations. It guarantees that for a given accuracy, a discretization level exists that allows to approximate the optimal value function of the original problem with that accuracy, independently of the parameter. If the sequence did not converge uniformly, this would mean that for a certain accuracy, for all discretization levels we could find a point in the time–interval, where this level would not be sufficient. Hence the uniform convergence is essential to guarantee that the approach be useful for numerical computations.

In this paper, we investigate the problem from the point of view of parametric optimization. The known sensitivity results from parametric optimization (see e.g. Lempio and Maurer, 1980; Gugat, 1994; Bonnans and Shapiro, 1998, and the references therein) cannot be applied since for our problem, not only one fixed space containing the control functions but for each controlling–time a different space occurs.

Our parametric auxiliary problem is different from the standard minimum norm problem since we allow for a more general objective function. A transformation of our objective function to the norm as the standard objective function yields a problem that differs form the standard minimum norm problem because the right hand sides of the moment equations depend on the controlling time. Also the functions that appear in the scalar products depend in a nontrivial way on the controlling time. This means that neither the results nor the corresponding methods of proof that are given in Krabs (1992) for the standard minimum norm problem are applicable. For example, for our problem the optimal value function need not be decreasing.

Our main assumption is a chain of inequalities for the functions defining the moment problem. For the problem of time–optimal control of an Euler–Bernoulli beam, we have a trigonometric moment problem (see Krabs, 1982). For these problems, the validity of our assumptions follows from a result of Ingham (see Ingham, 1936).

For the standard–minimum norm problem, we examine the regularity of the optimal value functions. We give conditions in terms of the regularity of the optimal solutions that guarantee the validity of Lipschitz and Hölder conditions with exponent 1/2 for the optimal value function. We give a similar result for the stability of the optimal solutions.

2. Notation and assumptions

Let l^2 denote the space of square summable sequences of real numbers.

For $c \in l^2$, let $\|c\|_{l^2} = \left(\sum_{i=1}^{\infty} c_i^2\right)^{1/2}$.

Let $\overline{T} > 0$ be given. For all T_1, $T_2 \in [0, \overline{T}]$, $T_1 \neq T_2$ let

$$Z(T_1, T_2) = Z(T_2, T_1) = L^2([\min\{T_1, T_2\}, \max\{T_1, T_2\}]),$$

the space of real–valued square integrable functions on the interval $[\min\{T_1, T_2\}, \max\{T_1, T_2\}]$. The usual scalar product in $Z(T_1, T_2)$ is denoted by $\langle \cdot, \cdot \rangle_{(T_1, T_2)}$ and the corresponding norm by $\| \cdot \|_{(T_1, T_2)}$. Let $\langle \cdot, \cdot \rangle_{(T_1, T_1)} = \| \cdot \|_{(T_1, T_1)} = 0$.

For $u \in Z(0, \overline{T})$, instead of $\|u|_{[\min\{T_1, T_2\}, \max\{T_1, T_2\}]}\|_{(T_1, T_2)}$ we write $\|u\|_{(T_1, T_2)}$; analogously, for u, $v \in Z(0, \overline{T})$ we use the notation $\langle u, v \rangle_{(T_1, T_2)}$.

For our analysis it is essential that we do not work in only one space, but use a whole (time–)parametric family of spaces.

For all $T \in (0, \overline{T}]$, let $S_T : Z(0, T) \longrightarrow Z(0, T)$ be a continuous linear map that is bijective and for which the following equation holds for all $u \in Z(0, \overline{T})$:

$$S_T(u|_{[0,T]}) = (S_{\overline{T}} u)|_{[0,T]}. \tag{1}$$

As an example for S_T consider the Volterra operator with a constant $\kappa > 0$ and kernel $K \in C(0, \overline{T})$ used in Krabs (1996):

$$(S_T u)(t) = \kappa u(t) - \int_0^t K(t - s) u(s)\, ds. \tag{2}$$

The adjoint operators of S_T, S_T^{-1} are denoted by S_T^*, $(S_T^{-1})^*$ respectively. In example (2) the adjoint operator is

$$(S_T^* u)(t) = \kappa u(t) - \int_t^T K(s-t)u(s)\,ds.$$

LEMMA 2.1 *For all $T \in [0, \overline{T}]$ and $y \in Z(0, \overline{T})$ the following equation is valid:*

$$S_T^{-1}(y|_{[0,T]}) = (S_{\overline{T}}^{-1} y)|_{[0,T]}.$$

Moreover $\|S_T^*\| = \|S_T\| \le \|S_{\overline{T}}\|$ *and* $\|(S_T^{-1})^*\| = \|S_T^{-1}\| \le \|S_{\overline{T}}^{-1}\|$.

Proof. Let $y \in Z(0, \overline{T})$, $T \in [0, \overline{T}]$. Let $u_1 = (S_{\overline{T}}^{-1} y)|_{[0,T]}$ and $u_2 = S_T^{-1}(y|_{[0,T]})$.

Then $S_T u_2 = y|_{[0,T]}$ and (1) implies

$$S_T u_1 = S_T((S_{\overline{T}}^{-1} y)|_{[0,T]}) = (S_{\overline{T}}(S_{\overline{T}}^{-1} y))|_{[0,T]} = y|_{[0,T]} = S_T u_2.$$

Hence $u_1 = u_2$.

For $u \in Z(0,T)$, define $\tilde{u} \in Z(0, \overline{T})$ by $\tilde{u}|_{[0,T]} := u$, $\tilde{u}|_{(T,\overline{T}]} := 0$. Then $\|u\|_{(0,T)} = \|\tilde{u}\|_{[0,\overline{T})}$. Hence (1) implies $\|S_T u\|_{(0,T)} = \|S_{\overline{T}} \tilde{u}\|_{(0,T)} \le \|S_{\overline{T}} \tilde{u}\|_{(0,\overline{T})} \le \|S_{\overline{T}}\| \|\tilde{u}\|_{(0,\overline{T})} = \|S_{\overline{T}}\|\, \|u\|_{(0,T)}$. Thus $\|S_T\| \le \|S_{\overline{T}}\|$. The inequality $\|S_T^{-1}\| \le \|S_{\overline{T}}^{-1}\|$ follows analogously.

The equation $\|S_T\| = \|S_T^*\|$ is always valid (see Pedersen, 1988, p. 90). □

We assume that for all $u \in Z(0, \overline{T})$ and $T_j \in (0, \overline{T}]$ ($j \in I\!N$) with $\lim_{j \longrightarrow \infty} T_j = T$ the following statement holds:

$$\lim_{j \longrightarrow \infty} \left\| \left(S_{T_j}^* (u(\cdot \frac{\overline{T}}{T_j})) \right) (\cdot \frac{T_j}{\overline{T}}) - \left(S_T^* (u(\cdot \frac{\overline{T}}{T})) \right) (\cdot \frac{T}{\overline{T}}) \right\|_{(0,\overline{T})} = 0. \tag{3}$$

For the example of the Volterra operator in (2) for all $u \in Z(0, \overline{T})$ and $t \in [0, T]$ the equation

$$\left(S_T^* (u(\cdot\, \overline{T}/T)) \right)(t) = \kappa u(t\overline{T}/T) - \int_t^T K(s-t) u(s\overline{T}/T)\, ds$$

$$= \kappa u(t\overline{T}/T) - \frac{T}{\overline{T}} \int_{t\overline{T}/T}^{\overline{T}} K(xT/\overline{T} - t) u(x)\, dx$$

is valid, hence we conclude that for all $y \in [0, \overline{T}]$ we have

$$\left(S_T^* (u(\cdot\, \overline{T}/T)) \right)(yT/\overline{T}) = \kappa u(y) - \frac{T}{\overline{T}} \int_y^{\overline{T}} K((x-y)\frac{T}{\overline{T}}) u(x)\, dx,$$

so it is easy to verify that (3) is valid.

Let $(z_j)_{j\in I\!N} \in (Z(0,\overline{T}))^{I\!N}$ be a sequence of functions, $b \in Z(0,\overline{T})$ and $c \in l^2$. For $\alpha \in I\!R \cup \{\infty\}$ and $e \in l^2$ we define the set

$$U(T,\alpha,e) = \{u \in Z(0,T) :$$

$$\|S_T u - b\|_{(0,T)}^2 \le \alpha^2 \text{ and } \langle u, z_j\rangle_{(0,T)} = e_j \text{ for all } j \in I\!N\}.$$

We make the following assumptions:

(A0) A number $\beta \in I\!R$ is given such that the set $U(\overline{T},\beta,c)$ is nonempty.

(A1) There exist constants $\underline{T}$, M, $P > 0$ such that for all $N \in I\!N$, $(a_1,...,a_N) \in I\!R^N$ we have

$$\begin{aligned}(1/M)(\sum_{i=1}^N a_i^2)^{1/2} &\le \|\sum_{i=1}^N a_i z_i\|_{(0,\underline{T})}\\ &\le \|\sum_{i=1}^N a_i z_i\|_{(0,\overline{T})}\\ &\le P(\sum_{i=1}^N a_i^2)^{1/2}.\end{aligned}$$

For trigonometric moment problems, the validity of the inequality in (A1) can sometimes be verified with the help of a result of Ingham (see Ingham, 1936). Usually (e.g. in Vasin and Ageev, 1995, Lemma 4.1, p.120, and Krabs, 1982, II.2.11) in the theory of moment problems a similar inequality for one fixed space is considered; in contrast to the present paper the parametric aspect is not taken into account. Condition (A1) is equivalent to the statement that for all $T \in [\underline{T},\overline{T}]$, the functions z_i form a Riesz–basis of the closure of their linear span. Condition (A1) is also equivalent to the statement that for all $T \in [\underline{T},\overline{T}]$, the Gram–matrix

$$\left(\langle z_i, z_j\rangle_{(0,T)}\right)_{i,j=1}$$

generates a linear bounded invertible operator on l^2. Riesz–bases can also be characterized in terms of biorthogonal sequences (see Young (1980), Theorem 9, p. 32).

Using Lemma 2.1, it is easy to prove that Assumption (A1) implies the following Lemma.

LEMMA 2.2 *Let $\hat M = M\,\|S_{\overline{T}}\|$, $\hat P = P\|S_{\underline{T}}^{-1}\|$. For $T \in [\underline{T},\overline{T}]$, $j \in I\!N$ define $H_j(T) = (S_T^*)^{-1} z_j$. Then for all $N \in I\!N$, $(a_1,...,a_N) \in I\!R^N$, $T \in [\underline{T},\overline{T}]$ the following inequality holds:*

$$(1/\hat M)(\sum_{i=1}^N a_i^2)^{1/2} \le \|\sum_{i=1}^N a_i H_i(T)\|_{(0,T)} \le \hat P(\sum_{i=1}^N a_i^2)^{1/2}.$$

3. The problem

We are interested in the minimal controlling time

$$T^* = \inf\{T \in [\underline{T}, \overline{T}] : \; U(T, \beta, c) \neq \emptyset\}.$$

The number T^* is the infimum of the set of points $T \in [\underline{T}, \overline{T}]$ for which there exists a control $u \in Z(0,T)$ that satisfies the moment equations, i.e. such that $\langle u, z_j \rangle_{(0,T)} = c_j$ for all $j \in I\!N$ and for which $\|S_T u - b\|^2_{(0,T)} \le \beta^2$.

The lower bound $\underline{T}$ is introduced since only for $T \ge \underline{T}$, (A1) implies that $U(T, \infty, c)$ is nonempty (see Guerre–Delabriere, 1992, Lemma I.6.2, where a result for the more general case of reflexive spaces is given).

For $T \in [\underline{T}, \overline{T}]$ define the parametric optimization problem $P_\infty(T)$:

$$\min \|S_T u - b\|^2_{(0,T)} - \beta^2 \;\text{ s.t. }\; \langle u, z_j \rangle_{(0,T)} = c_j \;\text{ for all } j \in I\!N.$$

Let $\omega(T)$ denote the value of $P_\infty(T)$.

Note that in the theory of moment problems (e.g. in Vasin and Ageev, 1995), usually instead of $\|S_T u - b\|^2_{(0,T)}$ the objective function $\|u\|^2_{(0,T)}$ is considered that yields so called normal solutions. For the special case of the control of a rotating beam with S_T as in (2), Krabs considers an objective function of the form $\|S_T \cdot - b\|^2_{(0,T)}$ (see Krabs, 1993), that is equal to the L^2–norm of the momentum at the axis of the beam.

In problem $P_\infty(T)$, the controlling time is fixed and the constraint function that is used to define the problem of time–minimal control is taken as the objective function.

4. The discretized problem

Since $P_\infty(T)$ has an infinite number of equality constraints, for numerical purposes it is necessary to examine a discretized problem $P_N(T)$, where only the first N equality constraints of problem $P_\infty(T)$ are considered.

For $T \in [\underline{T}, \overline{T}]$, $N \in I\!N$ define the parametric optimization problem $P_N(T)$:

$$\min \|S_T u - b\|^2_{(0,T)} - \beta^2 \;\text{ s.t.}$$

$$\langle u, z_j \rangle_{(0,T)} = c_j \;\text{ for all } j \in \{1, ..., N\}.$$

Let $\omega_N(T)$ denote the value of $P_N(T)$. Then for all $T \in [\underline{T}, \overline{T}]$, the inequality $\omega_{N+1}(T) \ge \omega_N(T)$ is valid.

In the following Lemma, the solution of problem $P_N(T)$ is characterized.

LEMMA 4.1 *Let $T \in [\underline{T}, \overline{T}]$, $N \in I\!N$. For $j \in \{1, ..., N\}$, define $H_j(T) = (S_T^*)^{-1} z_j$. Define $\eta_N(T) = (\eta_i^N(T))_{i=1}^N \in I\!R^N$ as the solution of the linear system*

$$\left(\langle H_i(T), H_j(T) \rangle_{(0,T)} \right)_{i,j=1}^N \eta_N(T) = \left(c_i - \langle b, H_i(T) \rangle_{(0,T)} \right)_{i=1}^N .$$

Then $u_N(T) = S_T^{-1}(\sum_{i=1}^N \eta_i^N(T)H_i(T) + b)$ *is the unique solution of problem* $P_N(T)$.

For the proof of Lemma 4.1, we need the following trivial statement.

STATEMENT 4.1 *Let* $S, T \in [0, \overline{T}]$. *Let* v, $w \in Z(S,T)$ *and* $\langle v - w, w\rangle_{(S,T)} = 0$. *Then* $\|w\|_{(S,T)} \leq \|v\|_{(S,T)}$.

Proof of Lemma 4.1. Define the symmetric matrix

$$G_N(T) = \left(\langle H_i(T), H_j(T)\rangle_{(0,T)}\right)_{i,j=1}^N.$$

Assumption (A1) implies that $G_N(T)$ is positive definite.

Define $\eta_N(T)$ as the solution of the linear system given in Lemma 4.1 and $v_N(T)$ by the equation

$$v_N(T) = \sum_{j=1}^N \eta_j^N(T)H_j(T).$$

Then, for $i \in \{1, ..., N\}$ the following equation holds:

$$\begin{aligned}\langle H_i(T), v_N(T)\rangle_{(0,T)} &= \sum_{j=1}^N \langle H_i(T), H_j(T)\rangle_{(0,T)}\eta_j^N(T)\\ &= c_i - \langle b, H_i(T)\rangle_{(0,T)}.\end{aligned}$$

Define the set $B_N(T) = \{v \in Z(0,T)$:

$$\langle v, H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)},\ i \in \{1, ..., N\}\}.$$

Since $v_N(T) \in \text{span}\{H_1(T), ..., H_N(T)\}$, for all $v \in B_N(T)$ we have

$$\langle v - v_N(T), v_N(T)\rangle_{(0,T)} = 0.$$

Thus, Statement 4.1 implies that $v_N(T)$ is the element of $B_N(T)$ with minimal norm.

For a point $u \in Z(0,T)$ the statement $\langle u, z_j\rangle_{(0,T)} = c_j$ $(j = 1, ..., N)$ holds if and only if $S_T u - b \in B_N(T)$. Hence $u_N(T) = S_T^{-1}(v_N(T) + b)$ is the solution of $P_N(T)$. The fact that the solution of $P_N(T)$ is uniquely determined follows from the strict convexity of $\|S_T \cdot -b\|_{(0,T)}$. □

5. Solvability of problem $P_\infty(T)$

To analyse the solvability of problem $P_\infty(T)$, we need an additional assumption. Assume that in the sequel, the following statement (A2) is valid:

(A2) For all $N \in I\!N$, $S \in [0, \overline{T}]$, $T \in [\underline{T}, \overline{T}]$, $S < T$ the functions $z_1|_{[S,T]}, ..., z_N|_{[S,T]}$ are linearly independent.

LEMMA 5.1 *For all $S \in [0, \overline{T}]$, $T \in [\underline{T}, \overline{T}]$, $S \leq T$, $u \in Z(S,T)$ the following inequality holds:*

$$\sum_{i=1}^{\infty} \left(\langle u, H_i(T) \rangle_{(S,T)} \right)^2 \leq \hat{P}^2 \|u\|^2_{(S,T)}.$$

Proof. If $S = T$, the assertion is trivial.

Assume now that $S < T$. For $N \in \mathbb{N}$ we define the symmetric matrix

$$G_N(S,T) = \left(\langle H_i(T), H_j(T) \rangle_{(S,T)} \right)_{i,j=1}^{N}.$$

Due to Assumption (A2), the functions $H_1(T)|_{[S,T]}, ..., H_N(T)|_{[S,T]}$ are linearly independent. Hence the matrix $G_N(S,T)$ is positive definite.

Let $u \in Z(S,T)$. Define

$$\begin{aligned} U_N &= \left(\langle u, H_i(T) \rangle_{(S,T)} \right)_{i=1}^{N}, \\ \alpha_N &= \left(G_N(S,T) \right)^{-1} U_N \text{ and} \\ u_N &= \sum_{i=1}^{N} \alpha_i^N H_i(T). \end{aligned}$$

Then we have $\langle u_N - u, u_N \rangle_{(S,T)} = 0$. Thus, Statement 4.1 implies

$$\|u_N\|_{(S,T)} \leq \|u\|_{(S,T)}.$$

Lemma 2.2 implies that for all $N \in \mathbb{N}$, $(a_1, ..., a_N) \in \mathbb{R}^N$, the following inequality holds:

$$\begin{aligned} \|\sum_{i=1}^{N} a_i H_i(T)\|_{(S,T)} &\leq \|\sum_{i=1}^{N} a_i H_i(T)\|_{(0,T)} \\ &\leq \hat{P} \left(\sum_{i=1}^{N} a_i^2 \right)^{1/2}. \end{aligned}$$

This implies that for all $y \in \mathbb{R}^N$, we have

$$y^T y \leq \hat{P}^2 y^T \left(G_N(S,T) \right)^{-1} y.$$

Thus the following statement is valid:

$$\begin{aligned} \sum_{i=1}^{N} \left(\langle u, H_i(T) \rangle_{(S,T)} \right)^2 &= U_N^T U_N \\ &\leq \hat{P}^2 U_N^T \left(G_N(S,T) \right)^{-1} U_N \\ &= \hat{P}^2 \alpha_N^T G_N(S,T) \alpha_N \\ &= \hat{P}^2 \|u_N\|^2_{(S,T)} \\ &\leq \hat{P}^2 \|u\|^2_{(S,T)}. \end{aligned}$$

Since this inequality holds for all $N \in \mathbb{N}$, the assertion follows. □

LEMMA 5.2 *For all $T \in [\underline{T}, \overline{T}]$ there exists an element $v_*(T)$ of the closure of span$\{H_i(T) : i \in \mathbb{N}\}$ such that for all $i \in \mathbb{N}$ the equality*

$$\langle v_*(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)} \tag{4}$$

is valid. Moreover, $u_*(T) = S_T^{-1}(v_*(T) + b)$ is the unique solution of problem $P_\infty(T)$.

Proof. Let $T \in [\underline{T}, \overline{T}]$, $N \in \mathbb{N}$ be given and $G_N(T)$, $v_N(T)$ as in the proof of Lemma 4.1. Define

$$V_N = (c_i - \langle b, H_i(T)\rangle_{(0,T)})_{i=1}^N \in \mathbb{R}^N.$$

As in Lemma 4.1, let $\eta_N(T)$ be defined as

$$\eta_N(T) = (G_N(T))^{-1} V_N.$$

On account of Lemma 4.1 and Lemma 2.2 we have the inequality

$$\begin{aligned} \|v_N(T)\|_{(0,T)}^2 &= \eta_N(T)^T G_N(T) \eta_N(T) \\ &= V_N^T (G_N(T))^{-1} V_N \\ &\leq \hat{M}^2 V_N^T V_N \\ &\leq \hat{M}^2 \gamma(T), \end{aligned}$$

$$\text{with } \gamma(T) = \sum_{i=1}^{\infty} (c_i - \langle b, H_i(T)\rangle_{(0,T)})^2.$$

Due to Lemma 5.1, $\gamma(T)$ is finite. Hence the sequence $(v_N(T))_{N\in\mathbb{N}}$ is bounded, and thus contains a weakly convergent subsequence. Let $v_*(T)$ denote a weak cluster point of $(v_N(T))_{N\in\mathbb{N}}$. For all i, $N \in \mathbb{N}$ with $i \leq N$ the following equation holds:

$$\langle v_N(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}.$$

Due to the definition of weak convergence, this implies for all $i \in \mathbb{N}$ the equation

$$\langle v_*(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}.$$

For all $N \in \mathbb{N}$, the function $v_N(T)$ is in span$\{H_1(T), ..., H_N(T)\}$ (see the proof of Lemma 4.1). Hence $v_*(T)$ is in the closure of span$\{H_i(T), i \in \mathbb{N}\}$.

Define the set $B(T)$

$$= \{v \in Z(0,T) : \langle v, H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)},\ i \in \mathbb{N}\}.$$

Since $v_*(T)$ is in the closure of span$\{H_i(T), i \in \mathbb{N}\}$, for all $w \in B(T)$ we have $\langle w - v_*(T), v_*(T)\rangle_{(0,T)} = 0$. Thus Statement 4.1 implies that $v_*(T)$ is the element of $B(T)$ with minimal norm.

For a point $u \in Z(0,T)$ the equation $\langle u, z_j\rangle_{(0,T)} = c_j$ ($j \in \mathbb{N}$) holds if and only if $S_T u - b \in B(T)$. Hence $u_*(T) = S_T^{-1}(v + b)$ is the solution of $P_\infty(T)$. The uniqueness follows from the strict convexity of $\|S_T \cdot - b\|_{(0,T)}$. □

6. Continuity of the value function for the original problem

In this section, we demonstrate the continuity of the optimal value function ω.

First we prove that the solutions of $P_\infty(T)$ for $T \in [\underline{T}, \overline{T}]$ are uniformly bounded. Then we use this fact to show that ω is lower semicontinuous.

We introduce a dual problem for $P_\infty(T)$ and show that the corresponding dual solutions are also uniformly bounded on $[\underline{T}, \overline{T}]$. We use this fact to show that ω is upper semicontinuous.

LEMMA 6.1 (UNIFORM BOUNDEDNESS OF PRIMAL SOLUTIONS) *The solutions of $P_\infty(T)$ are uniformly bounded on $[\underline{T}, \overline{T}]$, that is there exists $r \in I\!R$, such that for all $T \in [\underline{T}, \overline{T}]$*

$$\|u_*(T)\|_{(0,T)} \leq r.$$

Proof. Let $T \in [\underline{T}, \overline{T}]$. Let $\gamma(T)$ and $v_*(T)$ be defined as in the proof of Lemma 5.2. Then due to Lemma 5.1 we have

$$\begin{aligned}
\sqrt{\gamma(T)} &= \left(\sum_{i=1}^{\infty}(c_i - \langle b, H_i(T)\rangle_{(0,T)})^2\right)^{1/2} \\
&\leq \|c\|_{l^2} + \left(\sum_{i=1}^{\infty}\langle b, H_i(T)\rangle^2_{(0,T)}\right)^{1/2} \\
&\leq \|c\|_{l^2} + \hat{P}\|b\|_{(0,\overline{T})} =: R.
\end{aligned}$$

The fact that $v_*(T)$ is a weak cluster point of the sequence $(v_N(T))_{N\in I\!N}$ implies

$$\|v_*(T)\|^2_{(0,T)} \leq \hat{M}^2\gamma(T) \leq \hat{M}^2 R^2.$$

According to Lemma 5.2, we have $u_*(T) = S_T^{-1}(v_*(T) + b)$. By Lemma 2.1, this yields the inequality

$$\begin{aligned}
\|u_*(T)\|_{(0,T)} &\leq \|S_T^{-1}\| \left(\|v_*(T)\|_{(0,T)} + \|b\|_{(0,T)}\right) \\
&\leq \|S_{\overline{T}}^{-1}\| \left(\hat{M}R + \|b\|_{(0,\overline{T})}\right) =: r,
\end{aligned}$$

and the assertion follows. □

LEMMA 6.2 (LOWER SEMICONTINUITY) *The function ω is lower semicontinuous on $[\underline{T}, \overline{T}]$.*

Proof. Let $T \in [\underline{T}, \overline{T}]$ and a sequence $(T_l)_{l\in I\!N} \in [\underline{T}, \overline{T}]^{I\!N}$ converging to T be given. For $k \in I\!N$, let $u_k = u_*(T_k)$. Due to Lemma 6.1 there is $r \in I\!R$ such that for all k we have: $\|u_k\|_{(0,T_k)} \leq r$.

Define $\tilde{u}_k(\cdot) = u_k(\cdot\, T_k/\overline{T}) \in Z(0,\overline{T})$. Then

$$\|\tilde{u}_k\|_{(0,\overline{T})} = (\overline{T}/T_k)^{1/2}\|u_k\|_{(0,T_k)} \le (\overline{T}/T_k)^{1/2} r.$$

Hence the sequence $(\tilde{u}_k)_{k\in I\!N}$ is bounded. Thus there exists a subsequence that converges weakly to a point $\tilde{u}_* \in Z(0,\overline{T})$. Assume without restriction that the whole sequence $(\tilde{u}_k)_{k\in I\!N}$ is weakly convergent.

The definition of $\tilde{u}_j$ implies $u_j(\cdot) = \tilde{u}_j(\cdot\overline{T}/T_j)$. Define $w_*(\cdot) = \tilde{u}_*(\cdot\overline{T}/T)$. Let $\tilde{z}_l^j = z_l(\cdot T_j/\overline{T})$. For all $l \in I\!N$ we have $c_l = \langle u_j, z_l\rangle_{(0,T_j)} = (T_j/\overline{T})\langle \tilde{u}_j, \tilde{z}_l^j\rangle_{(0,\overline{T})}$.

Let $\tilde{z}_l^*(\cdot) = z_l(\cdot T/\overline{T})$. Then

$$\lim_{j\to\infty} \|\tilde{z}_l^j - \tilde{z}_l^*\|_{(0,\overline{T})} = 0.$$

Therefore for all $l \in I\!N$ the following equation holds:

$$\langle \tilde{u}_*, \tilde{z}_l^*\rangle_{(0,\overline{T})} = \lim_{j\longrightarrow\infty} \langle \tilde{u}_j, \tilde{z}_l^*\rangle_{(0,\overline{T})}$$

$$= \lim_{j\longrightarrow\infty} \langle \tilde{u}_j, \tilde{z}_l^j\rangle_{(0,\overline{T})} = \lim_{j\longrightarrow\infty} (\overline{T}/T_j)c_l = (\overline{T}/T)c_l.$$

Hence we get

$$\langle w_*, z_l\rangle_{(0,T)} = (T/\overline{T})\langle \tilde{u}_*, \tilde{z}_l^*\rangle_{(0,\overline{T})} = (T/\overline{T})(\overline{T}/T)c_l = c_l.$$

Thus we have $w_* \in U(T,\infty,c)$ and so $\omega(T) \le \|S_T w_* - b\|^2_{(0,T)} - \beta^2$.

The function $u \mapsto \|u\|_{(0,T)}$, $Z(0,T) \to I\!R$ is sequentially weakly lower semicontinuous (as the supremum of sequentially weakly continuous functions, see Pedersen, 1988, Prop. 1.5.12).

Let $\tilde{b}(\cdot) = b(\cdot T/\overline{T})$. Let $v_j = S_{T_j}u_j - b \in Z(0,T_j)$ and $\tilde{v}_j(\cdot) = v_j(\cdot\, T_j/\overline{T}) \in Z(0,\overline{T})$. Let $v_* = S_T w_* - b$ and $\tilde{v_*}(\cdot) = v_*(\cdot\, T/\overline{T}) \in Z(0,\overline{T})$. For $f \in Z(0,\overline{T})$, let $\hat{f}_j(\cdot) = f(\cdot\,\overline{T}/T_j) \in Z(0,T_j)$ and $\hat{f}(\cdot) = f(\cdot\,\overline{T}/T) \in Z(0,T)$. Then

$$\langle f, \tilde{v}_j\rangle_{(0,\overline{T})} = \langle f, (S_{T_j}u_j)(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})} - \langle f, b(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})}.$$

Our definitions imply the equation

$$\begin{aligned}
\langle f, (S_{T_j}u_j)(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})} &= \langle \hat{f}_j, (S_{T_j}u_j)\rangle_{(0,T_j)}(\overline{T}/T_j) \\
&= \langle S_{T_j}^*\hat{f}_j, u_j\rangle_{(0,T_j)}(\overline{T}/T_j) \\
&= \langle (S_{T_j}^*\hat{f}_j)(\cdot\, T_j/\overline{T}), \tilde{u}_j\rangle_{(0,\overline{T})}.
\end{aligned}$$

Then, assumption (3) and the weak convergence of the sequence $(\tilde{u}_j)_{j\in I\!N}$ imply

$$\begin{aligned}
\lim_{j\longrightarrow\infty} \langle f, (S_{T_j}u_j)(\cdot\, T_j/\overline{T})\rangle_{(0,\overline{T})} &= \langle (S_T^*\hat{f})(\cdot\, T/\overline{T}), \tilde{u}_*\rangle_{(0,\overline{T})} \\
&= \langle S_T^*\hat{f}, w_*\rangle_{(0,T)}(\overline{T}/T) \\
&= \langle \hat{f}, S_T w_*\rangle_{(0,T)}(\overline{T}/T) \\
&= \langle f, (S_T w_*)(\cdot\, T/\overline{T})\rangle_{(0,\overline{T})}.
\end{aligned}$$

Moreover, since $\|b(\cdot T_j/\overline{T}) - \tilde{b}(\cdot)\|_{(0,\overline{T})} \longrightarrow 0$ $(j \longrightarrow \infty)$ we have

$$\lim_{j \longrightarrow \infty} \langle f, b(\cdot\, T_j/T)\rangle_{(0,\overline{T})} = \langle f, \tilde{b}\rangle_{(0,\overline{T})}.$$

Thus we can conclude that

$$\lim_{j \longrightarrow \infty} \langle f, \tilde{v}_j\rangle_{(0,\overline{T})} = \langle f, (S_T w_* - b)(\cdot\, T/\overline{T})\rangle_{(0,\overline{T})} = \langle f, \tilde{v_*}\rangle_{(0,\overline{T})},$$

so the sequence $(\tilde{v}_j)_{j \in I\!N}$ converges weakly to $\tilde{v_*}$.

So we obtain the statement

$$\begin{aligned}
\omega(T) + \beta^2 &\leq \|v_*\|^2_{(0,T)} \\
&= (T/\overline{T}) \|\tilde{v_*}\|^2_{(0,\overline{T})} \\
&\leq (T/\overline{T}) \liminf_{j \longrightarrow \infty} \|\tilde{v}_j\|^2_{(0,\overline{T})} \\
&= \liminf_{j \longrightarrow \infty} (T_j/\overline{T}) \|\tilde{v}_j\|^2_{(0,\overline{T})} \\
&= \liminf_{j \longrightarrow \infty} \|v_j\|^2_{(0,T_j)} \\
&= \liminf_{j \longrightarrow \infty} \omega(T_j) + \beta^2,
\end{aligned}$$

which implies $\omega(T) \leq \liminf_{k \to \infty} \omega(T_k)$, that is, ω is lower semicontinuous in T. □

To show the upper semicontinuity of ω, we use the coefficients of $v_*(T)$ written as a linear combination of the functions $H_i(T)$.

These coefficients form a sequence in l^2 and can be used to express the optimal value $\omega(T)$.

LEMMA 6.3 *Let $T \in [\underline{T}, \overline{T}]$. Then there exist $(\alpha_i(T))_{i \in I\!N} \in l^2$ such that*

$$\begin{aligned}
v_*(T) &= \sum_{i=1}^{\infty} \alpha_i(T) H_i(T) \text{ and} \\
\omega(T) + \beta^2 &= \sum_{i=1}^{\infty} \alpha_i(T) (c_i - \langle b, H_i(T)\rangle_{(0,T)}).
\end{aligned}$$

Moreover, for all $i \in I\!N$ the following equation is valid:

$$\sum_{j=1}^{\infty} \alpha_j(T) \langle H_i(T), H_j(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}. \tag{5}$$

Proof. Lemma 5.2 implies that the function $v_*(T)$ is contained in the closure of *span*$\{H_i(T), i \in I\!N\}$. Hence there exists a sequence $(\alpha_i(T))_{i \in I\!N}$ such that

$$v_*(T) = \sum_{i=1}^{\infty} \alpha_i(T) H_i(T).$$

Lemma 2.2 implies that the sequence $(\alpha_i(T))_{i\in\mathbb{N}}$ is an element of l^2.

Since $v_*(T) = S_T u_*(T) - b$, we have

$$\begin{aligned}
\omega(T) + \beta^2 &= \|v_*(T)\|^2_{(0,T)} \\
&= \langle \sum_{i=1}^{\infty} \alpha_i(T) H_i(T), v_*(T)\rangle_{(0,T)} \\
&= \sum_{i=1}^{\infty} \alpha_i(T) \langle H_i(T), v_*(T)\rangle_{(0,T)} \\
&= \sum_{i=1}^{\infty} \alpha_i(T)(c_i - \langle b, H_i(T)\rangle_{(0,T)},
\end{aligned}$$

where the last equality follows from equation (4), which also implies equation (5). □

In the next Lemma, we introduce a maximization problem with value $\omega(T)+\beta^2$, i.e. a dual problem for $P_\infty(T)$.

LEMMA 6.4 (DUAL PROBLEM) *For all $T \in [\underline{T}, \overline{T}]$ the following equation holds:*

$$\omega(T) + \beta^2 = \sup_{\alpha\in l^2} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)}$$

$$+2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right).$$

Proof. For $T \in [\underline{T}, \overline{T}]$, $\alpha \in l^2$, define

$$h(T,\alpha) = - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)}$$

$$+2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right).$$

Let $\alpha(T) = (\alpha_i(T))_{i\in\mathbb{N}}$ be as in Lemma 6.3. Then, Lemma 6.3 implies

$$\begin{aligned}
h(T,\alpha(T)) &= -\|v_*(T)\|^2 + 2\sum_{j=1}^{\infty} \alpha_j(T)\left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right) \\
&= -(\omega(T)+\beta^2) + 2(\omega(T)+\beta^2) \\
&= \omega(T)+\beta^2. \qquad (6)
\end{aligned}$$

This implies the inequality

$$\omega(T) + \beta^2 \le \sup_{\alpha\in l^2} h(T,\alpha).$$

For $\alpha \in l^2$, $v \in Z(0,T)$ define $\phi(T,v,\alpha)$

$$= \|v\|^2_{(0,T)} + 2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)} - \langle v, H_j(T)\rangle_{(0,T)}\right) \qquad (7)$$

Lemma 5.1 implies that $\phi(T,v,\alpha)$ is well-defined.

According to Lemma 5.2, we have

$$\|v_*(T)\|^2_{(0,T)} = \omega(T) + \beta^2$$

and thus equation (5) implies that for all $\alpha \in l^2$

$$\phi(T, v_*(T), \alpha) = \|v_*(T)\|^2_{(0,T)} = \omega(T) + \beta^2.$$

For all $\alpha \in l^2$, the map $\phi(T,\cdot,\alpha)$ is coercive and strictly convex, hence the set

$$M_{\min}(T) = \{v \in Z(0,T) : \phi(T,v,\alpha) = \inf_{w \in Z(0,T)} \phi(T,w,\alpha)\}$$

is nonempty and consists of a single element.

Let $\alpha \in l^2$ be fixed and $M_{\min}(T) = \{w_*\}$. Since the map $\phi(T,\cdot,\alpha) : Z(0,T) \longrightarrow I\!R$ is Fréchet-differentiable, we can derive the equation

$$w_* = \sum_{j=1}^{\infty} \alpha_j H_j(T).$$

Thus, the following equation holds:

$$\begin{aligned}
&\phi(T, w_*, \alpha) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i \alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)} \\
&+ 2\sum_{j=1}^{\infty} \alpha_j(T) \left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right) \\
&- 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i \alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)} \\
&= h(T,\alpha).
\end{aligned}$$

Hence for all $\alpha \in l^2$ we have

$$\begin{aligned}
h(T,\alpha) &= \inf_{v \in Z(0,T)} \phi(T,v,\alpha) \qquad (8)\\
&\leq \phi(T, v_*(T), \alpha) \\
&= \omega(T) + \beta^2.
\end{aligned}$$

This implies

$$\sup_{\alpha \in l^2} h(T,\alpha) \leq \omega(T) + \beta^2,$$

and the assertion follows. □

LEMMA 6.5 (UNIQUENESS OF THE DUAL SOLUTIONS) *For all* $T \in [\underline{T}, \overline{T}]$, *the point* $(\alpha_i(T))_{i \in \mathbb{N}} \in l^2$ *as defined in Lemma 6.3 is uniquely determined and the unique solution of the dual problem stated in Lemma 6.4.*

Proof. Let $\alpha(T) = (\alpha_i(T))_{i \in \mathbb{N}}$ be as in Lemma 6.3. Equation (6) implies that $\alpha(T)$ solves the dual problem.

Lemma 2.2 implies that the function $h(T, \cdot) : l^2 \longrightarrow \mathbb{R}$ is strictly concave, hence the dual solution is unique.

Therefore $\alpha(T)$ is uniquely determined. □

Note that for all $T \in [\underline{T}, \overline{T}]$, the dual solution is an element of the space l^2 that is independent of T. This fact is very convenient for our analysis.

LEMMA 6.6 (UNIFORM BOUNDEDNESS OF THE DUAL SOLUTIONS) *Let* $T \in [\underline{T}, \overline{T}]$ *and* $(\alpha_i(T))_{i \in \mathbb{N}}$ *be as in Lemma 6.3. There exists* $r \in \mathbb{R}$, *such that for all* $T \in [\underline{T}, \overline{T}]$

$$\sum_{i=1}^{\infty} (\alpha_i(T))^2 \leq r.$$

Proof. According to Lemma 2.2, for all $T \in [\underline{T}, \overline{T}]$ we have

$$\begin{aligned}\left(\sum_{i=1}^{\infty} \alpha_i(T)^2\right)^{1/2} &\leq \hat{M} \| \sum_{i=1}^{\infty} \alpha_i(T) H_i(T) \|_{(0,T)} \\ &= \hat{M} \| v_*(T) \|_{(0,T)} \\ &\leq \hat{M} R\end{aligned}$$

with R as defined in the proof of Lemma 6.1. The assertion follows with $r = \hat{M}R$. □

LEMMA 6.7 *Let* $u \in Z(0, \overline{T})$. *For* $T \in [\underline{T}, \overline{T}]$, $i \in \mathbb{N}$ *define*

$$d_i(T) = \langle u, H_i(T) \rangle_{(0,T)}.$$

Then for all $T \in [\underline{T}, \overline{T}]$, *the following equation holds:*

$$\lim_{t \longrightarrow T, t \in [\underline{T}, \overline{T}]} \sum_{i=1}^{\infty} (d_i(t) - d_i(T))^2 = 0.$$

Proof. Due to Lemma 5.1, for all $t \in [\underline{T}, \overline{T}]$, we have $(d_i(t))_{i \in \mathbb{N}} \in l^2$. The definition of $d_i(T)$ and $H_i(T)$ imply

$$d_i(T) = \langle u, (S_T^*)^{-1} z_i \rangle_{(0,T)} = \langle S_T^{-1} u, z_i \rangle_{(0,T)}.$$

Let $T_1, T_2 \in [\underline{T}, \overline{T}]$, $T_1 < T_2$. Then Lemma 2.1 implies

$$\begin{aligned}d_i(T_2) - d_i(T_1) &= \langle S_{T_2}^{-1} u, z_i \rangle_{(0,T_2)} - \langle S_{T_2}^{-1} u, z_i \rangle_{(0,T_1)} \\ &= \langle S_{T_2}^{-1} u, z_i \rangle_{(T_1,T_2)}.\end{aligned}$$

Analogously to Lemma 5.1 we can prove (by replacing $H_i(T)$ by z_i) that for all $v \in Z(T_1, T_2)$ we have

$$\sum_{i=1}^{\infty} \langle v, z_i \rangle_{(T_1,T_2)}^2 \leq P^2 \|v\|_{(T_1,T_2)}.$$

This implies

$$\begin{aligned}\sum_{i=1}^{\infty} (d_i(T_2) - d_i(T_1))^2 &= \sum_{i=1}^{\infty} \langle S_{T_2}^{-1} u, z_i \rangle_{(T_1,T_2)}^2 \\ &\leq \hat{P}^2 \|u\|_{(T_1,T_2)}^2.\end{aligned}$$

On account of

$$\lim_{t \longrightarrow T, t \in [\underline{T}, \overline{T}]} \|u\|_{(t,T)} = 0,$$

the assertion follows. □

LEMMA 6.8 (UPPER SEMICONTINUITY) *The function ω is upper semicontinuous on $[\underline{T}, \overline{T}]$.*

Proof. Let $T \in [\underline{T}, \overline{T}]$ and a sequence $(T_j)_{j \in \mathbb{N}} \in [\underline{T}, \overline{T}]^{\mathbb{N}}$ converging to T be given. Then for all $u \in Z(0, \overline{T})$, the following statement holds:

$$\lim_{j \longrightarrow \infty} \|u\|_{(0,T_j)} = \|u\|_{(0,T)}.$$

Moreover, Lemma 6.7 implies

$$\begin{aligned}\lim_{k \longrightarrow \infty} \sum_{j=1}^{\infty} \left(\langle b, H_j(T_k) \rangle_{(0,T_k)} - \langle b, H_j(T) \rangle_{(0,T)} \right)^2 &= 0 \text{ and} \\ \lim_{k \longrightarrow \infty} \sum_{j=1}^{\infty} \left(\langle u, H_j(T_k) \rangle_{(0,T_k)} - \langle u, H_j(T) \rangle_{(0,T)} \right)^2 &= 0.\end{aligned}$$

Let $(\nu^j)_{j \in \mathbb{N}} \in (l^2)^{\mathbb{N}}$ be a weakly convergent sequence converging to the limit ν^*.

Then for ϕ as defined in (7) we have

$$\begin{aligned}&\lim_{k \longrightarrow \infty} \phi(T_k, u|_{[0,T_k]}, \nu^k) = \lim_{k \longrightarrow \infty} \|u\|_{(0,T_k)}^2 \\ &+2 \sum_{j=1}^{\infty} \nu_j^k \left(c_j - \langle b, H_j(T_k) \rangle_{(0,T_k)} - \langle u, H_j(T_k) \rangle_{(0,T_k)} \right) \\ &= \phi(T, u|_{(0,T)}, \nu^*),\end{aligned}$$

i.e. the map

$$(T, \nu) \mapsto \phi(T, u|_{(0,T)}, \nu), \ [\underline{T}, \overline{T}] \times l^2 \longrightarrow \mathbb{R}$$

is sequentially weakly continuous. Statement (8) implies

$$h(T,\nu) = \inf_{u \in Z(0,\overline{T})} \phi(T, u|_{(0,T)}, \nu).$$

Hence h is the infimum of sequentially weakly continuous maps. Thus Proposition 1.5.12 in Pedersen (1988) implies that h is sequentially weakly upper semicontinuous, i.e.

$$\limsup_{j \longrightarrow \infty} h(T_j, \nu^j) \leq h(T, \nu^*).$$

For $t \in [\underline{T}, \overline{T}]$, let $\alpha(t) = (\alpha_i(t))_{i \in I\!N}$. According to Lemma 6.6 there exists $r \in I\!R$ such that for all k we have

$$\sum_{i=1}^{\infty} (\alpha_i(T_k))^2 \leq r.$$

Hence there exists a subsequence $(t_j)_{j \in I\!N}$ of $(T_j)_{j \in I\!N}$ for which

$$\limsup_{k \longrightarrow \infty} h(T_k, \alpha(T_k)) = \lim_{k \longrightarrow \infty} h(t_k, \alpha(t_k))$$

and such that the sequence $(\alpha(t_k))_{k \in I\!N} \in (l^2)^{I\!N}$ converges weakly to a point $\alpha^* \in l^2$. Then, due to Lemma 6.4 we have

$$\begin{aligned} \limsup_{k \longrightarrow \infty} \omega(T_k) + \beta^2 &= \limsup_{k \longrightarrow \infty} h(T_k, \alpha(T_k)) \\ &= \lim_{k \longrightarrow \infty} h(t_k, \alpha(t_k)) \\ &\leq h(T, \alpha^*) \\ &\leq \omega(T) + \beta^2. \end{aligned}$$

Hence $\limsup_{k \longrightarrow \infty} \omega(T_k) \leq \omega(T)$, i.e. ω is upper semicontinuous on $[\underline{T}, \overline{T}]$. □

Now we state the main result of this section.

THEOREM 6.1 (CONTINUITY) *The function ω is continuous on the interval $[\underline{T}, \overline{T}]$.*

Proof. Lemma 6.2 and Lemma 6.8 together yield the assertion. □

LEMMA 6.9 *If $T^* > \underline{T}$, then $\omega(T^*) = 0$.*

Proof. Assumption (A0) implies $T^* \leq \overline{T}$.

By Lemma 5.2 the set $U(T, \beta, c)$ is nonempty if and only if

$$\omega(T) = \|S_T u_*(T) - b\|^2_{(0,T)} - \beta^2 = \|v_*(T)\|_{(0,T)} - \beta^2 \leq 0.$$

Hence the definition of T^* implies

$$T^* = \inf\{T \in [\underline{T}, \overline{T}] : \omega(T) \leq 0\}. \tag{9}$$

Thus, if $T^* > \underline{T}$, the continuity of ω implies $\omega(T^*) = 0$. □

7. Continuity of the value function for the discretized problem

LEMMA 7.1 *For all $N \in \mathbb{N}$, the function ω_N is continuous on the interval $[\underline{T}, \overline{T}]$.*

Proof. The assertion follows analogously to Theorem 6.1, by replacing the infinite series by finite sums and the infinite systems of moment equations by the corresponding finite systems. The dual solutions of problem $P_N(T)$ are elements of $\mathbb{R}^N$. □

8. Uniform convergence of the value functions for the discretized problems

In this section we present the result that is announced in the title of the present paper, a theorem about uniform convergence of the optimal value functions for the discretized problems. This theorem shows that if the discretization level is large enough, the discretized problem yields an arbitrarily good approximation for the optimal value function ω, uniformly on the whole interval $[\underline{T}, \overline{T}]$.

THEOREM 8.1 (UNIFORM CONVERGENCE) *The sequence $(\omega_N)_{N\in\mathbb{N}}$ converges uniformly and monotone to ω on $[\underline{T}, \overline{T}]$.*

Proof. The definitions of $P_\infty(T)$ and $P_N(T)$ imply that for all $N \in \mathbb{N}$ the following inequality holds:

$$\omega_N(T) \le \omega_{N+1}(T) \le \omega(T).$$

Hence for all $T \in [\underline{T}, \overline{T}]$, the sequence $(\omega_N(T))_{N\in\mathbb{N}}$ is convergent and

$$\lim_{N\longrightarrow\infty} \omega_N(T) \le \omega(T).$$

The proof of Lemma 5.2 implies that

$$\begin{aligned}\lim_{N\longrightarrow\infty} \omega_N(T) - \beta^2 &= \liminf_{N\longrightarrow\infty} \|v_N(T)\|^2_{(0,T)} \\ &\ge \|v_*(T)\|^2_{(0,T)} \\ &= \omega(T) - \beta^2,\end{aligned}$$

where we have used the fact that the function $\|\cdot\|_{(0,T)}$ is sequentially weakly lower semicontinuous. Hence for all $T \in [\underline{T}, \overline{T}]$, we have

$$\lim_{N\longrightarrow\infty} \omega_N(T) = \omega(T).$$

Thus the sequence of functions $(\omega_N)_{N\in\mathbb{N}}$ converges pointwise to the function ω. By Lemma 7.1, for all $N \in \mathbb{N}$ the functions ω_N are continuous. By Theorem 6.1, the limit function ω is also continuous. Hence Dini's Theorem (see Pedersen, 1988) implies the uniform convergence. □.

In the last theorem, we summarize our results.

THEOREM 8.2 *For all $N \in I\!N$, the optimal value functions ω_N of the discretized problems are continuous. The value function ω of the original problem is also continuous.*

The sequence $(\omega_N)_{N\in I\!N}$ converges uniformly and monotone to ω on $[\underline{T}, \overline{T}]$.

REMARK 8.1 For $N \in I\!N$, $T \in [\underline{T}, \overline{T}]$ define $\Omega(N,T) = \omega_N(T)$ and let $\Omega(\infty, T) = \omega(T)$. Then Theorem 8.2 implies that for all sequences $(N_k)_{k\in I\!N}$ with $N_k \in I\!N \cup \{\infty\}$, $(T_k)_{k\in I\!N}$ where $T_k \in [\underline{T}, \overline{T}]$ with

$$\lim_{k\longrightarrow\infty} (N_k, T_k) = (M, S) \in (I\!N \cup \{\infty\}) \times [\underline{T}, \overline{T}]$$

the statement

$$\lim_{k\longrightarrow\infty} \Omega(N_k, T_k) = \Omega(M, S), \tag{10}$$

holds, that is the function Ω is continuous on $(I\!N \cup \{\infty\}) \times [\underline{T}, \overline{T}]$. For $M \in I\!N$, (10) is equivalent to the continuity of ω_M. If $N_k = \infty$ for all $k \in I\!N$, (10) is equivalent to the continuity of ω. Using the compactness of $[\underline{T}, \overline{T}]$, we can also deduce from (10) the equation

$$\lim_{k\longrightarrow\infty} \max_{T\in[\underline{T},\overline{T}]} |\Omega(N_k, T) - \Omega(\infty, T)| = 0,$$

i.e. the uniform convergence of the sequence $(\omega_N)_{N\in I\!N}$ to ω.

Hence except for the statement about monotone convergence, Theorem 8.2 is equivalent to the statement that the function Ω is continuous on $(I\!N\cup\{\infty\}) \times [\underline{T}, \overline{T}]$. Note, however, that in the proof of Theorem 8.1 Dini's Theorem can only be applied due to the fact that for fixed $T \in [\underline{T}, \overline{T}]$, the sequence of numbers $(\omega_N(T))_{N\in I\!N}$ is increasing. Moreover, in the proof of the continuity of ω, we have used the fact that for fixed $T \in [\underline{T}, \overline{T}]$, $P_\infty(T)$ is a convex problem.

Continuity results of the type of Theorem 8.2 are well–known in different settings, for example Theorem 5.5.1 from Rolewicz (1987). This theorem basically states that with feasible sets that give a continuous set–valued map of the parameter, the corresponding optimal value function is continuous.

To show that the feasible set map is continuous, both lower and upper semicontinuity of the set–valued map has to be shown. This approach requires at least as much work as to show that the optimal value function is both upper and lower semicontinuous, as we have done.

The purpose of this paper is to examine the behaviour of optimal value functions that occur if the method of moments is used so that the moment equations appear as constraints. This problem is important since the method of moments is suitable for a numerical treatment of problems of time–optimal control and in this approach the optimal value functions that we consider occur in a natural way.

To compute T^* numerically, we consider the sequence $(T_N^*)_{N\in I\!N}$ defined as follows. For $N \in I\!N$, let

$$T_N^* = \inf\{T \in [\underline{T}, \overline{T}] : \omega_N(T) \leq 0\}.$$

Since $\omega_N \leq \omega_{N+1} \leq \omega$, for all $N \in I\!N$ we have $T_N^* \leq T_{N+1}^* \leq T^*$. Hence $\lim_{N\longrightarrow\infty} T_N^* \leq T^*$.

Analogously to Lemma 6.9 we can prove the following: If $T_N^* > \underline{T}$, then the equation $\omega_N(T_N^*) = 0$ holds. Hence if there exists $N_0 \in I\!N$ such that $T_{N_0}^* > \underline{T}$, Theorem 8.1 yields

$$\omega(\lim_{N\longrightarrow\infty} T_N^*) = \lim_{N\longrightarrow\infty} \omega_N(T_N^*) = 0;$$

since $\lim_{N\longrightarrow\infty} T_N^* \leq T^*$, by (9) this yields

$$\lim_{N\longrightarrow\infty} T_N^* = T^*.$$

This implies the following Lemma.

LEMMA 8.1 *If $T^* > \underline{T}$ the sequence $(T_N^*)_{N\in I\!N}$ converges monotonically to T^* and for N large enough, we have $\omega_N(T_N^*) = 0$.*

For the problem of time–minimal control of an Euler–Bernoulli beam, Lemma 8.1 has been stated in Krabs (1996).

9. Lipschitz and Hölder conditions

In this section, we consider the standard minimum norm problem

$$Q_\infty(T): \quad \min \|u\|_{(0,T)}^2 \ \text{ s.t.}$$

$$\langle u, z_j\rangle_{(0,T)} = c_j \quad (j \in I\!N)$$

for $T \in [\underline{T}, \overline{T}]$ with optimal value $\varphi(T)$:

$$\varphi(T) = \min\{\|u\|_{(0,T)}^2 : u \in Z(0,T),\ \langle u, z_j\rangle_{(0,T)} = c_j \ \ (j \in I\!N)\}.$$

We give an assumption that ensures that the optimal value function satisfies a certain Hölder condition with exponent 1/2. We also present an assumption that implies a certain Lipschitz condition. Our assumptions are regularity conditions for the solutions of problem $Q_\infty(T)$.

We need some additional notation. Let a sequence $(\lambda_j)_{j\in I\!N}$ of numbers greater than or equal to 1 be given. Assume that there is a number $s > 0$ such that

$$\sum_{i=1}^\infty \frac{1}{\lambda_i^s} < \infty.$$

For a sequence $(a_j)_{j\in\mathbb{N}}$ of real numbers and $r \in \mathbb{R}$ let

$$\|c\|_r = \left(\sum_{j=1}^{\infty} |a_j|^2 (\lambda_j)^r\right)^{1/2}.$$

Define the space of sequences

$$l_r^2 = \{(a_j)_{j\in\mathbb{N}} : \|a\|_r < \infty\}.$$

For $t \in [\underline{T}, \overline{T}]$, define the linear operator $A(t) : l^2 \longrightarrow l^2$, $A(t)\alpha = \left(\sum_{j=1}^{\infty} \alpha_j \langle z_i, z_j\rangle_{(0,t)}\right)_{i\in\mathbb{N}}$.

Up to now we have studied the optimal value function. The following lemma contains a result about the sensitivity of the optimal solutions with respect to the parameter t.

LEMMA 9.1 *Let $c \in l^2$. For $t \in [\underline{T}, \overline{T}]$, let $\eta(t) = A(t)^{-1}c$. As before, assume that A1 and A2 hold. Then for all t_1, $t_2 \in [\underline{T}, \overline{T}]$, the following inequality is valid:*

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \le M^2 P \|\sum_{j=1}^{\infty} \eta_j(t_1) z_j\|_{(t_1,t_2)}.$$

In particular, this implies

$$\lim_{t_2 \longrightarrow t_1} \|\eta(t_1) - \eta(t_2)\|_{l^2} = 0.$$

Moreover, if the functions z_i are continuous and

$$\max_{t\in[0,\overline{T}]} |z_i(t)| \le 1, \quad i \in \mathbb{N}, \tag{11}$$

and for some $r \ge s$ the sequence c is in $A(t_1)(l_r^2)$, then the following inequalities hold:

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \le \sqrt{|t_1 - t_2|} M^2 P \|\eta(t_1)\|_{l_r^2} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r}\right)^{1/2} \tag{12}$$

$$|\varphi(t_1) - \varphi(t_2)| \le \sqrt{|t_1 - t_2|} M^2 P \|c\|_{l^2} \|\eta(t_1)\|_{l_r^2} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r}\right)^{1/2} \tag{13}$$

Inequality (13) shows that the optimal value function φ satisfies a Hölder condition with exponent $1/2$.

Proof. The definition of $\eta(t)$ implies

$$\begin{aligned} & A(t_2)(\eta(t_1) - \eta(t_2)) \\ = \; & A(t_2)\eta(t_1) - c \\ = \; & (A(t_2) - A(t_1))\eta(t_1) \\ = \; & \left(\langle z_i, \sum_{j=1}^{\infty} \eta_j(t_1) z_j \rangle_{(t_1,t_2)} \right)_{i \in I\!N} . \end{aligned}$$

Let $u = \sum_{j=1}^{\infty} \eta_j(t_1) z_j \in L^2[0, \overline{T}]$. Then Lemma 5.1 implies

$$\|A(t_2)(\eta(t_1) - \eta(t_2))\|_{l^2}^2 = \sum_{i=1}^{\infty} \langle z_i, u \rangle_{(t_1,t_2)}^2 \leq P^2 \|u\|_{(t_1,t_2)}^2 .$$

Hence the following inequality holds:

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \leq M^2 \|A(t_2)(\eta(t_1) - \eta(t_2)\|_{l^2} \leq M^2 P \|u\|_{(t_1,t_2)},$$

and the first assertion follows. Due to (11) we have

$$\begin{aligned} \|u\|_{(t_1,t_2)}^2 & = |\int_{t_1}^{t_2} u(s)^2 \, ds| \\ & \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| \eta_i(t_1) \langle z_i, z_j \rangle_{(t_1,t_2)} \eta_j(t_1) \right| \\ & \leq \left(\sum_{i=1}^{\infty} |\eta_i(t_1)| \right) \left(\sum_{j=1}^{\infty} |\eta_j(t_1)| \right) |t_1 - t_2| \\ & \leq \left(\sum_{i=1}^{\infty} |\eta_i(t_1)| \lambda_i^{r/2} \lambda_i^{-r/2} \right)^2 |t_1 - t_2| \\ & \leq \|\eta(t_1)\|_{l_r^2}^2 \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r} \right) |t_1 - t_2|, \end{aligned}$$

hence if $\eta(t_1) \in l_r^2$, (12) follows. Now (13) is a consequence of the equation $\varphi(t_1) - \varphi(t_2) = c^T(\eta(t_1) - \eta(t_2))$ and the Cauchy–Schwarz inequality. □

The proof of Lemma 9.1 only works for the standard minimum norm problem $Q_\infty(T)$ and not for problem $P_\infty(T)$.

Note that the dual space of l_r^2 is l_{-r}^2.

LEMMA 9.2 *For* $t \in [\underline{T}, \overline{T}]$, $\alpha \in l_r^2$, *let*

$$(D(t)\alpha)_i = \sum_{i=1}^{\infty} z_i(t) z_j(t) \alpha_j .$$

Assume that (11) holds. Let $r \geq s$. Then $D(t)$ is a continuous linear map from l^2_r into l^2_{-r} and for all $\alpha \in l^2_r$

$$\|D(t)\alpha\|_{l^2_{-r}} \leq \|\alpha\|_{l^2_r} \left(\sum_{i=1}^{\infty} \lambda_i^{-r} \right). \tag{14}$$

Proof. Let $\beta \in l^2_r$. Then

$$\begin{aligned}
|\beta^T D(t)\alpha| &= |\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \beta_i z_i(t) z_j(t) \alpha_j| \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\beta_i \alpha_j| \\
&= \left(\sum_{i=1} |\beta_i|\right)\left(\sum_{j=1}^{\infty} |\alpha_j|\right) \\
&= \left(\sum_{i=1}^{\infty} |\beta_i| \lambda_i^{r/2} \lambda_i^{-r/2}\right)\left(\sum_{j=1}^{\infty} |\alpha_j| \lambda_j^{r/2} \lambda_j^{-r/2}\right) \\
&\leq \|\beta\|_{l^2_r} \left(\sum_{i=1}^{\infty} \lambda_i^{-r}\right) \|\alpha\|_{l^2_r},
\end{aligned}$$

where for the last line we have applied the Cauchy–Schwarz inequality twice. Hence the inequality (14) follows. □

LEMMA 9.3 *Assume that (A1) and (11) hold. Assume that the functions z_i are continuously differentiable with*

$$\max_{t \in [0,\overline{T}]} |z_i'(t)| \leq \sqrt{\lambda_i}, \quad i \in I\!N. \tag{15}$$

Let $r \geq s+1$. For $t \in [\underline{T}, \overline{T}]$, $\alpha \in l^2_r$, let

$$\overline{A}(t)\alpha = \left(\sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t)} \alpha_j \right)_{i \in I\!N}.$$

Then $\overline{A}(t)$ is a bounded linear operator from l^2_r into l^2_{-r}. $\overline{A}(t)$ is Fréchet-differentiable with respect to t, and

$$\left(\overline{A}'(t)\alpha\right)_i = \sum_{i=1}^{\infty} z_i(t) z_j(t) \alpha_j = (D(t)\alpha)_i.$$

Proof. Due to (A1), for $\alpha \in l_r^2$ we have

$$\|\overline{A}(t)\alpha\|_{l^2_{-r}} \leq \|\overline{A}(t)\alpha\|_{l^2} \leq P^2\|\alpha\|_{l^2} \leq P^2\|\alpha\|_{l^2_r}.$$

Let $h \neq 0$ be such that $t + h \in [\underline{T}, \overline{T}]$. The Taylor–expansion implies the existence of numbers $\xi_{ij} \in (0, \overline{T})$ such that

$$\left|\frac{1}{h}\langle z_i, z_j\rangle_{(t,t+h)} - z_i(t)z_j(t)\right|$$

$$= \frac{|h|}{2}\left|z_i(\xi_{ij})z_j'(\xi_{ij}) + z_i'(\xi_{ij})z_j(\xi_{ij})\right| \leq \frac{|h|}{2}(\sqrt{\lambda_i} + \sqrt{\lambda_j}).$$

Let $\alpha \in l_r^2$. Define $a_i = \sum_{j=1}^{\infty}(\sqrt{\lambda_i} + \sqrt{\lambda_j})\alpha_j$. Then for all $\beta \in l_r^2$, the following inequality holds:

$$\begin{aligned}\left|\sum_{i=1}^{\infty} a_i\beta_i\right| &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\beta_i|(\sqrt{\lambda_i} + \sqrt{\lambda_j})|\alpha_j| \\ &\leq \left(\sum_{i=1}^{\infty}|\beta_i|\sqrt{\lambda_i}\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\right) \\ &\quad + \left(\sum_{i=1}^{\infty}|\beta_i|\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\sqrt{\lambda_j}\right).\end{aligned}$$

For $q \geq s$, we define a positive number C_q by the equation

$$C_q = \left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^q}\right)^{1/2} < \infty. \tag{16}$$

For $\gamma \in l_r^2$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}|\gamma_i|\sqrt{\lambda_i} &= \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{r/2}\lambda_i^{(1-r)/2} \\ &\leq \|\gamma\|_{l_r^2}\left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{r-1}}\right) = \|\gamma\|_{l_r^2}C_{r-1}.\end{aligned}$$

Moreover,

$$\sum_{i=1}^{\infty}|\gamma_i| \leq \|\gamma\|_{l_r^2}C_r.$$

Hence for all $\beta \in l_r^2$ we have the inequality

$$\left|\sum_{i=1}^{\infty} a_i\beta_i\right| \leq \|\beta\|_{l_r^2}\|\alpha\|_{l_r^2}2C_{r-1}C_r.$$

Thus we conclude that

$$\|(a_i)_i\|_{l^2_{-r}} \le 2\|\alpha\|_{l^2_r} C_{r-1} C_r.$$

Then we obtain the statement

$$\begin{aligned}
& \| \left[\frac{\overline{A}(t+h) - \overline{A}(t)}{h} - D(t) \right] \alpha \|_{l^2_{-r}} \\
= & \| \left(\sum_{j=1}^{\infty} [\frac{1}{h} \langle z_i, z_j \rangle_{(t,t+h)} - z_i(t) z_j(t)] \alpha_j \right)_i \|_{l^2_{-r}} \\
\le & \| \left(\frac{h}{2} \sum_{j=1}^{\infty} (\sqrt{\lambda_i} + \sqrt{\lambda_j}) \alpha_j \right)_i \|_{l^2_{-r}} \\
= & \frac{|h|}{2} \|(a_i)_i\|_{l^2_{-r}} \\
\le & |h| \; \|\alpha\|_{l^2_r} C_{r-1} C_r.
\end{aligned} \tag{17}$$

So for $h \longrightarrow 0$ the assertion that $\overline{A}$ is Fréchet-differentiable in t follows. □

The following theorem contains a sufficient condition for a kind of Lipschitz condition for φ.

THEOREM 9.1 *Let $r \ge s+1$. Assume that (A1), (11) and (15) hold. Let $t \in [\underline{T}, \overline{T})$ be such that $c \in A(t)(l^2_r)$. Then there exists a constant $L(t) > 0$ such that for all $t_2 \in (t, \overline{T}]$, the following inequality is valid:*

$$\varphi(t) \ge \varphi(t_2) \ge \varphi(t) - L(t)\,(t_2 - t).$$

Proof. Let $t_2 \in (t, \overline{T}]$ and $h = t_2 - t > 0$. Let u_* be the solution of $Q_\infty(t)$. Define $\hat{u}(s) := u_*(s)$, if $s \in [0,t]$, $\hat{u}(s) := 0$ if $s \in (t, t_2]$. Then for all $i \in I\!N$ we have

$$\langle \hat{u}, z_i \rangle_{(0,t_2)} = \langle u_*, z_i \rangle_{(0,t)} = c_i,$$

hence

$$\varphi(t_2) \le \|\hat{u}\|^2_{(0,t+h)} = \|u_*\|^2_{(0,t)} = \varphi(t).$$

Moreover, due to Lemma 6.4 we obtain the statement

$$\begin{aligned}
\frac{\varphi(t+h) - \varphi(t)}{h} = & \frac{1}{h} \left(\sup_{\alpha \in l^2} - \sum_{i,j=1}^{\infty} \alpha_i \alpha_j \langle z_i, z_j \rangle_{(0,t+h)} + 2 \sum_{j=1}^{\infty} \alpha_j c_j \right. \\
& \left. + \sum_{i,j=1}^{\infty} \eta_i(t) \eta_j(t) \langle z_i, z_j \rangle_{(0,t)} - 2 \sum_{j=1}^{\infty} \eta_j(t) c_j \right)
\end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{1}{h}\sum_{i,j=1}^{\infty}\eta_i(t)\eta_j(t)\langle z_i, z_j\rangle_{(t,t+h)}\\
&= -\eta(t)^T\frac{A(t+h)-A(t)}{h}\eta(t)\\
&= -\eta(t)^T\left(\frac{A(t+h)-A(t)}{h} - D(t)\right)\eta(t)\\
&\quad -\eta(t)^T D(t)\eta(t)\\
&\geq -\eta(t)^T D(t)\eta(t) - \|\eta(t)\|_{l_r^2}^2 C_{r-1}C_r|h|,
\end{aligned}
$$

where the last line follows from (17).

Let $L(t) = \eta(t)^T D(t)\eta(t) + \|\eta(t)\|_{l_r^2}^2 C_{r-1}C_r[\overline{T} - \underline{T}] > 0$. Then

$$\varphi(t+h) \geq \varphi(t) - L(t)\, h,$$

and the assertion follows. $\square$

REMARK 9.1 The fact that φ is decreasing is well-known, but the lower bound for $\varphi(t_2)$ in Theorem 9.1 appears to be new.

Conditions (11) and (15) hold for trigonometric moment problems of the form

$$
\begin{aligned}
\int_0^T u(t)\sin(\sqrt{\lambda_i}t)\,dt &= c_{2i-1},\\
\int_0^T u(t)\cos(\sqrt{\lambda_i}t)\,dt &= c_{2i},\ i \in I\!N
\end{aligned}
$$

that appear for example in the characterization of the set of feasible controls for the exact control of hyperbolic partial differential equations (see, for example Krabs, 1982).

References

BONNANS, J.F. and SHAPIRO, A. (1998) Optimization Problems with Perturbations: A guided tour. *SIAM Review*, **40**, 228–264.

GUERRE-DELABRIERE, S. (1992) *Classical Sequences in Banach Spaces.* Marcel Dekker, New York.

GUGAT, M. (1994) One-sided derivatives for the value function in convex parametric programming. *Optimization*, **28**, 301–314.

INGHAM, A.E. (1936) Some trigonometrical inequalities with applications to the theory of series. *Math. Z.*, **41**, 367–379.

KRABS, W. (1981) Time-minimal contollability in the view of optimization. In: *Optimization and Optimal Control*, A. Auslender, W. Oettli, J. Stoer, eds., Springer-Verlag, 211–217.

KRABS, W. (1982) Optimal control of processes goverend by partial differential equations part II: Vibrations. *Zeitschrift für Operations Research*, **26**, 63–86.

KRABS, W. (1992) *On Moment Theory and Controllability of One-Dimensional Vibrating Systems and Heating Processes.* Lecture Notes in Control and Information Sciences, Springer, New York.

KRABS, W. (1993) Controllability of a rotating beam. *Lecture Notes in Control and Information Sciences*, **185**, 447–458.

KRABS, W. (1996) Time-minimal control of rotating beams. In: *Control of partial differential equations and applications*, E. Casas, ed., Marcel Dekker Inc., New York, 147–156.

LEMPIO, F. and MAURER, H. (1980) Differential stability in infinite dimensional nonlinear programming, *Appl. Math. Optim.*, **6**, 139–152.

PEDERSEN, G.K. (1988) *Analysis Now.* Springer-Verlag, New York.

ROLEWICZ, S. (1987) *Functional Analysis and Control Theory.* PWN and D. Reidel Publ. Company, Warszawa–Dordrecht.

RUSSEL, D.L. (1967) Nonharmonic fourier series in the control theory of distributed parameter systems. *Journal of Mathematical Analysis and Applications*, **18**, 542–560.

VASIN, V.V. and AGEEV, A.L. (1995) *Ill-posed problems with a priori information.* VSP, Utrecht.

YOUNG, R.M. (1980) *An Introduction to Nonharmonic Fourier Series.* Academic Press, New York.